Spectral Theory of Non-Markovian Dissipative Phase Transitions

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Motivations

- Dissipative phase transitions (DPTs) are mostly studied under the Markovian approximation.
- However, recent studies [1, 2] suggest that a Lindblad generator cannot always capture DPTs.
- Here, we propose a general theory and method to describe DPTs in non-Markovian systems.

Theoretical framework: HEOM

We consider an arbitrary quantum system S linearly coupled to a bosonic bath B at zero temperature. The total Hamiltonian takes the form ($\hbar = 1$)

$$H = H_S + H_B + H_{\text{int}} = H_S + \sum_k \omega_k a_k^\dagger a_k + \sum_k (g_k a_k L_k^\dagger + g_k^* a_k^\dagger L_k).$$

Assuming an initial product state $\rho(0) = \rho_S(0) \otimes \rho_B(0)$ and a correlation function that can be written as

$$\alpha(\tau) = \sum_k |g_k|^2 \langle a_k(t) a_k^\dagger(t+\tau)\rangle = \sum_{j=1}^{M} G_j e^{-i\omega_j \tau - \kappa_j |\tau|},$$

the dynamics of the system can be described by an exact method: the Hierarchical Equations Of Motion (HEOM)[3]

$$\frac{d\rho^{(\vec{n},\vec{m})}}{dt} = -i[H_S, \rho^{(\vec{n},\vec{m})}] - (\vec{w}\cdot\vec{n} + \vec{w}^*\cdot\vec{m})\rho^{(\vec{n},\vec{m})} + \sum_{j=1}^{M}\Big(G_j\left(n_j L_j \rho^{(\vec{n}-\vec{e_j},\vec{m})} + m_j \rho^{(\vec{n},\vec{m}-\vec{e_j})} L_j^\dagger\right) + [\rho^{(\vec{n}+\vec{e_k},\vec{m})}, L_j^\dagger] + [L_j, \rho^{(\vec{n},\vec{m}+\vec{e_j})}]\Big),$$

with $\vec{w} = (\kappa_j + i\omega_j) \in \mathbb{C}^M$, $\vec{n} = (n_j), \vec{m} = (m_j) \in \mathbb{N}^M$, $\rho^{(\vec{0},\vec{0})}$ the physical state.

An effective non-Markovian Liouvillian matrix can then be derived by vectorizing the matrices $\rho^{(\vec{n},\vec{m})} \to |\rho^{(\vec{n},\vec{m})}\rangle\rangle$ s.t.

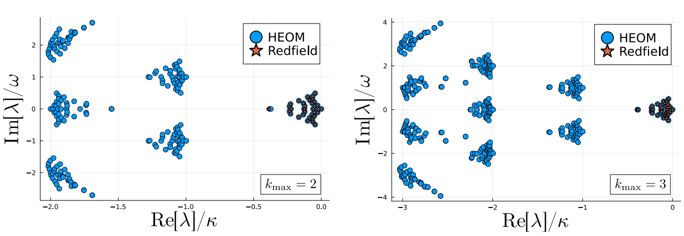
$$\frac{d|\rho\rangle\rangle}{dt} = \mathcal{L}_{\text{HEOM}}(k_{\max})|\rho\rangle\rangle,$$

where $|\rho\rangle\rangle$ is a stacked vector made of all the $|\rho^{(\vec{n},\vec{m})}\rangle\rangle$ and $k_{\max}$ is s.t. $|\vec{n}| + |\vec{m}| \le k_{\max}$

Spectral properties of $\mathcal{L}_{\text{HEOM}}$

- The eigenvalues λ_i are symmetric w.r.t the real axis.
- The eigenvalue $\lambda = 0$ is always in the spectrum of $\mathcal{L}_{\text{HEOM}}$.
- For $k_{\max} \to +\infty$, $\text{Re}[\lambda_i] \le 0 \ \forall \lambda_i$.

Illustration for the driven-dissipative Dicke model



The parameters, given by $\omega_0 = 0.1\kappa$, $\omega = 1.0\kappa$ and $g = 0.1\kappa$, are choosen in the domain of validity of the Redfield approach.

Definition of DPT and spectral gap

A DPT of order M is a non-analytical change in a g-independent system observable O when the parameter g tends to a critical value g_c for $N \to \infty$ [4], *i.e.*

$$\lim_{g\to g_c}\left|\frac{\partial^M}{\partial g^M}\lim_{N\to+\infty}\langle O\rangle_{ss}\right| = +\infty.$$

The spectral gap Δ is the maximum of $\text{Re}[\lambda_i]\ \forall \lambda_i \neq 0$.

$$\text{DPT} \implies \Delta \to 0 \text{ as } N \to +\infty$$

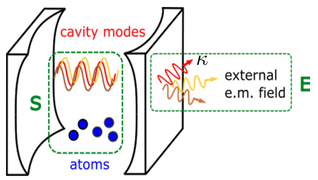
Second order DPT in the Dicke model

The Dicke Hamiltonian is given by

$$H_D = \omega_0 S_z + \omega a^\dagger a + \frac{2g}{\sqrt{N}}(a + a^\dagger)S_x,$$

where $S_i = \frac{1}{2}\sum_{k=1}^{N}\sigma_k^i$ are the collective spin operators, and can be realized in cavity QED systems.

There are two main choices for the open system description of atoms in a cavity.

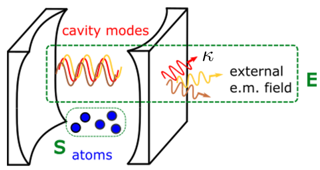


$H_S = \omega_0 S_z + \omega a^\dagger a + \frac{2g}{\sqrt{N}}(a + a^\dagger)S_x$

$L = a$

$\alpha(\tau) = \kappa\delta(\tau)$

→ Lindblad master equation (system = atoms+cavity)



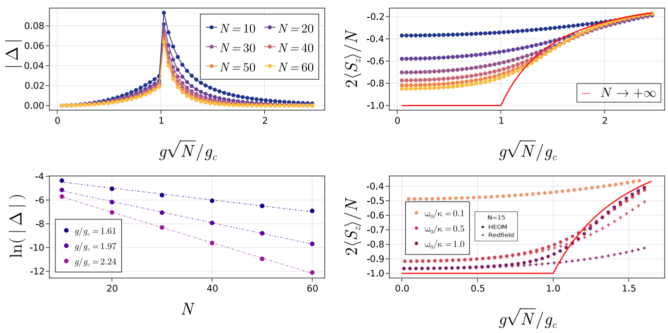
$H_S = \omega_0 S_z$

$L = 2S_x$

$\alpha(\tau) = g^2 e^{i\omega\tau - \kappa|\tau|}$

→ non-Markovian description (system = atoms)

Our exact method captures all the signatures of the DPT even beyond the weak coupling regime.



Conclusion and outlook

Conclusion

We showed that a HEOM-based approach of DPTs successfully captures DPTs in non-Markovian quantum systems.

Outlook

- Application of our method to other models (*e.g.* a two-mode U(1)-symmetric Dicke model [2]).
- Study in general the impact of non-Markovianity on DPTs.

References

[1] F. Damanet, A. J. Daley and J. Keeling, Phys. Rev. A **99**, 033845 (2019).
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[3] Y. Tanimura and R. Kubo, J. Phys. Soc. Jpn. 58 **58**, 1199 (1989).
[4] F. Minganti *et al.*, Phys. Rev. A **98**, 042118 (2018).