Entanglement classification schemes : comparison between Majorana representation and algebraic geometry approaches

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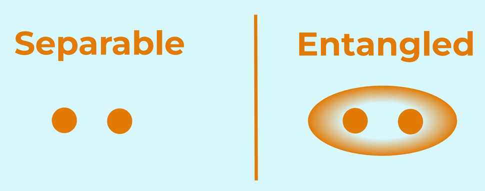
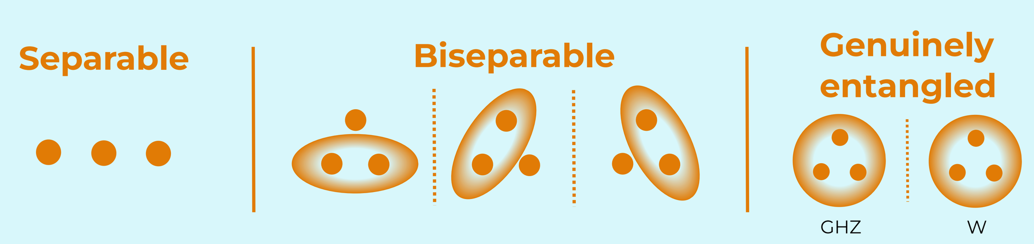
1 - Background

- Not all N-qubit entangled states are entangled in the same way with respect to SLOCC, where SLOCC stands for **S**tochastic **L**ocal **O**peration with **C**lassical **C**ommunication.
- Two N-qubit states $|\psi\rangle$ and $|\phi\rangle$ are said to be **SLOCC-equivalent** if one can switch from one to the other via local transformations on the subparts, potentially assisted by classical communication between those subparts. SLOCC-equivalent states form a SLOCC-class.
- For N=2, there are **two** entanglement classes [1] : separable and entangled classes.



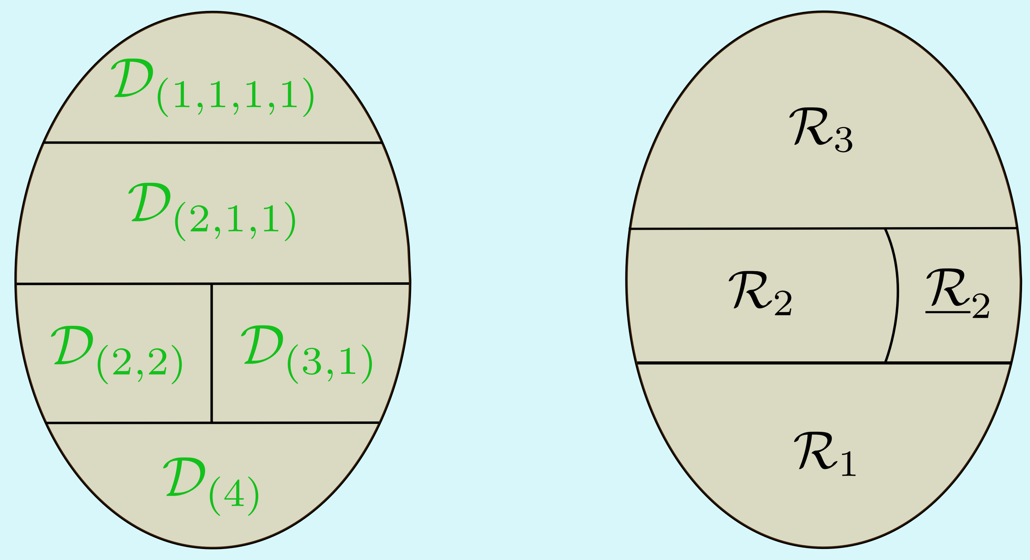
- For N=3, there are **six** entanglement classes [1] : separable(1), biseparable(3) and genuinely entangled(2) classes.



- For $N \geq 4$ there is an **infinity** of entanglement classes [1].

⟶ Given an N-qubit state, it is hard to identify its SLOCC class if $N \geq 4$.

- We can gather the infinite number of SLOCC classes into a **finite number of families** of SLOCC classes. Depending on the methodology used to do so, **different classification structures may** emerge (See [2] for a review of existing structures).
- Gathering SLOCC classes is simpler in the particular case of **symmetric** qubit states, i.e. states invariant under any permutation of their subparts. **We work here exclusively with symmetric states** (Hilbert space $\mathcal{H}_S \simeq S^N\mathbb{C}^2$).
- Here we focus on two specific classification schemes [3,4] presented as **partition structures** :



Majorana representation (MR) based structure for N=4 [3]

Symmetric rank (SR) based structure for N=4 [4]

- Any N-qubit symmetric state $|\psi_S\rangle$ can be written
$$|\psi_S\rangle = \mathcal{N} \sum_{1\leq i_1 \neq \ldots \neq i_N \leq N} |\epsilon_{i_1} \cdots \epsilon_{i_N}\rangle$$
where the $|\epsilon_i\rangle$'s are single qubit states.
- Several $|\epsilon_i\rangle$ states may be identical and we call their number the **degeneracy number** λ_i.
- We define the **degeneracy configuration** $\mathcal{D}_\lambda$ of $|\psi_S\rangle$ as the list of its degeneracy numbers λ_i, that is a **partition** λ of N.
- SLOCC-equivalent states have the same degeneracy configuration. The degeneracy configuration can be used to define **families** of SLOCC classes of states.

- The **symmetric rank** R_S of $|\psi_S\rangle$ is the smallest integer r s.t.
$$|\psi_S\rangle = c_1|\epsilon_1\rangle^{\otimes N} + \cdots + c_r|\epsilon_r\rangle^{\otimes N}$$
where the $|\epsilon_i\rangle$'s are single qubit states.
- The **symmetric border rank** $\underline{R}_S$ of $|\psi_S\rangle$ is the smallest integer r such that $|\psi_S\rangle$ is a limit of symmetric states of symmetric tensor rank r.
- SLOCC equivalent states have the same symmetric rank and symmetric border rank and can be used to define **families** of SLOCC classes of states.
- The families of SLOCC classes $\mathcal{R}_k$ are the sets of classes that gather all states of **symmetric rank k**.
- The families of SLOCC classes $\underline{\mathcal{R}}_k$ are the sets of classes that gather all states of **symmetric border rank k** with **symmetric rank $> k$**.

⟶ SLOCC classes are gathered into $\mathcal{D}_\lambda$ families in the MR structure, and into $\mathcal{R}_k$ and $\underline{\mathcal{R}}_k$ families in the SR structure.

2 - First question

Can MR and SR structures be embedded into one other ?

- **Challenge** : The **concepts** underlying these two structures are **distinct**.

⟶ We need to express one structure with the concepts of the other.

MR structure SLOCC-invariant is the degeneracy configuration.

SR structure SLOCC-invariants are the symmetric (border) rank.

What is the **link** between the degeneracy configuration and the symmetric (border) rank ?

3 - Answer : No

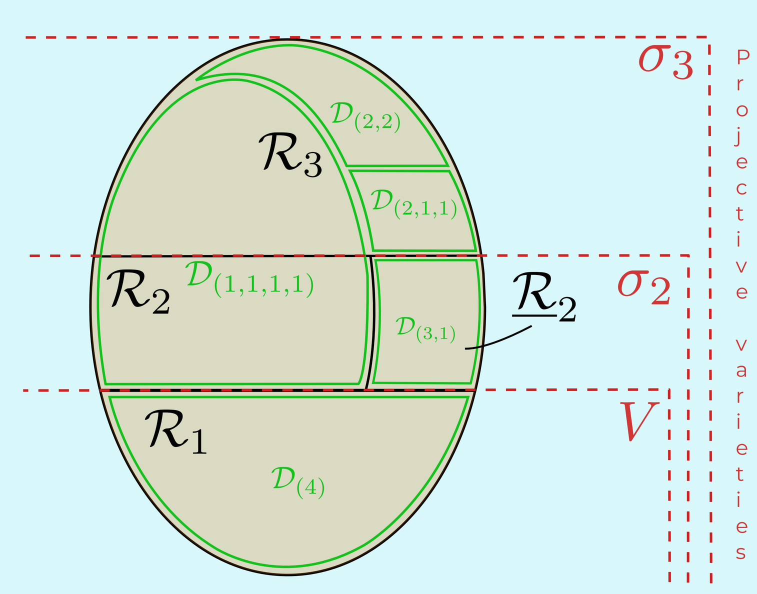
Method

- For each $\mathcal{D}_\lambda$ family, take a representative.
- Calculate the symmetric (border) rank of the representative.
- The SLOCC class of the representative belongs to the $\mathcal{R}_k$ or $\underline{\mathcal{R}}_k$ family according to its symmetric (border) rank k value.

Results

- $R_S(\psi_S) \leq N, \ \forall |\psi_S\rangle \in \mathcal{H}_S$ [5]
- $\underline{R}_S(\psi_S) \leq \lceil \frac{N+1}{2} \rceil, \ \forall |\psi_S\rangle \in \mathcal{H}_S$ [5]
- $\underline{R}_S(\psi_S) \leq R_S(\psi_S)$ [5]
- Either $R_S(\psi_S) = \underline{R}_S(\psi_S)$ or $R_S(\psi_S) = N - \underline{R}_S(\psi_S) + 2$ [5]
- $\underline{R}_S(D_N^{(k)}) = k+1, \ R_S(D_N^{(k)}) = N-k+1, \ \forall k \leq \lceil \frac{N}{2} \rceil$ [5]
- $\underline{R}_S(GHZ_N) = R_S(GHZ_N) = 2$

For N=4



⟶ **None** of the MR and SR structures is a refinement of the other.

4 - Second question

Can the MR structure be expressed in terms of projective varieties ?

- In the context of algebraic geometry, families of the SR structure can be expressed in terms of **projective varieties**, *i.e.* closed subsets of the projective space $\mathbb{P}(\mathcal{H})$, the space whose elements are lines of the Hilbert space $\mathcal{H}$.

- The **Veronese variety** V corresponds to the subspace of states $|\psi_S\rangle$ with $R_S(\psi_S) = 1$, *i.e.* symmetric separable states.
- The **k-secant** of the Veronese variety $\sigma_k(V) \equiv \sigma_k$ corresponds to the **closure** of the subset of states whose **symmetric rank is smaller or equal to k.**

Can this approach be extended to the MR structure ?

5 - Answer : Yes

Method

$$|\psi_S\rangle \xleftrightarrow{\text{Isomorphism}} P_{\psi_S}(x,y)$$

- The replacement
$$|1\rangle \to x$$
$$|0\rangle \to y$$
defines an **isomorphism** between the set of symmetric N-qubit states $\mathcal{H}_S$ and the set of homogeneous bivariate polynomials of degree N $\mathbb{C}[x,y]_N$.

$$|D_N^{(k)}\rangle \ \xrightarrow{|1\rangle \to x,\ |0\rangle \to y} \ \sqrt{\tbinom{N}{k}}\, x^k y^{N-k}$$

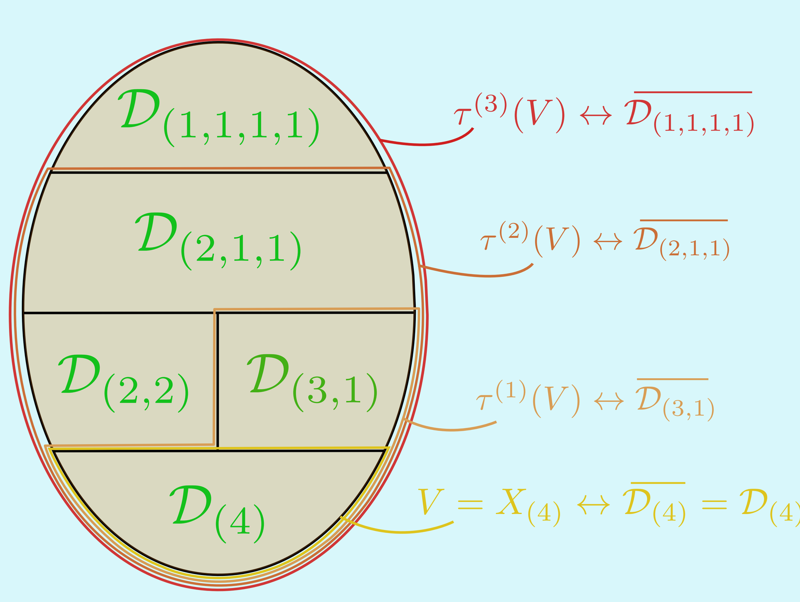
- Any homogeneous bivariate polynomial P of degree N is **decomposable** into a product of homogeneous bivariate polynomials of degree 1 :
$$P(x,y) = \prod_{i=1}^{n} P_i^{\lambda_i}(x,y) \qquad (1)$$
The number of occurrences λ_i of each polynomial P_i of degree 1 in this decomposition is called its **multiplicity.** We gather all multiplicities into a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of N.
- The **coincident root locus X_λ** is defined as the projective variety [6] that corresponds to the set of polynomials $P(x,y)$ that can be written in the form (1).
- The set of states corresponding to X_λ is $\overline{\mathcal{D}_\lambda}$:
$$X_\lambda \longleftrightarrow \overline{\mathcal{D}_\lambda}$$
- **Definitions** :
  1) Let X be a subset of bivariate polynomials. $\forall P \in X$, $T_P X$ is the span of all bivariate polynomials obtained as derivative at P of a smooth parametrized curve in X.
  2) Definition 1) can be extended to projective varieties X and enables to define the **tangent** variety $\tau(X) = \overline{\bigcup_{P \in X} T_P X}$.
  3) We define $\tau^{(k)}(X) = \underbrace{\tau(\ldots\tau}_{k \text{ times}}(X)\ldots)$.

Results

- **Theorem** [6] : $\tau(X_{(N-r,1^r)}) = X_{(N-r-1,1^{r+1})}, \ \forall r \in \{0, \ldots, N-2\}$
- **Corollary** : $\tau^{(k)}(X_{(N)}) = X_{(N-k,1^k)}, \ \forall k \in \{1, \ldots, N-1\}$

$$\longrightarrow \quad \boldsymbol{X_{(N-k,1^k)} = \tau^{(k)}(X_{(N)}) \longleftrightarrow \overline{\mathcal{D}_{(N-k,1^k)}}}$$

For N=4



6 - Conclusion

None of the MR and SR structures is a refinement of the other. Similarly to the SR structure, it is possible to formulate the MR structure in terms of projective varieties. However, in contrast to the SR structure, these varieties are obtained as ***tangent varieties*** rather than as ***secant varieties***.

References

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