LIÈGE université Mathematics

fnrs FREEDOM TO RESEARCH

Magic Numbers in Periodic Sequences

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Notation and Vocabulary

- period length of s smallest $\ell \geq 1$ s.t. $s(n+\ell) = s(n) \quad \forall n \geq 0$
- $\mathsf{Per}(\ell) = \{\text{periodic sequences with period length } \ell\}$
- $(s(0), s(1), \ldots, s(\ell-1))$ period of $s \in \mathsf{Per}(\ell)$

Example: $s \in \mathsf{Per}(12)$ period $(0, 4, 5, 1, 4, 6, 2, 4, 5, 3, 4, 6)$

	period	period length
$s(n)$	$(0, 4, 5, 1, 4, 6, 2, 4, 5, 3, 4, 6)$	12
$s(3n)$	$(0, 1, 2, 3)$	4
$s(3n+1)$	(4)	1 (constant)
$s(3n+2)$	$(5, 6)$	2
$s(9n)$	$(0, 3, 2, 1)$	4

Lemma

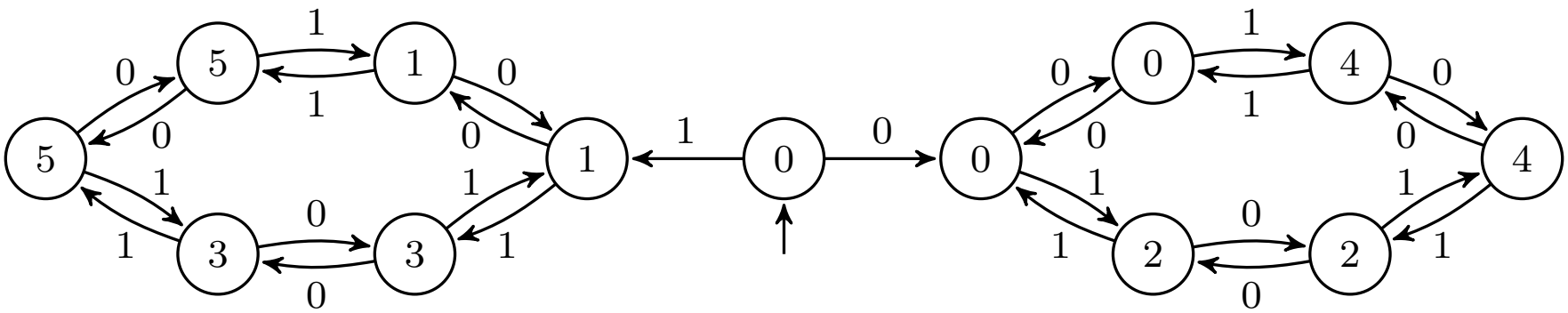
If the period length of s divides ℓ, that of $s(cn+r)_{n\geq 0}$ divides $\frac{\ell}{\gcd(c,\ell)}$

Theorem (Büchi, 1960)

Every periodic sequence is k-automatic

- s is k-automatic if there exists a DFAO producing s in base k

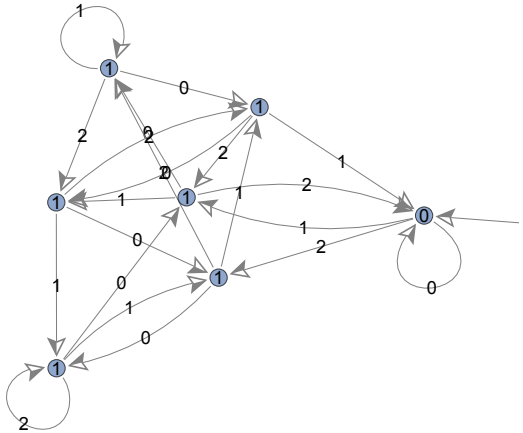
Example: $s \in \mathsf{Per}(6)$ period $(0, 1, 2, 3, 4, 5)$ 2-automatic



n	base-2 digits	output
0	ε	0
1	1	1
2	10	2
3	11	3
42	101010	0

Per(7) versus 3-Automatic

$(0, 1, 1, 1, 1, 1, 1)$



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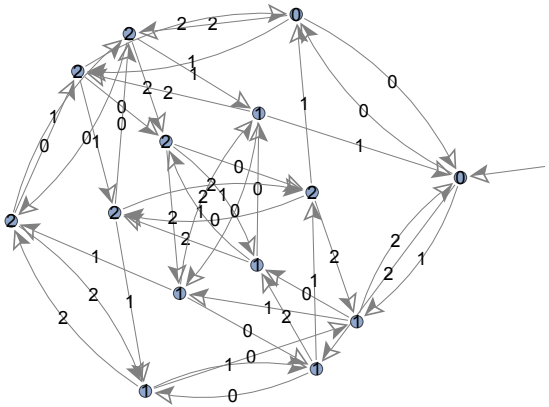
$(0, 1, 1, 2, 1, 2, 2)$



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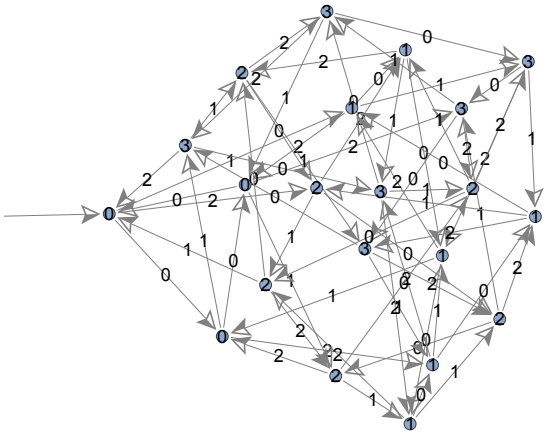
$(0, 1, 2, 3, 3, 2, 1)$



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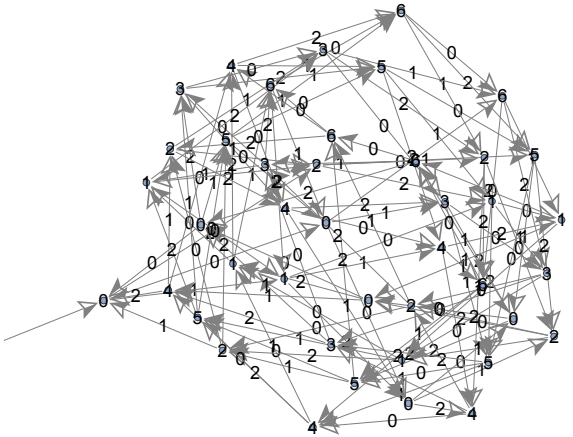
$(0, 1, 2, 3, 4, 5, 6)$



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Question: Can we characterize these sizes?

Wizardry: Magic Number Problem

- integers $k, \ell \geq 2$
- family $\mathsf{Per}(\ell)$
- n satisfies property $\mathrm{P_a}(k, \ell)$ if $\exists s \in \mathsf{Per}(\ell)$ s.t. $n =$ size DFAO for s

Muggle number

$\mathrm{P_a}(k, \ell)$ ✓

Magic number

$\mathrm{P_a}(k, \ell)$ ✗

- range smallest interval $\supseteq$ all muggle numbers

Goal: for $\mathrm{P_a}(k, \ell)$, determine

- the range
- the muggle and magic numbers

Kernel

set of states in the minimal DFAO for $s \leftrightharpoons \text{kernel}_k(s)$

- $\text{kernel}_k(s) = \{s(k^e n + j)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq j \leq k^e - 1\}$
- $\text{rank}_k(s) = |\text{kernel}_k(s)|$

Example: $s \in \mathsf{Per}(6)$ period $(0, 1, 2, 3, 4, 5)$

$\text{kernel}_2(s)$

(e, j)	period of $s(2^e n + j)_{n \geq 0}$
$(0, 0)$	$(0, 1, 2, 3, 4, 5)$
$(1, 0), (3, 0), (3, 6)$	$(0, 2, 4)$
$(1, 1), (3, 1), (3, 7)$	$(1, 3, 5)$
$(2, 0), (4, 0), (4, 6), (4, 12)$	$(0, 4, 2)$
$(2, 1), (4, 1), (4, 7), (4, 13)$	$(1, 5, 3)$
$(2, 2), (4, 2), (4, 8), (4, 14)$	$(2, 0, 4)$
$(2, 3), (4, 3), (4, 9), (4, 15)$	$(3, 1, 5)$
$(3, 2)$	$(2, 4, 0)$
$(3, 3)$	$(3, 5, 1)$
$(3, 4)$	$(4, 0, 2)$
$(3, 5)$	$(5, 1, 3)$
$(4, 4), (4, 10)$	$(4, 2, 0)$
$(4, 5), (4, 11)$	$(5, 3, 1)$

$\text{rank}_2(s) = 13$

Range for $\mathrm{P_a}(k, \ell)$

$(k^e \bmod \ell)_{e \geq 0}$ preperiod length $\mathrm{pre}_\ell(k)$ period length $\mathrm{ord}_\ell(k)$

Example: $\ell = 7$ $k = 3$ $(3^e \bmod 7)_{e \geq 0} = 1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, \ldots$

case	bound
general	upper $B_\ell(k) = \ell \cdot \mathrm{ord}_\ell(k) + \sum_{e=0}^{\mathrm{pre}_\ell(k)-1} \min(k^e, \ell)$ Ex: $(0, 1, \ldots, \ell - 1)$
coprime	lower ℓ Ex: $(0, 0, \ldots, 0, 1)$

Proposition

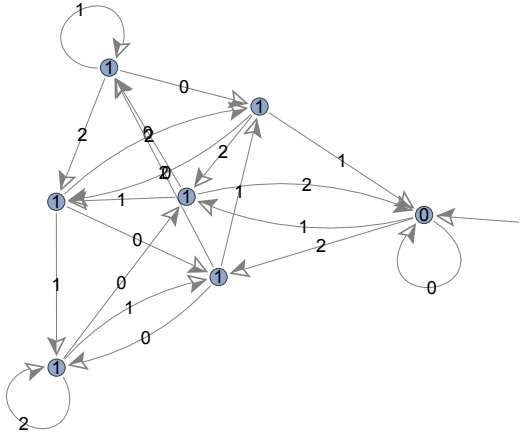
All muggle numbers belong to $[\ell, B_\ell(k)]$ if k and ℓ are coprime

Example:

		k					
		2	3	4	5	6	7
	2		$[2, 2]$		$[2, 2]$		$[2, 2]$
	3	$[3, 6]$		$[3, 3]$	$[3, 6]$		$[3, 3]$
ℓ	4		$[4, 8]$	$[4, 4]$			$[4, 8]$
	5	$[5, 20]$	$[5, 20]$	$[5, 10]$		$[5, 5]$	$[5, 20]$
	6				$[6, 12]$		$[6, 6]$
	7	$[7, 21]$	$[7, 42]$	$[7, 21]$	$[7, 42]$	$[7, 14]$	

3-Automatic versus Per(7)

$(0, 1, 1, 1, 1, 1, 1)$



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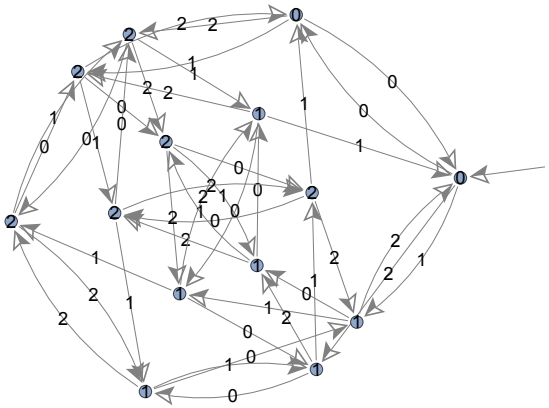
$(0, 1, 1, 2, 1, 2, 2)$



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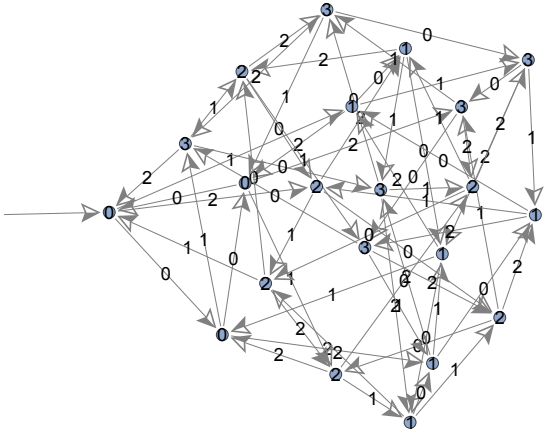
$(0, 1, 2, 3, 3, 2, 1)$



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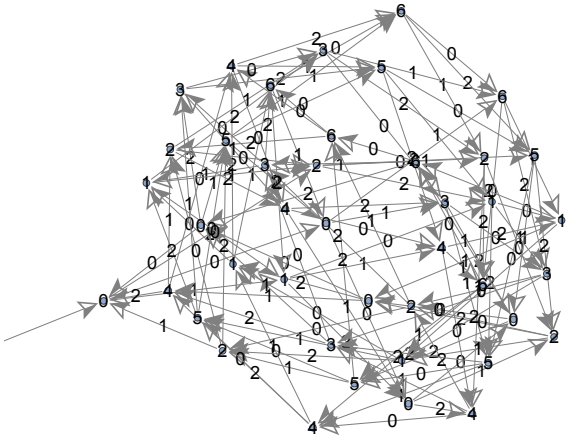
$(0, 1, 2, 3, 4, 5, 6)$



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Muggle and Magic Numbers for $\mathrm{P_a}(k, \ell)$

Theorem

Let k and ℓ be coprime. For $\mathrm{P_a}(k, \ell)$

$$A = \{d\ell : d \in \mathsf{Div}(\mathrm{ord}_\ell(k))\}$$

$$\mathbb{N} \setminus A$$

constructive proof

Example: $\ell = 7 \;\; k = 3 \;\; (3^e \bmod 7)_{e \geq 0} = 1, 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, \ldots$

$$\mathrm{pre}_3(7) = 0 \quad \mathrm{ord}_3(7) = 6$$
$$A = \{7d : d \in \mathsf{Div}(6)\}$$

d	$\mathrm{rank}_3(s)$	period of s
6	42	$(0, 1, 2, 3, 4, 5, 6)$
3	21	$(0, 1, 2, 3, 3, 2, 1)$
2	14	$(0, 1, 1, 2, 1, 2, 2)$
1	7	$(0, 1, 1, 1, 1, 1, 1)$

Constant-Recursive Sequences

Remark

Every periodic sequence is constant-recursive

- s is constant recursive if $\exists\, c_0, c_1, \ldots, c_{d-1} \in \mathbb{Z}$ with $c_0 \neq 0$ s.t.

$$s(n+d) = c_{d-1}s(n+d-1) + \cdots + c_1 s(n+1) + c_0 s(n) \quad \forall n \geq 0$$

Example:

$s \in \mathsf{Per}(6)$ period $(0,1,2,3,4,5)$	$s \in \mathsf{Per}(3)$ period $(-1,0,1)$
$s(n+6) = s(n)$	$s(n+3) = s(n)$ $s(n+2) = -s(n+1) - s(n)$

- $\mathrm{rank}(s) =$ smallest d

$\mathrm{rank}(s) = |\ker_1(s)| \quad \ker_1(s) = \{s(n+j)_{n\geq 0} : j \geq 0\}$

Example:

$\mathrm{rank}(s) = 6$	$\mathrm{rank}(s) = 2$

- integer $\ell \geq 2$
- family $\mathsf{Per}(\ell)$
- n satisfies property $\mathrm{P_{cr}}(\ell)$ if $\exists s \in \mathsf{Per}(\ell)$ s.t. $n = \mathrm{rank}(s)$

Muggle number

$\mathrm{P_{cr}}(\ell)$ ✓

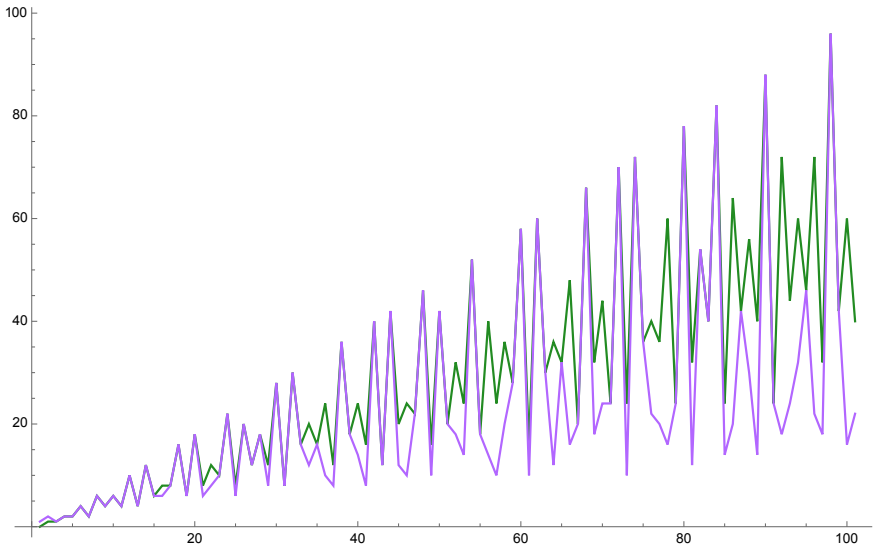
Magic number

$\mathrm{P_{cr}}(\ell)$ ✗

Goal: for $\mathrm{P_{cr}}(\ell)$, determine

- the range
- the muggle and magic numbers

Range for $P_{cr}(\ell)$



Euler totient function ϕ

additive version ψ

- p prime, $k \geq 1$: $\psi(p^k) = \phi(p^k)$
- n odd: $\psi(2n) = \phi(n)$
- m, n relatively prime, $\neq 2$: $\psi(mn) = \phi(m)+\phi(n)$

Proposition

All muggle numbers belong to $[\psi(\ell), \ell]$

Example:

ℓ	2	3	4	5	6	7	8
$[\psi(\ell), \ell]$	$[1, 2]$	$[2, 3]$	$[2, 4]$	$[4, 5]$	$[2, 6]$	$[6, 7]$	$[4, 8]$
ℓ	9	10	11	12	13	14	15
$[\psi(\ell), \ell]$	$[6, 9]$	$[4, 10]$	$[10, 11]$	$[4, 12]$	$[12, 13]$	$[6, 14]$	$[6, 15]$

Muggle and Magic Numbers for $\mathrm{P}_{\mathrm{cr}}(\ell)$

Theorem

S: $\emptyset \neq \{d_1, \ldots, d_j\} \subset \mathbb{N}$ pairwise distinct s.t. $\mathrm{lcm}(d_1, \ldots, d_j) = \ell$
For $\mathrm{P}_{\mathrm{cr}}(\ell)$

$R = \{\sum_{i=1}^{j} \phi(d_i) : \{d_1, \ldots, d_j\} \in S\}$

$\mathbb{N} \setminus R$

constructive proof

Example: $\ell = 6$

$\{d_1, \ldots, d_j\}$	rank(s)	char. poly	period of s
$\{6\}$	2	$x^2 - x + 1$	$(0, 1, 1, 0, -1, -1)$
$\{2, 3\}$	3	$x^3 + 2x^2 + 2x + 1$	$(0, 0, 1, -2, 2, -1)$
$\{3, 6\}$	4	$x^4 + x^2 + 1$	$(0, 0, 0, 1, 0, -1)$
$\{2, 3, 6\}$	5	$x^5 + x^4 + x^3 + x^2 + x + 1$	$(0, 0, 0, 0, 1, -1)$
$\{1, 2, 3, 6\}$	6	$x^6 - 1$	$(0, 0, 0, 0, 0, 1)$

Remark

Automatic sequences are regular $\rightsquigarrow$ Periodic sequences are regular

- s is k-regular if the $\mathbb{Q}$-vector space generated by $\text{kernel}_k(s)$ is finitely generated
- $\text{rank}_k(s)$ = dimension of the vector space

Example: $s \in \mathsf{Per}(4)$ period $(0,1,1,1)$

$$\underbrace{(0,1,1,1),(0,1),(1)}_{\text{lin. indep.}},\ (0) \qquad \text{kernel}_2(s)$$

	2-regular	2-automatic
$\text{rank}_2(s)$	3	4

Range, Muggle and Magic Numbers for $\mathrm{P_r}(k, \ell)$

- integers $k, \ell \geq 2$
- family $\mathsf{Per}(\ell)$
- n satisfies property $\mathrm{P_r}(k, \ell)$ if $\exists s \in \mathsf{Per}(\ell)$ s.t. $n = \mathrm{rank}_k(s)$

Theorem

All muggle numbers belong to $[\psi(\ell), \ell]$ if k and ℓ are coprime

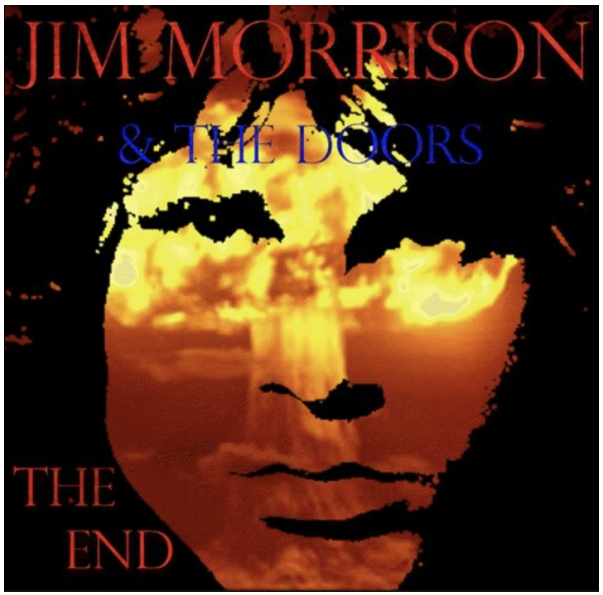
S: $\emptyset \neq \{d_1, \ldots, d_j\} \subset \mathbb{N}$ pairwise distinct s.t. $\mathrm{lcm}(d_1, \ldots, d_j) = \ell$
For $\mathrm{P_r}(k, \ell)$

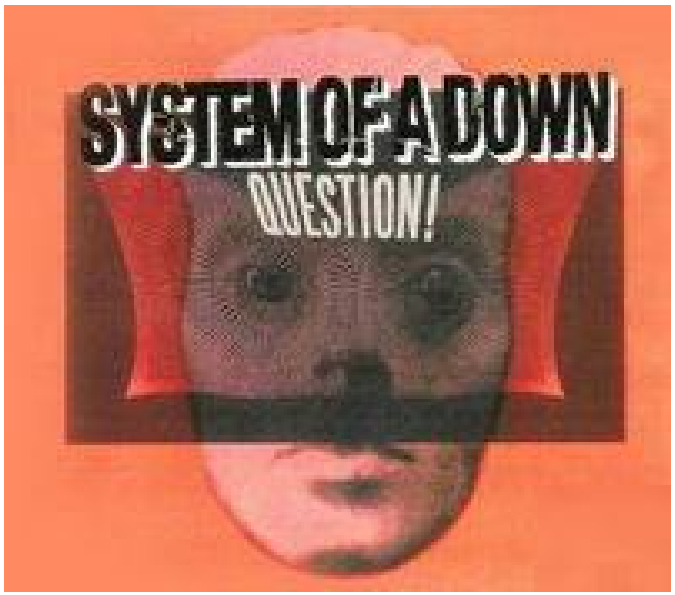
$R = \{\sum_{i=1}^{j} \phi(d_i) : \{d_1, \ldots, d_j\} \in S\}$

$\mathbb{N} \setminus R$

Conclusion

		automatic	constant-recursive	regular
range	general	?	✓	?
	coprime	✓	✓	✓
charact.	general	?	✓	?
	coprime	✓	✓	✓





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