Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

Toward oscillations inhibition by mean-field feedback in Kuramoto oscillators

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Abstract: This note shows that the oscillations in a network of all-to-all coupled Kuramoto oscillators can be inhibited by a scalar output feedback. More precisely, by injecting an input proportional to the oscillators mean-field, a set of isolated equilibria is shown to be almost globally attractive when natural frequencies are zero. The normal hyperbolicity of all relevant equilibria let us conjecture that this property persists in the presence of natural frequencies that are sufficiently small with respect to the coupling gain. This work constitutes a first step in the direction of testing the possible neuronal inhibition arising in deep brain stimulation treatment for neurological diseases.

Keywords: Kuramoto Oscillators, Mean-field Feedback, Phase-Locking, Deep Brain Stimulation

1. INTRODUCTION

The aim of this paper is to provide preliminary insights on how the oscillations of a network of nonlinear oscillators can be inhibited by relying on their average behavior only. The motivations for this study stand in the development of Deep Brain Stimulation (DBS), which is a treatment for neurological diseases such as Parkinson Disease (PD). DBS consists in permanently stimulating specific cerebral zones thanks to implanted electrodes. Despite its success and generalization, little is still known on its functioning. There are clear evidences (Alberts et al., 1969; Volkmann et al., 1996) that intense and coherent neuronal activity is present in the subthalamic nucleus (STN) of PD patients, which is the zone stimulated by the electrodes. This neuronal synchronization is absent in healthy subjects (Nini et al., 1995; Sarma et al., 2010). DBS would hence attenuate PD symptoms by somehow alter this pathological coherent activity. Yet, two main hypotheses still divide practitioners (McIntyre et al., 2004). One is that electrical stimulation restores the STN non-pathological activity, either by *desynchronizing* the interested neurons, or by *modulating* the network output activity (Carlson et al., 2010). Many attempts to formally support this hypothesis, based on simplified models, were recently made in the literature of dynamical systems (Rosenblum and Pikovsky, 2004; Tukhlina et al., 2007; Hauptmann et al., 2005; Pyragas et al., 2007; Popovych et al., 2006; Franci et al., 2010). The other hypothesis is that DBS *inactivates* STN neurons, by producing a functional lesion, meaning that the stimulation acts by impeding the pathological bursting and spiking (Filali et al., 2004). This hypothesis is also supported by the fact that, before the invention of DBS, the surgical PD treatment consisted in an ablation of the cerebral zone under concern (Benabid et al., 1996), which corresponds to a radical neuronal inhibition. Recently, some work has been devoted to the development of feedback approaches to DBS: by exploiting real-time information on the cerebral activity, a more adequate and parsimonious stimulation is expected (cf. (Tass, 2003; Hammond et al., 2008; Tarsy et al., 2008) and references therein). A strong constraint on these closed-loop approaches is due to the number and size of implanted electrodes, which impose to measure only collective information (*i.e.* the mean-field, that is the mean membrane voltages of the STN neuronal population) and to generate a limited number of stimulation signals.

In this note, we rely on a simplified model of the neuronal population of the STN to provide preliminary theoretical justifications on how mean-field feedback DBS may yield neuronal inhibition in the STN. We exploit the model recently introduced in (Franci et al., 2010) in which the effect of proportional mean-field feedback is explicitly taken into account. Despite its simplistic nature, this model owns the advantage to share similarities with Kuramoto oscillators (Kuramoto, 1984; Winfree, 1980) for which a wide literature exists (Sepulchre et al., 2007; Aeyels and Rogge, 2004; Dörfler and Bullo, 2010; Jadbabaie et al., 2004; Sarlette, 2009). We show that, assuming all-to-all coupling and neglecting the natural frequencies of the agents, Kuramoto oscillators can be inhibited by proportional mean-field feedback. More precisely we show that, with a proper choice of the feedback gain, the closed-loop system boils down to a gradient system. The corresponding potential function is shown to have isolated global extrema, and all other isolated fixed points are shown to be saddles. The main result then follows by showing that all non-isolated fixed points constitute an unstable manifold.

We are fully aware that the assumptions of this paper are strong and of little neurological relevance: Kuramoto oscillators constitute an over-simplified neuron model, neglecting natural frequencies means neglecting the neurons internal behavior, and all-to-all coupling is far from being biologically realistic. While some arguments may partly justify these assumptions (*e.g.* Kuramoto oscillators still model the neurons rhythm), we see the present work as a first step for future studies. In particular, due to the hyperbolicity of all relevant equilibria, we conjecture that inhibition persists in the presence of sufficiently small natural frequencies.

The paper is organized as follows. In Section 2, we recall the Kuramoto model under mean-field feedback and illustrate it through numerical simulations. In Section 3 we derive the main results of the paper. A discussion on the influence of natural frequencies is provided in Section 4. Proofs are given in Section 5.

Notations. For all $x, y \in \mathbb{R}$, $z = x \bmod y$ if $z = x + ky$ for some $k \in \mathbb{Z}$. $\mathbf{1}_{n\times m} \in \mathbb{R}^{n\times m}$ denotes the $n \times m$ matrix with all unitary entries, and $\mathbf{1}_n := \mathbf{1}_{n\times 1}$. I_n denotes the identity matrix in dimension n. For $x \in \mathbb{R}^n$ and $\epsilon > 0$, $\mathcal{B}(x, \epsilon)$ denotes the closed ball centered at x of radius ϵ in the Euclidean norm, that is $\mathcal{B}(x, \epsilon) := \{y \in \mathbb{R}^n : \sqrt{\sum_{i=1}^n (y_i - x_i)^2} \leq \epsilon\}$. μ denotes the Lebesgue measure in the space of interest. If $\mathcal{I} \subset \mathbb{Z}$, $\#\mathcal{I}$ denotes the number of its elements. Given a set $A \subset \mathbb{R}^N$, we define its stable set with respect to a given dynamics $\dot{x} = f(x)$ as $A^s := \{x_0 \in \mathbb{R} : \lim_{t\to\infty} |x(t; x_0)|_A = 0\}$, where $|x|_A := \inf_{z\in A} |x - z|$.

2. THE KURAMOTO SYSTEM UNDER MEAN-FIELD FEEDBACK

The complexity of neuronal cells dynamics impedes an analytical treatment of their behavior when interconnected. However, some neuronal behaviors can be analyzed based on much simpler models. In particular, their spiking rhythm and the synchronization of their behavior can be modeled by phase dynamics oscillators. Instead of using celebrated biology inspired models, such as (Hodgkin and Huxley, 1952), we therefore focus on interconnected Landau-Stuart oscillators. These possess the combined advantages to be analytically tractable and to provide a natural modeling of the effect of exogenous inputs. More precisely, we have recently shown in (Franci et al., 2010) that the spiking rhythm of N interconnected neurons under the influence of a proportional mean-field [1] feedback representing the closed-loop DBS can be modeled as

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} (k_{ij} + \gamma_{ij}) \sin(\theta_j - \theta_i) - \sum_{j=1}^{N} \gamma_{ij} \sin(\theta_i + \theta_j), \quad (1)$$

for all $i = 1, \ldots, N$, where $\omega_i \in \mathbb{R}$ denotes the natural frequency of the i-th neuron, $k_{ij} \in \mathbb{R}$ is the coupling strength from neuron j to neuron i, and $\gamma_{ij} \in \mathbb{R}$ is the feedback gain from neuron j to neuron i. The above model results from the interconnection of Landau-Stuart oscillators under simplifying assumptions: we refer the reader to (Franci et al., 2010) for details. Note that (1) encompasses the standard Kuramoto model (Kuramoto, 1984) in the case of zero feedback gains γ_{ij}.

In order to drive the synchronous STN activity to a non-pathological state, the feedback gain γ_{ij}, can be tuned to reduce or eliminate the effective closed-loop diffusive coupling $k_{ij}+\gamma_{ij}$. In the following, we consider the particular case of the all-to-all coupling, that is $k_{ij} = k_0$ and $\gamma_{ij} = \gamma_0$ for all $i, j = 1, \ldots, N$ where $k_0 > 0$ and $\gamma_0 \in \mathbb{R}$. In this case (1), boils down to

$$\dot{\theta}_i = \omega_i + (k_0 + \gamma_0) \sum_{j=1}^{N} \sin(\theta_j - \theta_i) - \gamma_0 \sum_{j=1}^{N} \sin(\theta_i + \theta_j). \quad (2)$$

In this particular situation, one intuitively expects that, if the resulting diffusing coupling $k_0 + \gamma_0$ is canceled, then synchronization of the network of oscillators is compromised, yielding either an oscillating desynchronized state or the end of oscillations. Numerical simulations support these expectations. They reveal that, when the natural frequencies ω_i are large with respect to the coupling strength k_0, the use of a mean-field feedback with gain $\gamma_0 = -k_0$ desynchronizes the network (cf. Fig. 1, in which the feedback is activated at $t = 20$).

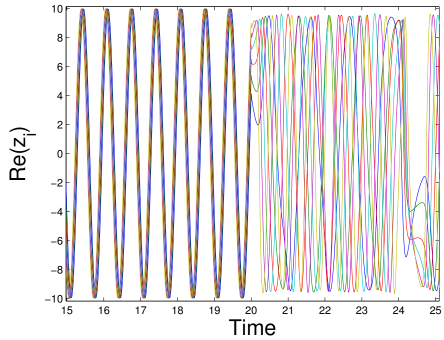

Fig. 1. Large natural frequencies: desynchronization.

On the contrary, when natural frequencies ω_i are small compared to the coupling strength k_0, mean-field feedback inhibits oscillations when γ_0 is picked as $-k_0$. The phase of each oscillator goes to a fixed point (Fig. 2), thus eventually stopping oscillations.

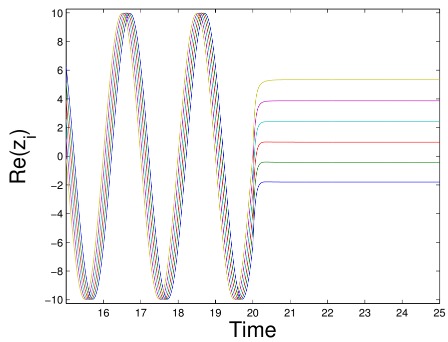

Fig. 2. Small natural frequencies: oscillations death.

The desynchronizing effect of mean-field feedback is currently under investigation, cf. (Franci et al., 2010). This note rather aims at providing theoretical justifications to the oscillators death (neuronal inhibition) described in Fig. 2. More precisely, in Section 3, we show that with the choice $\gamma_0 = -k_0$ and considering zero natural frequencies (*i.e.* $\omega_i = 0$ for all $i = 1, \ldots, N$) the presence of mean-field feedback almost globally asymptotically stabilizes a set of equilibria for (2). In Section 4, we conjecture the extension of this result to the case of non-zero natural frequencies and the non-full compensation of the diffusive coupling.

[1] The mean-field of a neurons population is the weighted mean of their membrane voltages.

3. THE UNPERTURBED CASE

In the case of zero natural frequencies and with the choice $\gamma_0 = -k_0$, (2) boils down to

$$\dot{\theta}_i = k_0 \sum_{j=1}^{N} \sin(\theta_i + \theta_j), \quad \forall i = 1, \dots, N. \tag{3}$$

We note that (3) can be equivalently written as the gradient system

$$\dot{\theta}_i = -\frac{\partial W}{\partial \theta_i}(\theta), \quad \forall i = 1, \dots, N,$$

where the function W is given, for all $\theta \in \mathbb{R}^N$, by

$$W(\theta) := -k_0 \sum_{i,j=1}^{N} \sin^2\left(\frac{\theta_i + \theta_j}{2}\right). \tag{4}$$

3.1 Fixed points identification

We start by computing the fixed points of (3) or, equivalently, the critical points of its potential function (4). To that aim, we define the following set:

$$\begin{aligned} A_0 := \Big\{ \theta \in \mathbb{R}^N : \theta_i &= \frac{\pi}{2} \bmod 2\pi \ \ \forall i \in \mathcal{I}_{\frac{\pi}{2}}, \\ \theta_i &= \frac{3\pi}{2} \bmod 2\pi \ \ \forall i \in \mathcal{I}_{\frac{3\pi}{2}}, \\ \mathcal{I}_{\frac{\pi}{2}} \cup \mathcal{I}_{\frac{3\pi}{2}} &= \{1, \dots, N\}, \ \#\mathcal{I}_{\frac{\pi}{2}} \neq \#\mathcal{I}_{\frac{3\pi}{2}} \Big\}. \end{aligned} \tag{5}$$

That is, A_0 is made of all the vectors $\theta \in \mathbb{R}^N$ whose components are either $\pi/2$ or $3\pi/2$ (modulo 2π), and for which the number of $\pi/2$ entries is different from the number of $3\pi/2$ entries. In the same way, we define

$$\begin{aligned} B_0 := \big\{ \theta \in \mathbb{R}^N : \theta_i &= 0 \bmod 2\pi \ \ \forall i \in \mathcal{I}_0, \\ \theta_i &= \pi \bmod 2\pi \ \ \forall i \in \mathcal{I}_\pi, \\ \mathcal{I}_0 \cup \mathcal{I}_\pi &= \{1, \dots, N\}, \ \#\mathcal{I}_0 \neq \#\mathcal{I}_\pi \big\}. \end{aligned} \tag{6}$$

In other words, all the vectors of B_0 are made only with 0 and π elements, and the number of their π's differ from the number of their 0's. Finally, we introduce

$$\mathcal{N} := \left\{ \theta \in \mathbb{R}^N : \sum_{i=1}^{N} \sin(\theta_i) = \sum_{i=1}^{N} \cos(\theta_i) = 0 \right\}. \tag{7}$$

The following lemma, whose proof is given in Section 5.1, shows that A_0, B_0 and $\mathcal{N}$ completely characterize the fixed points of (3).

Lemma 1. (Fixed points identification). Given any $k_0 > 0$, the set F_0 of fixed points of (3) is given by $F_0 = A_0 \cup B_0 \cup \mathcal{N}$, where $A_0, B_0, \mathcal{N}$ are given in (5)-(7).

We stress that, since $A_0, B_0, \mathcal{N}$ are disjoint, they form a partition of F_0. Lemma 1 states in particular that the critical points of W can be divided into two families. The critical points contained in the sets A_0 and B_0 are *isolated* by their definitions (5)-(6). Their stability can then be easily studied by analyzing the sign definiteness of the Hessian of W at these points. This will be achieved by Lemma 2. On the contrary, as we show in the sequel, fixed points belonging to $\mathcal{N}$ are not isolated. Noticing that $\mathcal{N}$ is defined as a level set of the function $(\sum_{i=1}^{N} \sin(\theta_i), \sum_{i=1}^{N} \cos(\theta_i))^T$, we argue that, at least locally, it defines an embedded submanifold. Its stability can then be analyzed through the linearization of (3) on the orthogonal subspace of this submanifold. This will be achieved by Lemma 3.

3.2 Analysis of isolated equilibria

In the following lemma, proved in Section 5.2 we address the stability of the fixed points of (3) belonging to $A_0 \cup B_0$.

Lemma 2. (Stability of the isolated fixed points). Let W be the function defined in (4), let

$$\begin{aligned} \mathcal{W}_m &:= \left\{ \theta \in \mathbb{R}^N : \theta = \left(\frac{\pi}{2} \bmod \pi\right) \mathbf{1}_N \right\} \\ \mathcal{W}_M &:= \left\{ \theta \in \mathbb{R}^N : \theta = (0 \bmod \pi) \mathbf{1}_N \right\}, \end{aligned} \tag{8}$$

and let k_0 be any given positive constant. Then the following holds true:

a) $\mathcal{W}_m$ contains all global minima of W and its points are hyperbolically asymptotically stable for (3).
b) $\mathcal{W}_M$ contains all global maxima of and all its points are hyperbolically unstable for (3).
c) All the critical points of W, which are not global extrema, that is all the points in $(A_0 \cup B_0) \setminus (\mathcal{W}_m \cup \mathcal{W}_M)$ where A_0 and B_0 are defined in (5)-(6), are non-degenerate saddles for (3).

3.3 Analysis of non-isolated equilibria

The following lemma, whose proof is given in Section 5.3, characterizes that the non-isolated critical points of the gradient function W contained in $\mathcal{N}$.

Lemma 3. (Normal hyperbolicity of non-isolated fixed points) Let k_0 be any positive constant and let $\mathcal{N}$ be defined as in (7). Then, the following holds true:

a) If N *is odd*, $\mathcal{N}$ is an embedded submanifold of codimension 2 that is normally hyperbolic for (3). In particular, for all $\theta \in \mathcal{N}$, the eigenvalues $\lambda_-(\theta), \lambda_+(\theta)$ of the linearization of (3) restricted to the orthogonal directions to $\mathcal{N}$ are such that $\lambda_-(\theta) < 0 < \lambda_+(\theta)$.
b) If N *is even*, there exists a normally hyperbolic submanifold $\tilde{\mathcal{N}}$ of codimension 2, and 2^N 1-dimensional submanifolds $\mathcal{N}_{0i}$, $i = 1, \dots, 2^N$, such that $\mathcal{N} = \tilde{\mathcal{N}} \cup \bigcup_{i=1}^{2^N} \mathcal{N}_{0i}$. Moreover, for all $\theta \in \tilde{\mathcal{N}}$, the eigenvalues $\lambda_-(\theta), \lambda_+(\theta)$ of the linearization of (3) restricted to the orthogonal directions to $\tilde{\mathcal{N}}$ are such that $\lambda_-(\theta) < 0 < \lambda_+(\theta)$, and, for all $i = 1, \dots, 2^N$, the stable set of $\mathcal{N}_{0i}$ is contained in a submanifold of dimension 2.

Lemma 3 states in particular that locally around almost all points θ in $\mathcal{N}$, the dynamics (3) can be decomposed in three behaviors: the null behavior tangent to $\mathcal{N}$; the convergent behavior toward $\mathcal{N}$ along the eigenvector associated to $\lambda_-(\theta)$; and the divergent behavior away from $\mathcal{N}$ along the eigenvector associated to $\lambda_+(\theta)$. In the even case, the set $\mathcal{N}$ cannot be globally described as a normally hyperbolic submanifold, due to the presence of the singularities $\mathcal{N}_{0i}$, where the equation $\sum_{i=1}^{N} \sin(\theta_i) = \sum_{i=1}^{N} \cos(\theta_i) = 0$ loses rank. The same phenomenon happens for the unstable set of standard all-to-all Kuramoto system, as shown in details in Sepulchre et al. (2007). The two sets are indeed described by the same algebraic relationships.

3.4 Main result

We have all the ingredients to prove the main result of this paper.

Proposition 4. (Almost global oscillations inhibition) Given any $k_0 > 0$, the set $\mathcal{W}_m$ defined in (8) is almost globally asymptotically stable for the dynamics (3).

We stress that the effect of oscillations inhibition is peculiar to the presence of mean-field feedback. Indeed, in the standard Kuramoto system this phenomenon is avoided by the T^1 symmetry of the coupled dynamics that let the system be invariant with respect to global phase-shifts ((Sarlette, 2009)).

Proof Let $\mathcal{N}$ be as in (7), let $\tilde{\mathcal{N}}$ be defined as in Lemma 3, and let $\tilde{\mathcal{N}}^s$ denote the stable manifold of $\tilde{\mathcal{N}}$. Since, by Lemma 3, $\tilde{\mathcal{N}}$ is normally hyperbolic with one unstable direction, it follows from (Hirsh et al., 1977, Theorem 4.1) that $\tilde{\mathcal{N}}^s$ has zero Lebesgue measure. Moreover, by Lemma 3, all the points of $\mathcal{N}$ which are not in $\tilde{\mathcal{N}}$ forms a finite set of 1-dimensional manifolds $\mathcal{N}_{0i}$, $i = 1, \ldots, 2^N$. The stable set of each of the $\mathcal{N}_{0i}$ is contained in a 2-dimensional submanifold $\mathcal{M}_i^{us}$. Define the set $\mathcal{C}_0 = \mathcal{N} \cup \mathcal{N}^s \cup \bigcup_{i=1}^{2^N} \mathcal{M}_i^{us}$. It follows that $\mu(\mathcal{C}_0) = 0$. Consider the domain $\mathcal{D} = \mathbb{R}^N \setminus \mathcal{C}_0$. By definition $\mathcal{D}$ is open, forward invariant for (3), and contains only isolated critical points. Hence by (Hirsch and Smale, 1974, Theorems 1 and 4 and Corrollary), the restriction of (3) to $\mathcal{D}$ is a well defined gradient dynamics that contains only isolated critical points, and, for almost all $\theta_0 \in \mathcal{D}$, the trajectory starting in θ_0 converges to the minima $\mathcal{W}_m$ of the potential function. Recalling that $\mu(\mathcal{C}_0) = 0$, we conclude that, for almost all $\theta_0 \in \mathbb{R}^N$, the trajectory starting at θ_0 converges to $\mathcal{W}_m$. □

4. CONJECTURE FOR THE PERTURBED CASE

Although Proposition 4 is stated in the ideal case of zero natural frequencies, the almost global stability and the local hyperbolicity of all the relevant equilibria let us conjecture the global robustness of the oscillations inhibition to small natural frequencies and diffusive coupling.
The idea behind this conjecture is mainly based on the fact that all the global extrema of W are hyperbolic for (3) hence they persist under small perturbations. The non-isolated fixed points define a normally hyperbolic invariant manifold that persists under small perturbations (possibly with non-zero dynamics on it) along with its stable and unstable manifolds (Hirsh et al., 1977, Theorem 4.1). Further work will aim at proving this conjecture.

5. PROOFS

5.1 Proof of Lemma 1

The fixed points $\theta^* \in \mathbb{R}^N$ of (3) satisfy

$$\sum_{j=1}^{N} \sin(\theta_i^* + \theta_j^*) = 0\,, \quad \forall i = 1, \ldots, N. \tag{9}$$

Using the trigonometric identity $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, this condition can be rewritten as

$$\sin(\theta_i^*)a(\theta^*) + \cos(\theta_i^*)b(\theta^*) = 0, \ \forall i = 1, \ldots, N, \tag{10}$$

where, for all $\theta \in \mathbb{R}^N$,

$$a(\theta) := \sum_{j=1}^{N} \cos(\theta_j), \quad b(\theta) := \sum_{j=1}^{N} \sin(\theta_j). \tag{11}$$

By contradiction, it easy to show that the equation (10) admits no solution whenever $a(\theta^*) \neq 0$ and $b(\theta^*) \neq 0$. Recalling (10)-(11), all solutions of (9) then necessarily belong to one of the following three sets:

$$\tilde{A}_0 := \{\theta \in \mathbb{R}^N : \ a(\theta) = 0, \ b(\theta) \neq 0, \\ \cos(\theta_i) = 0, \ \forall i = 1, \ldots, N\},$$

$$\tilde{B}_0 := \{\theta \in \mathbb{R}^N : \ b(\theta) = 0, \ a(\theta) \neq 0, \\ \sin(\theta_i) = 0, \ \forall i = 1, \ldots, N\},$$

$$\mathcal{N} = \{\theta \in \mathbb{R} : \ a(\theta) = b(\theta) = 0\}$$

as defined as in (7). Through elementary computation, it can be shown that $\tilde{A}_0 = A_0$ and $\tilde{B}_0 = B_0$, where A_0 and B_0 are defined in (5) and (6) respectively, which proves the lemma.

5.2 Proof of Lemma 2

We start by computing the Hessian of W, $H(\theta) := \frac{\partial^2 W}{\partial \theta^2}(\theta)$. Basic computations reveal that $H = [H_{ij}]_{i,j=1,\ldots,N}$ with, for all $i, j = 1, \ldots, N$,

$$H_{ii}(\theta) = -k_0(\cos(2\theta_i) + s_i(\theta)) \\ H_{ij}(\theta) = -k_0 \cos(\theta_i + \theta_j), \quad \forall\ i \neq j, \tag{12}$$

where

$$s_i(\theta) := \sum_{j=1}^{N} \cos(\theta_i + \theta_j) \quad \forall i = 1, \ldots, N. \tag{13}$$

Item a): Global minima. Noticing that $W(\theta) \geq -k_0 n^2$ for all $\theta \in \mathbb{R}^N$, the global minimum of W is attained when $\sin^2\left(\frac{\theta_i^* + \theta_j^*}{2}\right) = 1$, for all $i, j = 1, \ldots, N$, that is, $\theta^* = \left(\frac{\pi}{2} \bmod \pi\right) \mathbf{1}_N$. This leads to $\cos(\theta_j^* + \theta_i^*) = -1$ for all $i, j = 1, \ldots, N$. Recalling the expression of the Hessian of W, given in (12), at the global minima we have

$$H|_{\mathcal{W}_m} = N I_N + \mathbf{1}_{N \times N}$$

Since $H|_{\mathcal{W}_m}$ is symmetric diagonally dominant with strictly positive diagonal entries, all its eigenvalues are strictly positive (Horn and Johnson, 1985, Theorem 6.1.10), that is all the points of $\mathcal{W}_m$ are hyperbolically asymptotically stable.
Item b): Global maxima. The proof of this item is omitted here as it follows along the same lines as for Item 2.
*Item c): Saddles.*In view of Lemma 1, the points $\theta^* \in B_0 \setminus \mathcal{W}_M$ can be picked such that $\theta_i = 0 \bmod 2\pi$, for $i = 1, \ldots, m_0$, and $\theta_i = \pi \bmod 2\pi$, for $i = m_0 + 1, \ldots, N$. $m_0 \in (0, N)$, after a reordering the phase indexes, where $m_0 := \#\mathcal{I}_0 \in \{1, \ldots, N\}$. Let $m_\pi := N - m_0 = \#\mathcal{I}_\pi$ and consider $i_0, i_0' \in \{1, \ldots, m_0\}$ and $i_\pi, i_\pi' \in \{m_0 + 1, \ldots, N\}$. Since $\cos(2\theta_{i_0}) = \cos(\theta_{i_0} + \theta_{i_0'}) = \cos(2\theta_{i_\pi}) = \cos(\theta_{i_\pi} + \theta_{i_\pi'}) = 1$, and $\cos(\theta_{i_0} + \theta_{i_0\pi}) = -1$, basic computations from (12) reveal that, for all $\theta^* \in B_0 \setminus \mathcal{W}_M$, the Hessian of W has the form

$$H(\theta^*) = -k_0 \left[\begin{array}{c|c} A & C \\ \hline C^T & B \end{array}\right]. \tag{14}$$

where $A \in \mathbb{R}^{m_0 \times m_0}$ and $B \in \mathbb{R}^{m_\pi \times m_\pi}$ are defined as

$$A := (m_0 - m_\pi) I_{m_0} + \mathbf{1}_{m_0 \times m_0} \tag{15}$$

$$B := (m_\pi - m_0) I_{m_\pi} + \mathbf{1}_{m_\pi \times m_\pi} \tag{16}$$

and

$$C := -\mathbf{1}_{m_0 \times m_\pi}. \tag{17}$$

Consider the two vectors $e_1 := (1, 1, 0, \ldots, 0)^T$ and $e_2 := (0, \ldots, 0, 1, 1)^T$. Since $e_1^T H(\theta^*) e_1 = -2k_0(m_0 - m_\pi)$ and $e_2^T H(\theta^*) e_2 = 2k_0(m_0 - m_\pi)$, and recalling that $m_0 \neq m_\pi$ in view of Lemma 1, $H(\theta^*)$ is sign indefinite. In particular, small variations $\delta\hat{e}_1$ and $\delta\hat{e}_2$ at this critical points change the values of W by δW_1 and δW_2 with $\delta W_1 \delta W_2 < 0$. Hence, all the points $\theta^* \in B_0 \setminus \mathcal{W}_m$ are non degenerate saddles. Through similar computations, the same result holds for the points $\theta^* \in A_0 \setminus \mathcal{W}_M$.

5.3 Proof of Lemma 3

The set $\mathcal{N}$ of fixed points, defined in (7), is the zero level set of the function

$$F(\cdot) := \begin{pmatrix} a(\cdot) \\ b(\cdot) \end{pmatrix} : \mathbb{R}^N \to \mathbb{R}^2. \tag{18}$$

The level set of a function defines a submanifold of codimension m if the function has constant rank m. In order to check this condition on F, we have to compute its Jacobian's rank in each point $\theta \in \mathcal{N}$. In view of (11) and (18), basic computations reveal that

$$J_F(\theta) := \frac{\partial F}{\partial \theta}(\theta) = \begin{pmatrix} -\sin\theta_1 & \dots & -\sin\theta_N \\ \cos\theta_1 & \dots & \cos\theta_N \end{pmatrix}. \tag{19}$$

This matrix has rank 2 if and only if it contains two independent columns. A necessary and sufficient condition is then that there exists $i, j \in \{1, \dots, N\}$, $i \neq j$, such that

$$\det \begin{pmatrix} -\sin\theta_i & -\sin\theta_j \\ \cos\theta_i & \cos\theta_j \end{pmatrix} = \sin(\theta_j - \theta_i) \neq 0.$$

In other words, the rank of F is strictly smaller than 2 at some point $\overline{\theta} \in \mathcal{N}$ if and only if

$$\sin\left(\overline{\theta}_j - \overline{\theta}_i\right) = 0, \quad \forall i, j = 1, \dots, N. \tag{20}$$

This implies[2] $\overline{\theta}_i - \overline{\theta}_j = 0$ or $\overline{\theta}_i - \overline{\theta}_j = \pi$, for all $i, j = 1, \dots, N$. That is, reordering the phase index, $\overline{\theta}_i = \theta_0$, $i = 1, \dots, q_0$, for some $\theta_0 \in [0, 2\pi)$, and $\overline{\theta}_i = \theta_0 + \pi$, $i = q_0 + 1, \dots, N$, where $0 \leq q_0 \leq N$.

Case 1: *N is odd.*

In the case N is an odd number (20) is not compatible with the condition $a(\overline{\theta}) = b(\overline{\theta}) = 0$ imposed in $\mathcal{N}$. Indeed, for any $0 \leq q_0 \leq N$, $a(\overline{\theta}) = q_0 \cos(\theta_0) - (N - q_0)\cos(\theta_0) \neq 0$, for all $\theta_0 \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. If $\theta_0 \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, then $b(\overline{\theta}) = q_0 \sin(\theta_0) - (N - q_0)\sin(\theta_0) \neq 0$, for all $0 \leq q_0 \leq N$. Hence (20) is not satisfied for all $\overline{\theta} \in \mathcal{N}$. We conclude that, in the case when N is odd, the Jacobian of F (19) has rank 2 on $\mathcal{N}$. Hence, $\mathcal{N}$ is a submanifold of codimension 2.

Since $\mathcal{N}$ is a submanifold of codimension 2, we can develop a stability analysis on its orthogonal subspace $\mathcal{N}^\perp$. In view of (7), a base for this subspace at $\theta \in \mathcal{N}$ is given by $(\nabla a(\theta)^T, \nabla b(\theta)^T)$.

We start by computing the expression of the Hessian of W restricted to $\mathcal{N}$ that we denote as $\mathcal{H}(\theta) := H|_{\mathcal{N}}(\theta)$, for all $\theta \in \mathcal{N}$. Basic computations show that, for all $\theta \in \mathcal{N}$,

$$\mathcal{H}(\theta)\nabla a(\theta)^T = \frac{k_0}{2}\left(\nabla a(\theta)^T \sum_{j=1}^{n} \sin^2\theta_j + \nabla b(\theta)^T \sum_{j=1}^{n} \sin\theta_j \cos\theta_j\right) \tag{21}$$

$$\mathcal{H}(\theta)\nabla b^T(\theta) = \frac{k_0}{2}\left(\nabla a^T(\theta) \sum_{j=1}^{n} \sin\theta_j \cos\theta_j + \nabla b^T(\theta) \sum_{j=1}^{n} \cos^2\theta_j\right). \tag{22}$$

Defining $\mathcal{H}^\perp(\theta) = \mathcal{H}|_{\mathcal{N}^\perp}(\theta)$, for all $\theta \in \mathcal{N}$, it follows from (21) and (22), that, in the basis $(\nabla a(\theta)^T, \nabla b(\theta)^T)$, $\mathcal{H}^\perp(\theta)$ is given by

$$\mathcal{H}^\perp(\theta) = \frac{k_0}{2}\begin{pmatrix} \alpha(\theta) & -\gamma(\theta) \\ \gamma(\theta) & -\beta(\theta) \end{pmatrix}, \tag{23}$$

where $\alpha(\theta) := \sum_{j=1}^N \sin^2\theta_j$, $\beta(\theta) := \sum_{j=1}^N \cos^2\theta_j$, and $\gamma(\theta) := \sum_{j=1}^N \sin\theta_j \cos\theta_j$. The eigenvalues of $\mathcal{H}^\perp$ are then given by

$$\lambda_\pm(\theta) = \left(\sum_{j=1}^N \sin^2\theta_j\right) - \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \left(\sum_{j=1}^N \sin\theta_j \cos\theta_j\right)^2}. \tag{24}$$

The following claim ends the proof the lemma in the case N is odd. Its proof is omitted due to lack of space but follows from basic computations.

Claim 5. The functions λ_- and λ_+ defined in (24) satisfy $\lambda_-(\theta) < 0 < \lambda_+(\theta)$ for all $\theta \in \mathcal{N}$.

Case 2: *N is even.*

In the case N is an even number, given the grouping of indexes $\mathcal{I}_0 = \{i_1, \dots, i_{\frac{N}{2}}\}, \mathcal{I}_\pi = \{i_{\frac{N}{2}+1}, \dots, i_N\}$, $i_j \in \{1, \dots, N\}$ for all $j = 1, \dots, N$, such that $\mathcal{I}_0 \cap \mathcal{I}_\pi = \emptyset$, let

$$\mathcal{N}_0 := \{\theta \in \mathbb{R}^N : \theta_i = \theta_0, \forall i \in \mathcal{I}_0, \\ \theta_i = \theta_0 + \pi, \forall i \in \mathcal{I}_\pi, \theta_0 \in \mathbb{R}\}. \tag{25}$$

$\mathcal{N}_0$ is a 1-dimensional manifold parametrized by θ_0. Condition (20) is satisfied for all point in $\mathcal{N}_0$. Moreover, since $a(\theta) = b(\theta) = 0$, for all $\theta \in \mathcal{N}_0$, it holds that $\mathcal{N}_0 \subset \mathcal{N}$. Note that there exists exactly 2^N different groupings $\{\mathcal{I}_0, \mathcal{I}_\pi\}$. Let $\mathcal{N}_{0i}$ be the set of the form (25) relative to the i-th grouping. With the same reasoning as in Case 1, no other sets than $\bigcup_{i=1}^{2^N} \mathcal{N}_{0i}$ in which (20) is satisfied are contained in $\mathcal{N}$. Define $\tilde{\mathcal{N}} := \mathcal{N} \setminus \mathcal{N}_0$. With the same computation as in Case 1, $\tilde{\mathcal{N}}$ is a normally hyperbolic submanifold of codimension 2 with normal eigenvalues $\lambda_\pm$ as defined in (24).

It remains to show that, for each $i = 1, \dots, 2^N$, the stable set of $\mathcal{N}_{0i}$ is contained in a submanifold of dimension 2. In the following, if no confusion can arise we omit the index i. Let $e_i \in \mathbb{R}^{N/2}$ be the vector with all zero entries apart 1 in the i-th position, that is $\{e_i\}_{i=1,\dots,\frac{N}{2}}$ forms the canonical base of $\mathbb{R}^{N/2}$. Recalling (12) and (25), the Jacobian of the dynamics (3) on $\mathcal{N}_0$ can be written, after a reordering of the indexes, as

$$H_0(\theta_0) = -k_0 \sin(2\theta_0)\begin{bmatrix} \mathbf{1}_{N/2\times N/2} & -\mathbf{1}_{N/2\times N/2} \\ -\mathbf{1}_{N/2\times N/2} & \mathbf{1}_{N/2\times N/2} \end{bmatrix}. \tag{26}$$

Clearly, for all $\theta_0 \in \mathbb{R}$, $H_0(\theta_0)\mathbf{1}_N = 0$. Moreover, for all $i, j = 1, \dots, N/2$ and all $\theta_0 \in \mathbb{R}$, $H_0(\theta_0)\begin{pmatrix} e_i \\ e_j \end{pmatrix} = 0$. Hence, for all $\theta \in \mathcal{N}_0$ the tangent space to $\mathcal{N}_0$ at θ contains an $N-1$-dimensional center space E_c, which is spanned by

$$\left\{\begin{pmatrix} e_1 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} e_1 \\ e_{N/2} \end{pmatrix}, \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}, \dots, \begin{pmatrix} e_{N/2} \\ e_1 \end{pmatrix}\right\}.$$

This center space is tangent to the $N-1$-dimensional submanifold (*center manifold*) $\mathcal{M}_c$, which can be locally described as $\mathcal{M}_c := \theta + E_c$. Noticing that, by construction, $\mathcal{M}_c \subset \mathcal{N}$, all the points in the center submanifold $\mathcal{M}_c$ are fixed point of (3), and, thus, they do not belong to the stable set of $\mathcal{N}_0$. From (Sijbrand, 1985, Theorem 3.2' - Case (ii)) this central manifold is unique. By the central manifold theorem (Guckenheimer and Holmes, 1983, Theorem 3.2.1), for all $\theta \in \mathcal{N}_0$, the only other invariant set that contains θ is given by a 1-dimensional submanifold

[2] All this reasoning holds modulo 2π. We omit to write the modulo operator for clarity.

$\mathcal{M}^{u,s}$ that is tangent to $E_c^\perp := \text{span}\begin{pmatrix} -\mathbf{1}_{N/2} \\ \mathbf{1}_{N/2} \end{pmatrix}$. Since the stable set of $\mathcal{N}_0$ is an invariant set, it is contained in the two dimensional submanifold $\mathcal{M}^{us} \times \mathcal{N}_0$.

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