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Phase-locking between Kuramoto oscillators: robustness to time-varying natural frequencies

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Abstract— **In this paper we analyze the robustness of phase-locking in the Kuramoto system with arbitrary bidirectional interconnection topology. We show that the effects of time-varying natural frequencies encompass the heterogeneity in the ensemble of oscillators, the presence of exogenous disturbances, and the influence of unmodeled dynamics. The analysis, based on a Lyapunov function for the incremental dynamics of the system, provides a general methodology to build explicit bounds on the region of attraction, on the size of admissible inputs, and on the input-to-state gains. As an illustrative application of this method, we show that, in the particular case of the all-to-all coupling, the synchronized state is exponentially input-to-state stable provided that all initial phase differences lie in the same half circle. The approach provides an explicit bound on the convergence rate, thus extending recent results on the exponential synchronization of the finite Kuramoto model. Furthermore, the proposed Lyapunov function for the incremental dynamics allows for a new characterization of the robust asymptotically stable phase-locked states of the unperturbed dynamics in terms of its isolated local minima.**

I. INTRODUCTION

Synchronization has recently found many applications in the modeling and control of physical [1], [2], [3], chemical [4], medical [5], biological [6], and engineering problems [7]. Roughly speaking, an ensemble of interacting agents is said to synchronize when their outputs tend to a common value [8, Chapter 5]. Examples of such a behavior can be found in interconnected neurons [6], [9], [10], chemical oscillators [4], coupled mechanical systems [11] and consensus algorithms [12], [13], [14]. Phase-locking, or frequency synchronization, is a particular type of synchronization that describes the ability of interconnected oscillators to tune themselves to the same frequency. One of the most widely used mathematical model to analyze this behavior is the Kuramoto model, which was first introduced in [4] to describe globally coupled chemical oscillators, as a generalization of the one originally proposed by Winfree [15]. Later on, many other works generalized these pioneer seminal works [12], [16], [17], [18], [19], [20], [21], [7], [14], [8], [22], [23]. In this paper we consider the Kuramoto system with time-varying natural frequencies and a general interconnection topology. Letting the signal ϖ_i denote the *time-varying natural frequency* of the oscillator $i \in \{1, \dots, N\}$, and $k = [k_{ij}]_{i,j=1,\dots,N} \in \mathbb{R}^{N\times N}_{\geq 0}$ represent the *coupling matrix*, each agent i is described by its phase θ_i ruled by:

$$\dot{\theta}_i(t) = \varpi_i(t) + \sum_{j=1}^{N} k_{ij} \sin(\theta_j(t) - \theta_i(t)), \ \forall t \geq 0. \qquad (1)$$

The analysis of robustness with respect to time-varying natural frequencies encompasses different types of perturbations, including agents heterogeneity, influence of exogenous inputs and imprecise modeling (cf. equation (6) below). In particular, it may include the effects of the feedback signals studied in the literature for their desynchronizing features [24], [25], [26], [27], [28], [29], as well as time-varying interconnection topologies and non-sinusoidal coupling. This issue is particularly relevant for the study of interconnected neuronal cells for which little is known on the interconnection topology and synaptic weights between neurons [30].

The robustness of phase-locking in the Kuramoto model has already been partially addressed in the literature both in the case of infinite and finite number of oscillators. On the one hand, the infinite dimensional Kuramoto model allows for an easier analytical treatment of the robustness analysis (see for example [17] for a complete survey). This approach has been used to analyze the effect of delayed [28] and multisite [25] mean-field feedback approach to desynchronization. In the case of stochastic inputs it allows to find the minimum coupling to guarantee phase-locking in the presence of noise [31]. However, this approach is feasible only in the case of the all-to-all interconnection. On the other hand, the finite dimensional case has been the object of both analytical and numerical studies. In particular, [32] proposes a complete numerical analysis of robustness to time-varying natural frequencies, time-varying interconnection topologies and non-sinusoidal coupling. It suggests that phase-locking exhibits some robustness to all these types of perturbations. Analytical studies on the robustness of phase-locking in the finite Kuramoto model have been addressed only for constant natural frequencies [19], [16], [23]. The existence and explicit expression of the fixed points describing stable and unstable phase-locked states is studied in [18], [12]. The Lyapunov approach proposed in [20] for an all-to-all coupling suggests that an analytical study of phase-locking robustness can be deepened. To the best of our knowledge the problem of the robustness of phase-locking with respect to time-varying natural frequencies has still not been analytically addressed in the finite Kuramoto model with arbitrary bidirectional interconnection topologies.

This paper establishes that phase-locking is locally input-to-state stable (ISS) with respect to small inputs (total stability). The proof is based on the existence of a local ISS-Lyapunov function for the incremental dynamics of the system. This analysis provides a general methodology to build explicit estimates on the size of the region of convergence, the ISS gain, and the tolerated input bound. It applies to general symmetric inter-

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connection topologies and to any asymptotically stable phase-locked state. As an illustrative application of the main theorem, we extend some results in [16], [23] to the time-varying case, by proving the exponential ISS of synchronization when all the initial phase differences lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and by giving explicit bounds on the convergence rate. The size of the region of convergence, the sufficient bound on the coupling strength and the convergence rate are compared to those obtained in [16], [23]. Furthermore, the Lyapunov function for the incremental dynamics allows for a new characterization of the phase-locked states of the unperturbed system. In particular, when restricted to a suitable invariant manifold, it allows to completely characterize the robust phase-locked states in terms of its isolated local minima.

Due to space limitations, only the proof of the main result is included in the present document. The interested reader can find all technical details in the extended versions [33], [29].

Notation. For a set $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, $A_{\geq a}$ denotes the set $\{x \in A : x \geq a\}$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm, that is $|x| := \sqrt{\sum_{i=1}^n x_i^2}$, while $|x|_\infty$ denotes its infinity norm, that is $|x|_\infty := \max_{i=1,\dots,n} |x_i|$. We adopt the notation $|x|_2 := |x|$, when we want to explicitly distinguish $|x|$ from $|x|_\infty$. For a set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |y - x|$ denotes the point-to-set distance from x to $\mathcal{A}$. $\mathcal{B}(x, R)$ refers to the closed ball of radius R centered at x in the Euclidean norm, *i.e.* $\mathcal{B}(x, R) := \{z \in \mathbb{R}^n : |x - z| \leq R\}$. T^n denotes the n-Torus. $\|u\|$ is the L^1 norm of the signal $u(\cdot)$, that is, if $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ denotes a measurable signal, locally essentially bounded, $\|u\| := \mathrm{esssup}_{t \geq 0} |u(t)|$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any fixed $t \geq 0$ and $\beta(s, \cdot)$ is continuous decreasing and tends to zero at infinity for any fixed $s \geq 0$. If $x \in \mathbb{R}^n$, ∇_x is the *gradient vector* with respect to x, *i.e.* $\nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$. Given $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$, ($x \mod a$) $:= [x_i \mod a]_{i=1,\dots,n}$, where $\mod$ denotes the modulo operator. The vector with all unitary components in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$.

II. Robustness of phase-locked solutions

A. Robustness analysis

Phase-locking can be formally defined based on the *incremental dynamics* $\dot{\theta}_i - \dot{\theta}_j$ associated to (1). Roughly speaking a phase-locked solution corresponds to a fixed point of the incremental dynamics. This formulation is equivalent to the one given in [8, Definition 5.1],[18], [7].

Definition 1 (Phase-locking / Exact synchronization) A solution θ^* to system (1) is said to be *phase-locked* iff

$$\dot{\theta}_j^*(t) - \dot{\theta}_i^*(t) = 0, \quad \forall\, i, j = 1, \dots, N,\ \forall t \geq 0.$$

It is said to be *exactly synchronized* if it is phase-locked with zero phase differences, that is

$$\theta_i^*(t) - \theta_j^*(t) = 0, \quad \forall\, i, j = 1, \dots, N,\ \forall t \geq 0.$$

In view of this definition, the robustness analysis of phase-locked solutions boils down to the analysis of the fixed points of the dynamics ruling the phase differences $\theta_i - \theta_j$. A similar approach has been exploited in [16], [23] in the case of all-to-all coupling and constant inputs. In contrast to [19], studying the incremental dynamics of the system avoids the use of the grounded Kuramoto model, in which the mean frequency of the ensemble is "grounded" to zero and synchronization corresponds to a fixed point. While the latter is a well defined mathematical object for constant perturbations, its extension to time-varying inputs, which is the subject of the present study, is not clear. Hence, we start by defining the *common drift* $\overline{\varpi}$ of the system (1) as

$$\overline{\varpi}(t) := \frac{1}{N} \sum_{j=1}^N \varpi_j(t), \quad \forall t \geq 0, \tag{2}$$

and the *grounded input* $\tilde{\omega}$ as $\tilde{\omega} := [\tilde{\omega}_i]_{i=1,\dots,N}$, where

$$\tilde{\omega}_i(t) := \varpi_i(t) - \overline{\varpi}(t), \quad \forall i = 1, \dots, N,\ \forall t \geq 0. \tag{3}$$

Noticing that $\varpi_i - \varpi_j = \tilde{\omega}_i - \tilde{\omega}_j$, the evolution equation of the *incremental dynamics* ruled by (1) reads

$$\begin{aligned} &\dot{\theta}_i(t) - \dot{\theta}_j(t) = \tilde{\omega}_i(t) - \tilde{\omega}_j(t) + \\ &\quad \textstyle\sum_{l=1}^N k_{il} \sin(\theta_l(t) - \theta_i(t)) - \sum_{l=1}^N k_{jl} \sin(\theta_l(t) - \theta_j(t)) \end{aligned} \tag{4}$$

for all $i, j = 1, \dots, N$, $i \neq j$, and all $t \geq 0$. In the sequel we use $\tilde{\theta}$ to denote the *incremental variable*:

$$\tilde{\theta} := [\theta_i - \theta_j]_{i,j=1,\dots,N, i \neq j} \in \mathrm{T}^{(N-1)^2}. \tag{5}$$

As expected, the incremental dynamics (4) is independent of $\overline{\varpi}$, meaning that it is invariant to common drifts among the oscillators. As stressed in the introduction, the system (1), and thus its incremental dynamics (4), encompasses both the heterogeneity between agents, the presence of exogenous disturbances and the uncertainties in the interconnection topology. To see this clearly, let ω_i denote the constant natural frequency of the agent i, let p_i represent its additive external perturbations, and let Δ_{ij} denote the uncertainty on each coupling gain k_{ij}. Then the effects of all these disturbances can be analyzed in a unified manner by letting, for all $t \geq 0$,

$$\varpi_i(t) = \omega_i + p_i(t) + \sum_{j=1}^N \Delta_{ij}(t) \sin(\theta_j(t) - \theta_i(t)). \tag{6}$$

When no inputs are applied, *i.e.* $\tilde{\omega} = 0$, we expect the solutions of (4) to converge to some asymptotically stable fixed point or, equivalently, the solution of (1) to converge to some asymptotically stable phase-locked solution at least for some coupling matrices k. To make this precise, we start by defining the notion of 0-*asymptically stable* (0-AS) *phase-locked solutions*, which are described by asymptotically stable fixed points of the incremental dynamics (4) when no inputs are applied. This concept is related to asymptotic synchronization (cf. [16]).

Definition 2 (0-AS phase-locked solutions) Given a coupling matrix $k \in \mathbb{R}_{\geq 0}^{N \times N}$, let $\mathcal{O}_k$ denote the set of all asymptotically stable fixed points of the unperturbed (*i.e.* $\tilde{\omega} \equiv 0$) incremental dynamics (4). A phase-locked solution θ^* of (1) is said to be 0-*asymptotically stable* if and only if the incremental state $\tilde{\theta}^* := \left[\theta_i^* - \theta_j^*\right]_{i,j=1,\dots,N, i \neq j}$ belongs to $\mathcal{O}_k$.

A characterization of 0-AS phase-locked solutions of (1) for general bidirectional interconnection topologies can be found in [34] and [14, Chapter 3]. In Section II-C, we characterize the set $\mathcal{O}_k$ in terms of the isolated local minima of a suitable Lyapunov function. The reason for considering only *asymptotically* stable fixed points of the incremental dynamics stands in the fact that only those are expected to provide some robustness properties (as asymptotic stability implies local robustness with respect to small inputs [35], [36]). On the contrary, (non 0-AS) stable fixed points may correspond to non-robust phase-locked state, as illustrated by the following example.

We next recall the definition of local Input-to-State Stability with respect to small inputs [37]. This concept is also referred to as *Total Stability* [36].

Definition 3 (LISS w.r.t. small inputs) For a system of the form $\dot{x} = f(x, u)$, a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *locally input-to-state stable (LISS) with respect to small inputs* iff there exist some constants $\delta_x, \delta_u > 0$, a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function ρ, such that, for all $|x_0|_{\mathcal{A}} \leq \delta_x$ and all u satisfying $\|u\| \leq \delta_u$, its solution satisfies

$$|x(t)|_{\mathcal{A}} \leq \beta(|x_0|_{\mathcal{A}}, t) + \rho(\|u\|), \quad \forall t \geq 0.$$

If this estimate holds with $\beta(r,s) = Cre^{-\frac{s}{\tau}}$, where C, τ are positive constants, then $\mathcal{A}$ is said to be *locally exponentially Input-to-State Stable with respect to small inputs*.

Remark 1 (Local Euclidean metric on the n-Torus) Definition 3 is given on $\mathbb{R}^n$, which is little adapted to the context of this article. Its extension to the n-Torus is natural since T^n is locally isometric to $\mathbb{R}^n$ through the identity map $\mathcal{I}$ (*i.e.* $|\theta|_{\mathrm{T}^n} := |\mathcal{I}(\theta)| = |\theta|$). In particular this means that the n-Torus can be provided with the local Euclidean metric and its induced norm. Hence, Definition 3 applies locally in the n-Torus.

The next theorem, whose proof is given in Section III, states the LISS of $\mathcal{O}_k$ with respect to small inputs $\tilde{\omega}$.

Theorem 1 (LISS of phase-locking w.r.t. small inputs) *Let $k \in \mathbb{R}^{N \times N}_{\geq 0}$ be any symmetric interconnection matrix. Suppose that the set $\mathcal{O}_k$ of Definition 3 is non-empty. Then the system (4) is locally input-to-state stable with respect to small $\tilde{\omega}$. In other words, there exist $\delta_{\tilde{\theta}}, \delta_\omega > 0$, $\beta \in \mathcal{KL}$ and $\rho \in \mathcal{K}_\infty$, such that, for all $\tilde{\omega}$ satisfying $\|\tilde{\omega}\| \leq \delta_\omega$ and all $|\tilde{\theta}_0|_{\mathcal{O}_k} \leq \delta_{\tilde{\theta}}$, its solution satisfies*

$$|\tilde{\theta}(t)|_{\mathcal{O}_k} \leq \beta(|\tilde{\theta}_0|_{\mathcal{O}_k}, t) + \rho(\|\tilde{\omega}\|), \quad \forall t \geq 0. \tag{7}$$

Theorem 1 guarantees that, if a given configuration is asymptotically stable for the unperturbed system, then solutions starting sufficiently near from that configuration remain near it at all time, in presence of sufficiently small perturbations $\tilde{\omega}$. Moreover, the steady-state distance of the incremental state $\tilde{\theta}$ from $\mathcal{O}_k$ is somehow "proportional" to the amplitude of $\tilde{\omega}$ through the nonlinear gain ρ. This means that the phase-locked states described by $\mathcal{O}_k$ are robust to time-varying natural frequencies, provided they are not too heterogeneous. We stress that, while local ISS with respect to small inputs is a natural consequence of asymptotic stability [35], the size of the constants δ_x and δ_u in Definition 3, defining the robustness domain in terms of initial conditions and inputs amplitude, are potentially very small. As we show explicitly in the next section in the special case of all-to-all coupling, the Lyapunov analysis used in the proof of Theorem 1 (cf. Section III) provides a general *methodology* to build these estimates explicitly. We stress in particular that, while the region of attraction depends on the geometric properties of the fixed points of the unperturbed system, the size of admissible inputs can be made arbitrarily large by taking a sufficiently large coupling strength. This is detailed in the sequel (cf. (19), (21), (25) and (26)).

Question 1 (Link with algebraic connectivity) *In related works on the existence and robustness of phase-locking [19], [23] or consensus [13] the algebraic connectivity of the underlying interconnection graph plays a crucial role. In the present work the estimate of the region of attraction and of the tolerated inputs relies on the geometrical properties of a graph-dependent global Lyapunov function. It is an open question to characterize the link between the two approaches.*

B. Robustness of the synchronized state in the case of all-to-all coupling

In this section we focus the Lyapunov analysis used in the proof of Theorem 1 to the case of the all-to-all coupling. In this case, it is known [34] that the only asymptotically stable phase-locked solution for the unperturbed dynamics is the *exact synchronization* corresponding to a zero phase difference between each pair of oscillators (cf. Definition 1). The following proposition states the local *exponential* input-to-state stability of the synchronized state with respect to small inputs, and provides explicit bounds on the region of convergence, the size of admissible inputs, the ISS gain, and the convergence rate. Its detailed proof can be found in [33, Section III-B].

Proposition 1 (Exponential LISS of synchronization) *Consider the system (1) with the all-to-all interconnection topology, i.e. $k_{ij} = K > 0$ for all $i, j = 1, \ldots, N$. Then, for all $0 \leq \epsilon \leq \frac{\pi}{2}$, and all $\tilde{\omega}$ satisfying*

$$\|\tilde{\omega}\| \leq \delta_\omega^\epsilon := \frac{K\sqrt{N}}{\pi^2}\left(\frac{\pi}{2} - \epsilon\right), \tag{8}$$

the following facts hold:

1) *the set $\mathcal{D}_\epsilon := \left\{\tilde{\theta} \in T^{(N-1)^2} : |\tilde{\theta}|_\infty \leq \frac{\pi}{2} - \epsilon\right\}$ is forward invariant for the system (4);*
2) *for all $\tilde{\theta}_0 \in \mathcal{D}_0$, the set $\mathcal{D}_\epsilon$ is attractive, and the solution of (4) satisfies*

$$|\tilde{\theta}(t)| \leq \frac{\pi}{2}|\tilde{\theta}_0|e^{-\frac{K}{\pi^2}t} + \frac{\pi^2}{K}\|\tilde{\omega}\|, \quad \forall t \geq 0.$$

Proposition 1 establishes the exponential ISS of the synchronized state in the all-to-all Kuramoto model with respect to time-varying inputs whose amplitudes are smaller than $\frac{K\sqrt{N}}{2\pi}$. It holds for any initial condition lying in $\mathcal{D}_0$, that is when all the initial phase differences lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Moreover, if the inputs amplitude is bounded by δ_ω^ϵ, for some $0 \leq \epsilon \leq \frac{\pi}{2}$, then the set $\mathcal{D}_\epsilon$ is forward invariant and all the solutions starting in $\mathcal{D}_0$ actually converge to $\mathcal{D}_\epsilon$.

Recently, necessary and sufficient conditions for the exponential synchronization of the Kuramoto system with all-to-all coupling and constant different natural frequencies were given in [16]. We stress that the estimated region of attraction provided by Proposition 1 is strictly larger than the one obtained in [16, Theorem 4.1], which does not allow ϵ to be picked as zero. For initial conditions lying in $\mathcal{D}_\epsilon$, with a strictly positive ϵ, it is interesting to compare the convergence rate obtained in Proposition 1, $\frac{K}{\pi^2}$, with the one obtained in [16, Theorem 3.1], $NK\sin(\epsilon)$. While the convergence rate of Proposition 1 is slower than the one obtained in [16, Theorem 3.1] for large ϵ, it provides a better estimate for small values of ϵ. Furthermore, for any fixed amplitude $\|\tilde{\omega}\| \le \delta_\omega^\epsilon$, the bound (8) allows to find the sufficient coupling strength K_ϵ which ensures the attractivity of $\mathcal{D}_\epsilon$. After some computations (see [33] for details) this bound reads

$$K_\epsilon \le \frac{\pi^3}{2\cos(\epsilon)} \max_{i,j=1,\dots,N} \|\varpi_i - \varpi_j\|.$$

This bound is of the same order of the one provided in [16, Proof of Theorem 4.1] K_{inv}, in the sense that, for $\epsilon \neq 0$, $\frac{K_\epsilon}{K_{inv}} < \pi^3$. For the same region of attraction, a tighter bound K_{suff} for the sufficient coupling strength has recently been given in [23], where this bound is inversely proportional to the number of oscillators, that is $\frac{K_\epsilon}{K_{suff}} \sim N$. Nonetheless, similarly to [16], their rate of convergence is proportional to $\sin(\epsilon)$, leading to a worse bound than ours for large regions of attraction.

In conclusion, Proposition 1 partially extends the main results of [16], [23] to time-varying inputs. On the one hand, it allows to consider sets of initial conditions larger than those of [16], and bounds the convergence rate by a strictly positive value, independently of the region of attraction. On the other hand the required coupling strength is comparable to the one given in [16], but more conservative than the lower bound in [23]. Finally for small regions of attraction, the bound on the convergence rate obtained in Proposition 1 is not as good as the one of [16], [23].

C. A Lyapunov function for the incremental dynamics

In this section, we introduce the Lyapunov function for the incremental dynamics (4) used in the proof of Theorem 1, that will be referred to as the *incremental Lyapunov function* in the sequel. We start by showing that the incremental dynamics (4) possesses an invariant manifold, that we characterize through some linear relations. This observation allows us to restrict the analysis of the critical points of the Lyapunov function to this manifold. Beyond its technical interest, this analysis shows that phase-locked solutions correspond to these critical points. In particular, it provides an analytic way of computing the set $\mathcal{O}_k$ of Definition 2, completely characterizing the set of robust asymptotically stable phase-locked solutions. Furthermore, we give some partial extensions on existing results on the robustness of phase-locking in the finite Kuramoto model.

The incremental Lyapunov function: We start by introducing the *normalized interconnection matrix* associated to k

$$E = [E_{ij}]_{i,j=1,\dots,N} := \frac{1}{K}\,[k_{ij}]_{i,j=1,\dots,N}\,, \tag{9}$$

where the scalar K is defined as

$$K = \max_{i,j=1,\dots,N} k_{ij}. \tag{10}$$

Inspired by [14, Chapter 3], we consider the *incremental* Lyapunov function $V_I : \mathrm{T}^{(N-1)^2} \to \mathbb{R}_{\ge 0}$ defined by

$$V_I(\tilde{\theta}) = 2\sum_{i=1}^{N}\sum_{j=1}^{N} E_{ij}\sin^2\left(\frac{\theta_i-\theta_j}{2}\right), \tag{11}$$

where the incremental variable $\tilde{\theta}$ is defined in (5). We stress that V_I is independent of the coupling strength K.

The invariant manifold: Before analyzing the behavior of the function V_I along the solutions of (4), we stress the existence and identify an invariant manifold for the dynamics of interest. The presence of an invariant manifold results from the fact that the components of the incremental variable $\tilde{\theta}$ are not linearly independent. Indeed, we can express $(N-1)(N-2)$ of them in terms of the other $N-1$ independent components. More precisely, by choosing $\varphi_i := \theta_i - \theta_N$, $i = 1, \dots, N-1$ as the independent variables, it is possible to write, for all $i = 1, \dots, N$,

$$\theta_i - \theta_N = \varphi_i, \tag{12a}$$
$$\theta_i - \theta_j = \varphi_i - \varphi_j, \quad \forall j = 1, \dots, N-1. \tag{12b}$$

These relations can be expressed in a compact form as

$$\tilde{\theta} = \tilde{B}(\varphi) := B\varphi \mod 2\pi, \quad \varphi \in \mathcal{M}, \tag{13}$$

where $\varphi := [\varphi_i]_{i=1,\dots,N-1}$, $B \in \mathbb{R}^{(N-1)^2\times(N-1)}$, rank$B = N-1$, $\tilde{B}$ is continuous and continuously differentiable, and $\mathcal{M} \subset \mathrm{T}^{(N-1)^2}$ is the submanifold defined by the embedding (13). The continuous differentiability of $\tilde{B} : \mathcal{M} \to \mathrm{T}^{(N-1)^2}$ comes from the fact that $\varphi_i \in \mathrm{T}^1$, for all $i = 1, \dots, N$, and the components of $\tilde{B}(\varphi)$ are linear functions of the form (12). Formally, this means that the system is evolving in the invariant submanifold $\mathcal{M} \subset \mathrm{T}^{(N-1)^2}$ of dimension $N-1$. In particular $\mathcal{M}$ is diffeomorphic to T^{N-1}.

Restriction to the invariant manifold: In order to conduct a Lyapunov analysis based on V_I it is important to identify its critical points. Since the system is evolving on the invariant manifold $\mathcal{M}$, only the critical points of the Lyapuonov function V_I restricted to this manifold are of interest. Hence we restrict our attention to the critical points of the restriction of V_I to $\mathcal{M}$, which is defined by the function $V_I|_{\mathcal{M}} : \mathrm{T}^{N-1} \to \mathbb{R}$ as

$$V_I|_{\mathcal{M}}(\varphi) := V_I(B\varphi), \quad \forall \varphi \in \mathcal{M}. \tag{14}$$

The analysis of the critical points of $V_I|_{\mathcal{M}}$ is not trivial. To simplify this problem, we exploit the fact that the variable φ can be expressed in terms of θ by means of a linear transformation $A \in \mathbb{R}^{(N-1)\times N}$, with rank$A = N-1$, in such a way that

$$\varphi = \tilde{A}(\theta) = A\theta \mod 2\pi. \tag{15}$$

Based on this, we define the function $V : \mathrm{T}^N \to \mathbb{R}$ as

$$V(\theta) = V_I|_{\mathcal{M}}(A\theta). \tag{16}$$

In contrast with $V_I|_{\mathcal{M}}$, the critical points of V are already widely studied in the synchronization literature, see for instance [7, Section III] and [14, Chapter 3]. The following lemma allows to reduce the analysis of the critical points of V_I on $\mathcal{M}$ to that of the critical points of V on T^N. Its proof is given in [33, Section IV-E].

Lemma 1 (Computation of the critical points on the invariant manifold) *Let $\mathcal{M}$, $V_I|_{\mathcal{M}}$, A and V be defined by (13)-(16). Then $\theta^* \in T^N$ is a critical point of V (i.e. $\nabla_\theta V(\theta^*) = 0$) if and only if $\varphi^* = A\theta^* \in \mathcal{M}$ is a critical point of $V_I|_{\mathcal{M}}$ (i.e. $\nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0$). Moreover if θ^* is a local maximum (resp. minimum) of V then φ^* is a local maximum (resp. minimum) of $V_I|_{\mathcal{M}}$. Finally the origin of $\mathcal{M}$ is a local minimum of $V_I|_{\mathcal{M}}$.*

Lyapunov characterization of robust phase-locking: The above development allows to characterize phase-locked states through the incremental Lyapunov function V_I. The following lemma, whose proof is given in [33, Section IV-F], states that the fixed points of the unperturbed incremental dynamics are the critical points of $V_I|_{\mathcal{M}}$, modulo the linear relations (12). That is, recalling Definition 1, the critical points of $V_I|_{\mathcal{M}}$ completely characterize phase-locked solutions.

Lemma 2 (Incremental Lyapunov characterization of phase-locking) *Let $k \in \mathbb{R}_{\geq 0}^{N\times N}$ be a symmetric interconnection matrix. Let B and $V_I|_{\mathcal{M}}$ be defined as in (13) and (14). Then $\varphi^* \in \mathcal{M}$ is a critical point of $V_I|_{\mathcal{M}}$ (i.e. $\nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0$) if and only if $B\varphi^*$ is a fixed point of the unperturbed (i.e. $\tilde{\omega} = 0$) incremental dynamics (4).*

Remark 2 (Incremental Lyapunov characterization of robust 0-AS phase-locked solutions) The Lyapunov function V_I is strictly decreasing along the trajectories of (4) if and only if the state does not belong to the set of critical points of $V_I|_{\mathcal{M}}$ (this will be rigorously shown by Claim 1 below for $\tilde{\omega} = 0$). It then follows directly from Lemma 2 that isolated local minima of $V_I|_{\mathcal{M}}$ correspond to asymptotically stable fixed points of (4). By and Theorem 1, we conclude that the robust asymptotically stable phase-locked states are completely characterized by the set of isolated local minima of $V_I|_{\mathcal{M}}$. The computation of this set is simplified through Lemma 1.

Consequence for the system without inputs: At the light of Lemma 2, we can state the following corollary, which recovers, and partially extends, the result of [14, Proposition 3.3.2] in terms of the incremental dynamics of the system. It states that, for a symmetric interconnection topology, any disturbance with zero grounded input (3) preserves the almost global asymptotic stability of phase-locking for (1).

Corollary 1 (Almost global asymptotic phase-locking) *Let $\varpi : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ be any signal satisfying $\tilde{\omega}(t) = 0$, for all $t \geq 0$, where $\tilde{\omega}$ is defined in (3). If the interconnection matrix $k \in \mathbb{R}_{\geq 0}^{N\times N}$ is symmetric, then almost all trajectories of (1) converge to a stable phase-locked solution.*

We stress that Corollary 1 is an *almost global* result. It follows from the fact that almost all trajectories converge to the set of local minima of $V_I|_{\mathcal{M}}$. From Lemma 2, this set corresponds to stable fixed points of the incremental dynamics, that is to stable phase-locked solutions. The precise proof is omitted here for lack of space.

III. Proof of Theorem 1

In order to develop our robustness analysis we consider the Lyapunov function defined in (11), where the incremental variable $\tilde{\theta}$ is defined in (5), and the normalized interconnection matrix E is defined in (9). The derivative of V_I along the trajectories of the incremental dynamics (4) yields $\dot{V}_I(\tilde{\theta}) = (\nabla_{\tilde{\theta}} V_I)^T \dot{\tilde{\theta}}$. The following claim, whose proof is given in [33, Section IV-A], provides an alternative expression for $\dot{V}_I$.

Claim 1 *If k is symmetric, then $\dot{V}_I = -2(K\chi^T\chi + \chi^T\tilde{\omega})$, where $\chi(\tilde{\theta}) := \nabla_\theta V(\tilde{\theta}) = \left[\sum_{j=1}^N E_{ij}\sin(\theta_j - \theta_i)\right]_{i=1,\dots,N}$.*

From Claim 1, we see that if the inputs are small, there are regions of the phase space where the derivative of V_I is negative even in the presence of perturbations. More precisely, it holds that:

$$|\chi| \geq \frac{2|\tilde{\omega}|}{K} \quad \Rightarrow \quad \dot{V}_I \leq -K\chi^T\chi.$$

However, LISS does not follow yet as these regions are given in terms of χ instead of the phase differences $\tilde{\theta}$. In order to estimate these region in terms of $\tilde{\theta}$, we define $\mathcal{F}$ as the set of critical points of $V_I|_{\mathcal{M}}$ (*i.e.* $\mathcal{F} := \{\varphi^* \in \mathcal{M} : \nabla_\varphi V_I|_{\mathcal{M}}(\varphi^*) = 0\}$), where $\mathcal{M}$ and $V_I|_{\mathcal{M}}$ are defined in (13) and (11), respectively. Then, from Lemma 1 and recalling that $\chi = \nabla_\theta V$, it holds that $|\chi| = 0$ if and only if $\tilde{\theta} \in \mathcal{F}$. Since $|\chi|$ is a positive definite function of $\tilde{\theta}$ defined on a compact set, [38, Lemma 4.3], guarantees the existence of a $\mathcal{K}_\infty$ function σ such that, for all $\tilde{\theta} \in \mathrm{T}^{(N-1)^2}$,

$$|\chi| \geq \sigma(|\tilde{\theta}|_{\mathcal{F}}). \tag{17}$$

Let $\mathcal{U} := \mathcal{F} \setminus \mathcal{O}_k$. In view of Lemma 2, $\mathcal{U}$ denotes the set of all the critical points of $V_I|_{\mathcal{M}}$ which are not asymptotically stable fixed points of the incremental dynamics. Since $\nabla V_I|_{\mathcal{M}}$ is a Lipschitz function defined on a compact space, it can be different from zero only on a finite collection of open sets. That is $\mathcal{U}$ and $\mathcal{O}_k$ can be expressed as the disjoint union of a finite family of closed sets:

$$\mathcal{U} = \bigcup_{i\in I_{\mathcal{U}}} \nu_i, \quad \mathcal{O}_k = \bigcup_{i\in I_{\mathcal{O}_k}} \{\phi_i\}, \tag{18}$$

where $I_{\mathcal{U}}, I_{\mathcal{O}_k} \subset \mathbb{N}$ are finite sets, $\{\nu_i,\ i \in I_{\mathcal{U}}\}$ is a family of closed subsets of $\mathcal{M}$, and $\left\{\{\phi_i\},\ i \in I_{\mathcal{O}_k}\right\}$ is a family of singletons of $\mathcal{M}$. We stress that $a \neq b$ implies $a \cap b = \emptyset$ for any $a, b \in \{\nu_i,\ i \in I_{\mathcal{U}}\} \bigcup \left\{\{\phi_i\},\ i \in I_{\mathcal{O}_k}\right\} =: \mathcal{F}_S$. Define

$$\delta := \min_{a,b\in\mathcal{F}_S, a\neq b}\ \inf_{\tilde{\theta}\in a} |\tilde{\theta}|_b, \tag{19}$$

which represents the minimum distance between two critical sets, and, at the light of Lemma 2, between two fixed points of the unpertubed incremental dynamics (1). Note that, since $\mathcal{F}_S$ is finite, $\delta > 0$. Define

$$\delta'_\omega = \frac{K}{2}\sigma\left(\frac{\delta}{2}\right), \tag{20}$$

and let

$$\delta_{\tilde{\theta}} := \frac{\delta}{2}, \tag{21}$$

which gives an estimate of the size of the region of attraction, modulo the shape of the level sets of the Lyapunov function V_I. Then the following claim holds true. Its proof is given in [33, Lemma IV-B].

Claim 2 *For all $i \in I_{\mathcal{O}_k}$, all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$, and all $|\tilde{\omega}| \leq \delta'_\omega$, it holds that*

$$|\tilde{\theta} - \phi_i| \geq \sigma^{-1}\left(\frac{2|\tilde{\omega}|}{K}\right) \quad \Rightarrow \quad \dot{V}_I \leq -K\sigma^2(|\tilde{\theta} - \phi_i|).$$

For all $i \in I_{\mathcal{O}_k}$, the function $V_I(\tilde{\theta}) - V_I(\phi_i)$ is zero for $\tilde{\theta} = \phi_i$, and strictly positive for all $\theta \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}}) \setminus \phi_i$. Hence it is positive definite on $\mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$. Noticing that $\mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$ is compact, [38, Lemma 4.3] guarantees the existence of $\mathcal{K}$ functions $\underline{\alpha}_i, \overline{\alpha}_i$ defined on $[0, \delta_{\tilde{\theta}}]$ such that, for all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$,

$$\underline{\alpha}_i(|\tilde{\theta} - \phi_i|) \leq V_I(\tilde{\theta}) - V_I(\phi_i) \leq \overline{\alpha}_i(|\tilde{\theta} - \phi_i|). \tag{22}$$

The two functions can then be picked as $\mathcal{K}_\infty$ by choosing a suitable prolongation on $\mathbb{R}_{\geq 0}$ Define the following two $\mathcal{K}_\infty$ functions

$$\underline{\alpha}(s) := \min_{i \in I_{\mathcal{O}_k}} \underline{\alpha}_i(s), \; \overline{\alpha}(s) := \max_{i \in I_{\mathcal{O}_k}} \overline{\alpha}_i(s), \; \forall s \geq 0. \tag{23}$$

It then holds that, for all $i \in I_{\mathcal{O}_k}$, and all $\tilde{\theta} \in \mathcal{B}(\phi_i, \delta_{\tilde{\theta}})$

$$\underline{\alpha}(|\tilde{\theta} - \phi_i|) \leq V_I(\tilde{\theta}) - V_I(\phi_i) \leq \overline{\alpha}(|\tilde{\theta} - \phi_i|). \tag{24}$$

In view of Claim 2 and (24), an estimates of the ISS gain is then given by

$$\rho(s) := \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \sigma^{-1}\left(\frac{2}{K}s\right), \; \forall s \geq 0 \tag{25}$$

where σ is defined in (17). In the same way, the tolerated input bound is given by

$$\delta_\omega := \rho^{-1}(\delta_{\tilde{\theta}}) \leq \delta'_\omega. \tag{26}$$

From [39, Section 10.4] and Claim 2, it follows that, for all $\|\tilde{\omega}\| \leq \delta_\omega$, the set $|\tilde{\theta}_0|_{\mathcal{O}_k} \leq \delta_{\tilde{\theta}}$ is forward invariant for the system (4) . Furthermore, invoking [40] and [39, Section 10.4], Claim 2 thus implies LISS with respect to small inputs satisfying $\|\tilde{\omega}\| \leq \delta_\omega$, meaning that there exists a class $\mathcal{KL}$ function β such that, for all $\|\tilde{\omega}\| \leq \delta_\omega$, and all $|\tilde{\theta}_0|_{\mathcal{O}_k} \leq \delta_{\tilde{\theta}}$, the trajectory of (4) satisfies $|\tilde{\theta}(t)| \leq \beta(|\tilde{\theta}_0|, t) + \rho(\|\tilde{\omega}\|)$, for all $t \geq 0$. □

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