About Boyd functions, admissible sequences and interpolation

Thomas Lamby

FARF4 - Porquerolles

September 8, 2022

Boyd functions

A function $\phi : (0, \infty) \to (0, \infty)$ is a *Boyd function* if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0, \infty)$. The *lower* and *upper Boyd indices* of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

Boyd functions

A function $\phi : (0,\infty) \to (0,\infty)$ is a *Boyd function* if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0,\infty)$.The *lower* and *upper Boyd indices* of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

Admissible sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. Let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively. Given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0 & \text{if } t \in (0, 1) \end{cases},$$

with $\sigma_0 = 1$ is a Boyd function.

Admissible sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. Let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively. Given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \frac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0 & \text{if } t \in (0,1) \end{cases},$$

with $\sigma_0 = 1$ is a Boyd function.

Admissible sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. Let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively. Given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0 & \text{if } t \in (0,1) \end{cases},$$

with $\sigma_0 = 1$ is a Boyd function.

We will denote by $\mathcal{B}^\infty$ the set of continuous functions $\phi : [1,\infty) \to I$ such that $\phi(1) = 1$ and

$$0 < \underline{\phi}(t) := \inf_{s \geq 1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s \geq 1} \frac{\phi(ts)}{\phi(s)} < \infty,$$

for any $t \geq 1$. Given $\phi \in \mathcal{B}$, we denote by ϕ_∞ the restriction of ϕ to $[1,\infty)$ and by ϕ_0 the restriction of ϕ to $(0,1]$.

Proposition

The application

$$\tau : \mathcal{B} \to \mathcal{B}^\infty \times \mathcal{B}^\infty \quad \phi \mapsto (t \mapsto \frac{1}{\phi_0(1/t)}, \phi_\infty)$$

is a bijection.

Some instructive examples

Consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define the admissible sequence σ by

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ 2^{j_{2n}} 4^{j - j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases}.$$

The sequence oscillates between $(j)_j$ and $(2^j)_j$ and we have $\underline{s}(\sigma) = 0$ and $\overline{s}(\sigma) = 1$.

Relations between Boyd functions and admissible sequences

Proposition

If $\phi \in \mathcal{B}$ and $\sigma_j = \phi(2^j)$ or $\sigma_j = 1/\phi(2^{-j})$ then we have $\underline{b}(\phi) \leq \underline{s}(\sigma) \leq \overline{s}(\sigma) \leq \overline{b}(\phi)$.

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Corollary

If ϕ belongs to $\mathcal{B}$, then we have $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ and $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.

Relations between Boyd functions and admissible sequences

Proposition

If $\phi \in \mathcal{B}$ *and* $\sigma_j = \phi(2^j)$ *or* $\sigma_j = 1/\phi(2^{-j})$ *then we have* $\underline{b}(\phi) \leq \underline{s}(\sigma) \leq \overline{s}(\sigma) \leq \overline{b}(\phi)$.

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ *and* $\theta_j = 1/\phi(2^{-j})$ *then*

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Corollary

If ϕ *belongs to* $\mathcal{B}$, *then we have* $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ *and* $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.

Relations between Boyd functions and admissible sequences

Proposition

If $\phi \in \mathcal{B}$ *and* $\sigma_j = \phi(2^j)$ *or* $\sigma_j = 1/\phi(2^{-j})$ *then we have* $\underline{b}(\phi) \leq \underline{s}(\sigma) \leq \overline{s}(\sigma) \leq \overline{b}(\phi)$.

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ *and* $\theta_j = 1/\phi(2^{-j})$ *then*

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Corollary

If ϕ *belongs to* $\mathcal{B}$, *then we have* $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ *and* $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.

Boyd function obtained from one admissible sequence

Some elementary examples :

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}), \\ \dfrac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}]. \end{cases}$$

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \\ \dfrac{1}{\phi(1/t)} & \text{if } t \in (0, 1) \end{cases}$$

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ t^s & \text{if } t \in (0, 1) \end{cases}$$

where s satisfies $\underline{s}(\sigma) \leq s \leq \overline{s}(\sigma)$.

Constructing a regular Boyd function from an admissible sequence

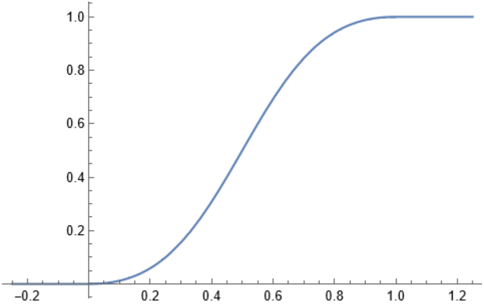
Let

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

to define

$$g : x \mapsto \frac{f(x)}{f(x) + f(1-x)}$$

on $[0, 1]$.



Constructing a regular Boyd function from an admissible sequence

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $j \in \mathbb{N}$, we set

$$\begin{cases} X_j = 2^j \cos\alpha + \sigma_j \sin\alpha \\ Y_j = -2^j \sin\alpha + \sigma_j \cos\alpha \end{cases},$$

$$\xi^{(j)}(X) = \frac{X - X_j}{X_{j+1} - X_j}$$

and

$$\tau^{(j)}(X) = Y_j + (Y_{j+1} - Y_j)X$$

to consider the curve

$$Y = \tau^{(j)}(g(\xi^{(j)}(X)))$$

on $[X_j, X_{j+1}]$.

Constructing a regular Boyd function from an admissible sequence

It gives rise to

$$Y(y) = \tau^{(j)}(g(\xi^{(j)}(X(x))))$$

on the original Euclidean plane.

Let $\eta_j^{(\alpha)}$ be the function $x \mapsto y$ on $[2^j, 2^{j+1}]$.

We can construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}), j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases} .$$

Constructing a regular Boyd function from an admissible sequence

It gives rise to

$$Y(y) = \tau^{(j)}(g(\xi^{(j)}(X(x))))$$

on the original Euclidean plane.
Let $\eta_j^{(\alpha)}$ be the function $x \mapsto y$ on $[2^j, 2^{j+1}]$.
We can construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases} .$$

Constructing a regular Boyd function from an admissible sequence

For $\alpha = 0$, we explicitly get

$$\eta_j^{(0)}(t) = \sigma_j + \frac{\sigma_{j+1} - \sigma_j}{1 + \left(\frac{t-2^{j+1}}{t-2^j}\right)^2}.$$

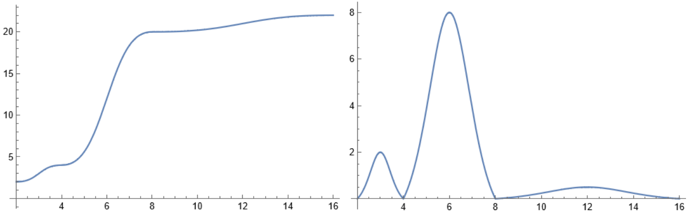


Figure: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

Constructing a regular Boyd function from an admissible sequence

If $\alpha > 0$ is small enough, we get a function $\eta_j^{(\alpha)}$ whose explicit form is far more complicated.

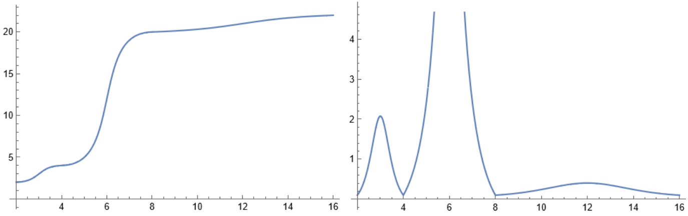


Figure: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0.1$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

Constructing a regular Boyd function from an admissible sequence

Let $\mathcal{B}'$ denote the set of functions $f : I \to I$ that belong to $C^1(I)$ with $f(1) = 1$ and satisfy

$$0 < \inf_{t>0} t\frac{|f'(t)|}{f(t)} \leq \sup_{t>0} t\frac{|f'(t)|}{f(t)} < \infty.$$

One can show that $\mathcal{B}'$ is a subset of $\mathcal{B}$. If $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ (resp. $\overline{b}(\phi) < 0$), then there exists a non-decreasing bijection (resp. a non-increasing bijection) $\psi \in \mathcal{B}'$ such that $\phi \sim \psi$ and $\psi^{-1} \in \mathcal{B}'$

Proposition

If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $\xi \sim \phi$.

The K-operator of interpolation is defined for $t > 0$ and $a \in \Sigma(\overline{A})$ by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

If $\theta \in (0,1)$ and $q \in [1,\infty]$, then a belongs to the interpolation space $K_{\theta,q}(A_0,A_1)$ if $a \in \Sigma(\overline{A})$ and

$$(2^{-\theta j}K(2^j,a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}).$$

This last condition is equivalent to $t \mapsto t^{-\theta}K(t,a) \in L^q_*$.

For example, $B^s_{p,q} = K_{\alpha,q}(H^t_p, H^u_p)$ for $s = (1-\alpha)t + \alpha u$.

The K-method

The K-operator of interpolation is defined for $t > 0$ and $a \in \Sigma(\overline{A})$ by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

If $\theta \in (0,1)$ and $q \in [1,\infty]$, then a belongs to the interpolation space $K_{\theta,q}(A_0,A_1)$ if $a \in \Sigma(\overline{A})$ and

$$(2^{-\theta j} K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}).$$

This last condition is equivalent to $t \mapsto t^{-\theta} K(t,a) \in L^q_*$.

For example, $B^s_{p,q} = K_{\alpha,q}(H^t_p, H^u_p)$ for $s = (1-\alpha)t + \alpha u$.

The K-method

Let $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we let $K_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{\phi,q,K} := \int_0^\infty (\frac{1}{\phi(t)} K(t,a))^q \frac{dt}{t} < \infty$$

holds.

Theorem

$K_{\phi,q}$ is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$. Moreover, we have

$$K(t,a) \leq C\, \phi(t)\, \|a\|_{\phi,q,K}.$$

The K-method

Let $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we let $K_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{\phi,q,K} := \int_0^\infty (\frac{1}{\phi(t)} K(t,a))^q \frac{dt}{t} < \infty$$

holds.

Theorem

$K_{\phi,q}$ is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$. Moreover, we have

$$K(t,a) \leq C\,\phi(t)\,\|a\|_{\phi,q,K}.$$

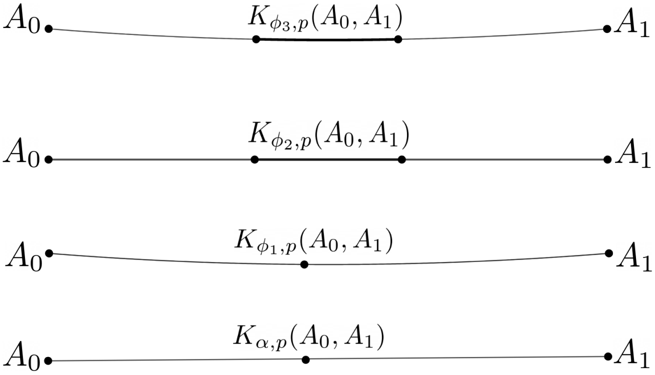


Figure: Differents interpolation spaces where for example $\phi_1(t) = t^\alpha \log(1/t)$, $\phi_2(t) = t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}$ and $\phi_3(t) = (t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}) \log(1/t)$.

The K-method

Given $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, a belongs to the generalized interpolation space $[A_0, A_1]^{\gamma}_{\phi,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\gamma}_{\phi,q}} := \|\phi(t)^{-1} K(\gamma(t), a)\|_{L^q_*} < \infty.$$

Proposition

If $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, then a belongs to $[A_0, A_1]^{\gamma}_{\phi,q}$ if and only if $\sum_{j \in \mathbb{Z}} \left(\frac{1}{\phi(2^j)} K(\gamma(2^j), a) \right)^q < \infty$.

Proposition

Let $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$; if $\underline{b}(\gamma) > 0$, then there exists $\xi \in \mathcal{B}'_+$ such that $\xi \sim \gamma$ and

$$[A_0, A_1]^{\gamma}_{\phi,q} = K_{\phi \circ \xi^{-1}, q}(A_0, A_1).$$

The K-method

Given $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, a belongs to the generalized interpolation space $[A_0, A_1]^{\gamma}_{\phi,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\gamma}_{\phi,q}} := \|\phi(t)^{-1} K(\gamma(t), a)\|_{L^q_*} < \infty.$$

Proposition

If $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, then a belongs to $[A_0, A_1]^{\gamma}_{\phi,q}$ if and only if $\sum_{j \in \mathbb{Z}} \left(\frac{1}{\phi(2^j)} K(\gamma(2^j), a) \right)^q < \infty$.

Proposition

Let $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$; if $\underline{b}(\gamma) > 0$, then there exists $\xi \in \mathcal{B}'_+$ such that $\xi \sim \gamma$ and

$$[A_0, A_1]^{\gamma}_{\phi,q} = K_{\phi \circ \xi^{-1}, q}(A_0, A_1).$$

The K-method

Let σ be an admissible sequence and $q \in [1, \infty]$; a belongs to the upper generalized interpolation space $[A_0, A_1]^{\wedge}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\wedge}_{\sigma,q}} := \sum_{j=1}^{\infty} \frac{1}{\sigma_j} K(2^j, a) < \infty.$$

In the same way, a belongs to the lower generalized interpolation space $[A_0, A_1]^{\vee}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\vee}_{K,\sigma,q}} := \sum_{j=1}^{\infty} \sigma_j K(2^{-j}, a) < \infty.$$

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then

$$K_{\phi,q}(A_0, A_1) = [A_0, A_1]^{\vee}_{\delta,q} \cap [A_0, A_1]^{\wedge}_{\sigma,q}.$$

The K-method

Let σ be an admissible sequence and $q \in [1, \infty]$; a belongs to the upper generalized interpolation space $[A_0, A_1]^{\wedge}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\wedge}_{\sigma,q}} := \sum_{j=1}^{\infty} \frac{1}{\sigma_j} K(2^j, a) < \infty.$$

In the same way, a belongs to the lower generalized interpolation space $[A_0, A_1]^{\vee}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\vee}_{K,\sigma,q}} := \sum_{j=1}^{\infty} \sigma_j K(2^{-j}, a) < \infty.$$

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then

$$K_{\phi,q}(A_0, A_1) = [A_0, A_1]^{\vee}_{\delta,q} \cap [A_0, A_1]^{\wedge}_{\sigma,q}.$$

Thank you for your attention !