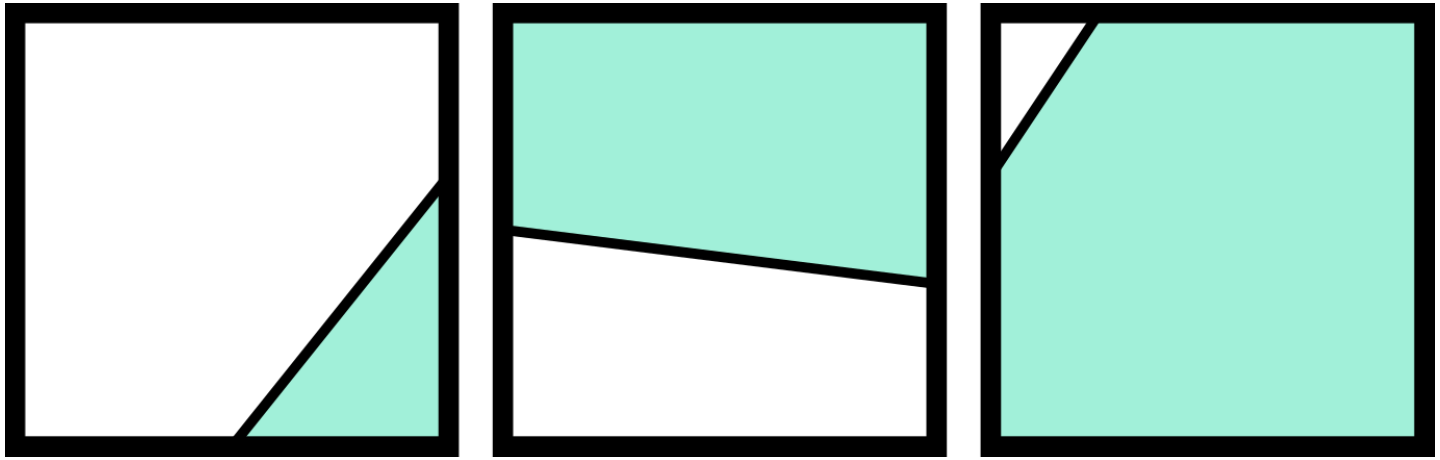
Interior Penalty Method for a Cartesian discontinuous Galerkin solver with immersed boundaries



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IACM Computational Fluids Conference 2023, nayan.levaux@uliege.be

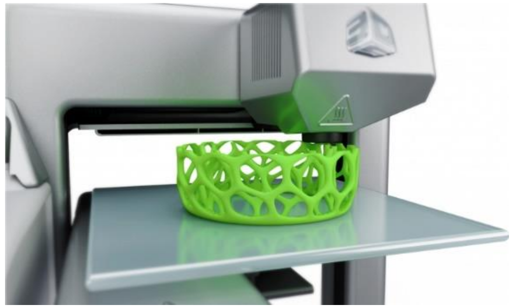
Motivation

Complex / evolving geometry

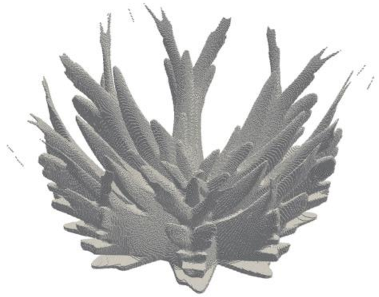
Multiphysics

High resolution

Additive manufacturing

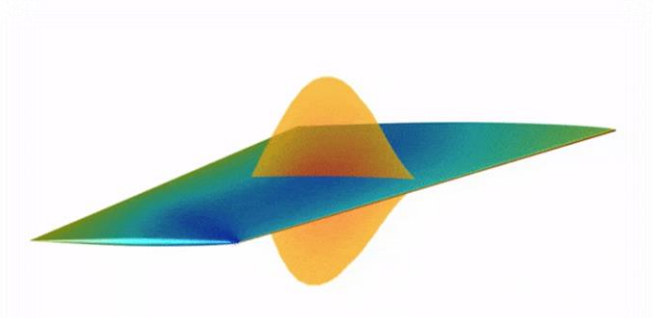


Topology optimization



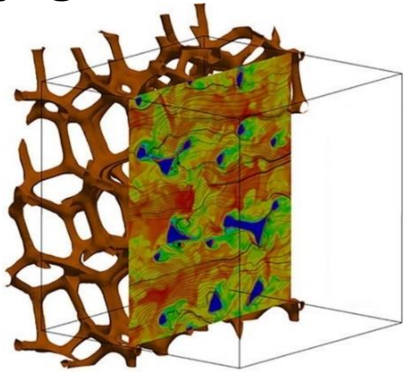
[Alexandersen, 2016]

Fluid-solid interactions

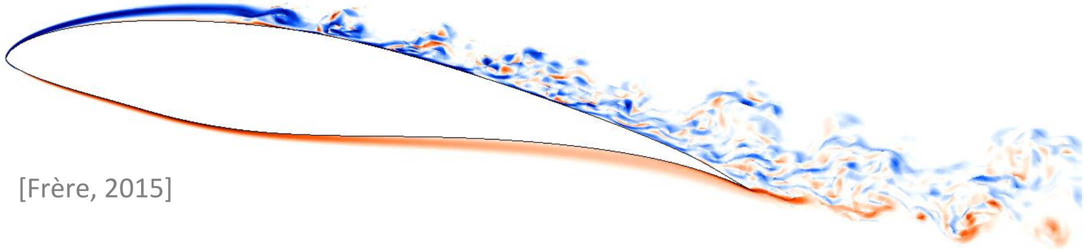


[Thomas, 2019]

Conjugate heat transfer

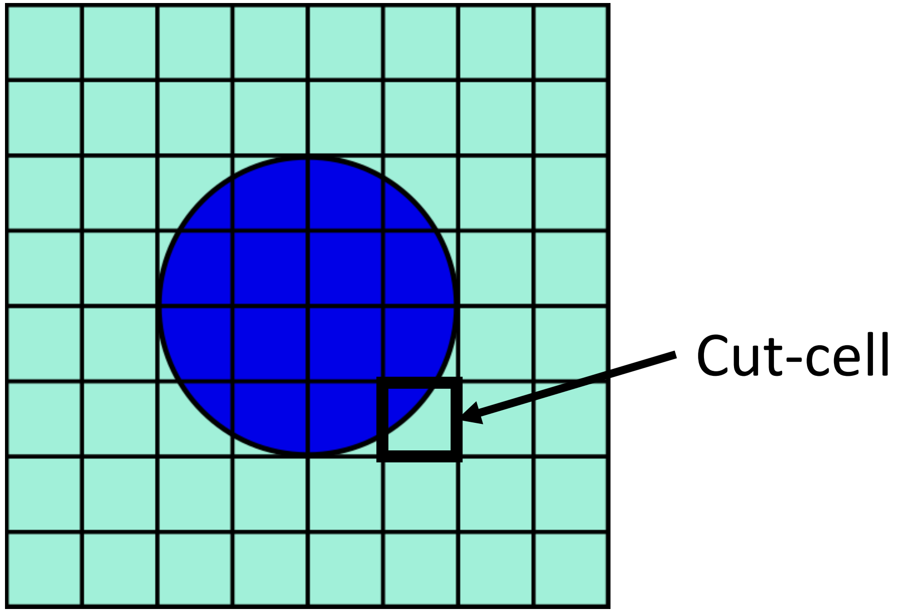


[Das, 2018]



[Frère, 2015]

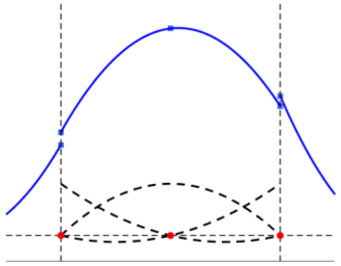
Towards high accuracy immersed



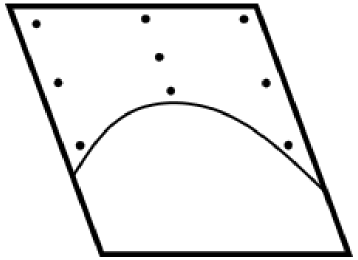
Immersed approach

- ✔ Simple mesh generation
- ✘ Numerical errors at **interface**
- ✘ Lack of resolution near interfaces

How to tackle lack of accuracy?



High-order DG
on cartesian grid

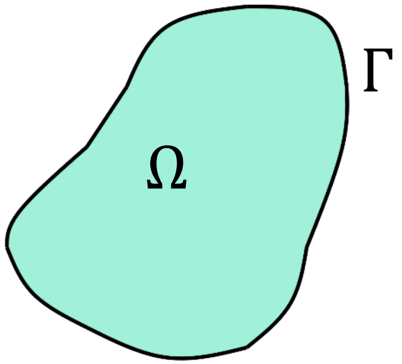


High-order
cut-cells

Adaptive mesh refinement

Discretization of elliptic equations

$$\nabla \cdot (\mu \nabla u) = 0, \qquad \forall\, x \in \Omega$$
$$\mathrm{u} = u^*, \qquad \forall\, x \in \Gamma$$



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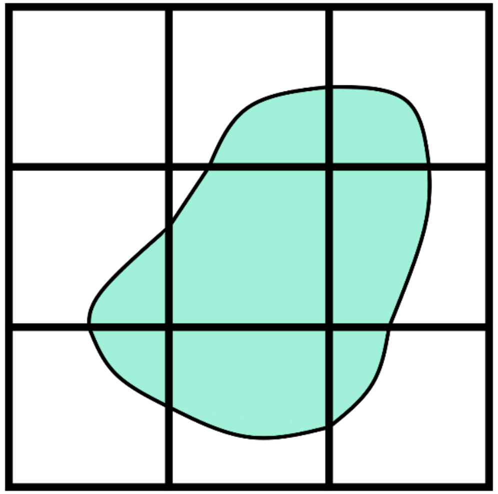

$\forall v \in \mathcal{V}$, find $u \in \mathcal{V}$:

$$\sum_e \int_e \nabla v \cdot \mu \nabla u \, dV - \sum_f \int_f \gamma(u^+, u^-, v^+, v^-) dS = \sum_{f_\Gamma} \int_{f_\Gamma} \gamma(u^+, u^*, v^+, 0) dS$$

$$\Leftrightarrow a(u, v) = b(v)$$

Discretization of elliptic equations

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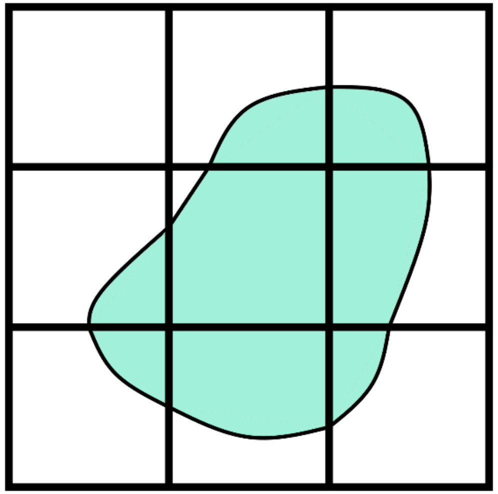

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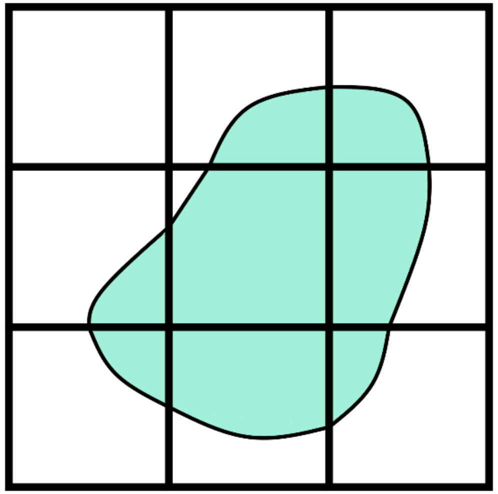

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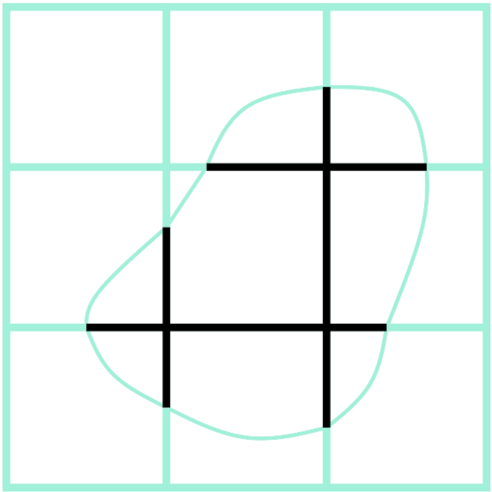

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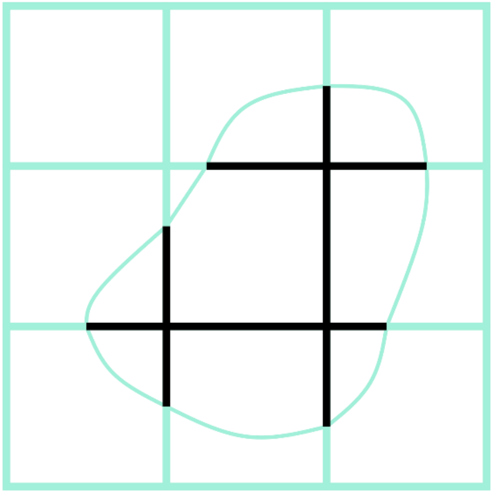

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Appropriate numerical flux ▶ Weakly impose BCs & solution continuity

Impose flux continuity

Interior penalty method

$$\gamma(u^+, u^-, v^+, v^-) = [\![v\, \mu\nabla u]\!] = [\![v]\!] \cdot \langle \mu\nabla u \rangle + \cancel{[\![\mu\nabla u]\!]} \cdot \langle v \rangle$$

Impose continuity of flux

$$= [\![v]\!] \cdot \langle \mu\nabla u \rangle + \theta [\![u]\!] \cdot \langle \mu\nabla v \rangle - \sigma_f [\![v]\!] \cdot [\![u]\!]$$

Penalty term

Weakly impose

- continuity of solution
- BC

Interior penalty method

$$\gamma(u^+, u^-, v^+, v^-) = [\![v\,\mu\nabla u]\!] = [\![v]\!] \cdot \langle \mu\nabla u \rangle + \cancel{[\![\mu\nabla u]\!]} \cdot \langle v \rangle$$

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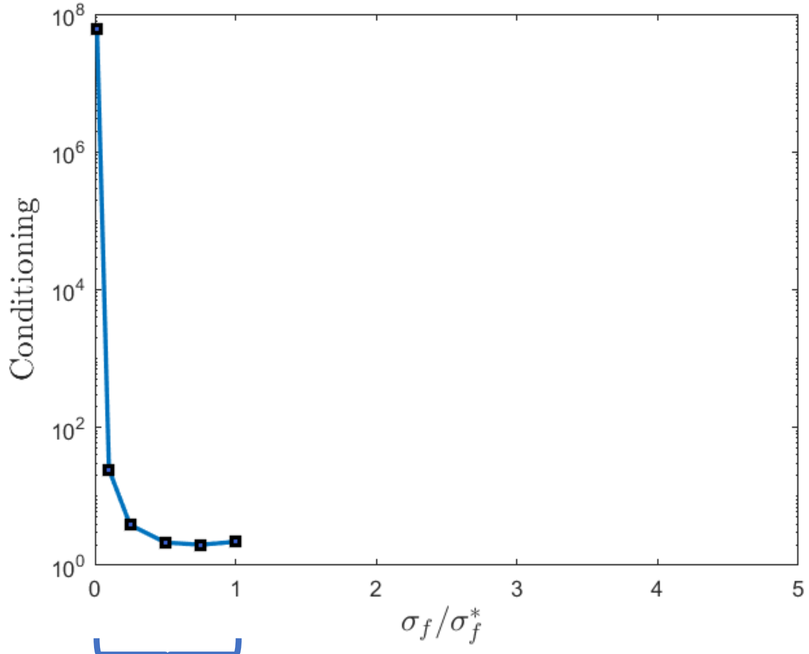
Penalty factor σ_f to ensure coercivity: $\exists C > 0 : a(v, v) \geq C\|v\|^2, \forall v \in \mathcal{V}$



Coercivity violated

Interior penalty method

$$\gamma(u^+, u^-, v^+, v^-) = [\![v\, \mu \nabla u]\!] = [\![v]\!] \cdot \langle \mu \nabla u \rangle + \cancel{[\![\mu \nabla u]\!]} \cdot \langle v \rangle$$

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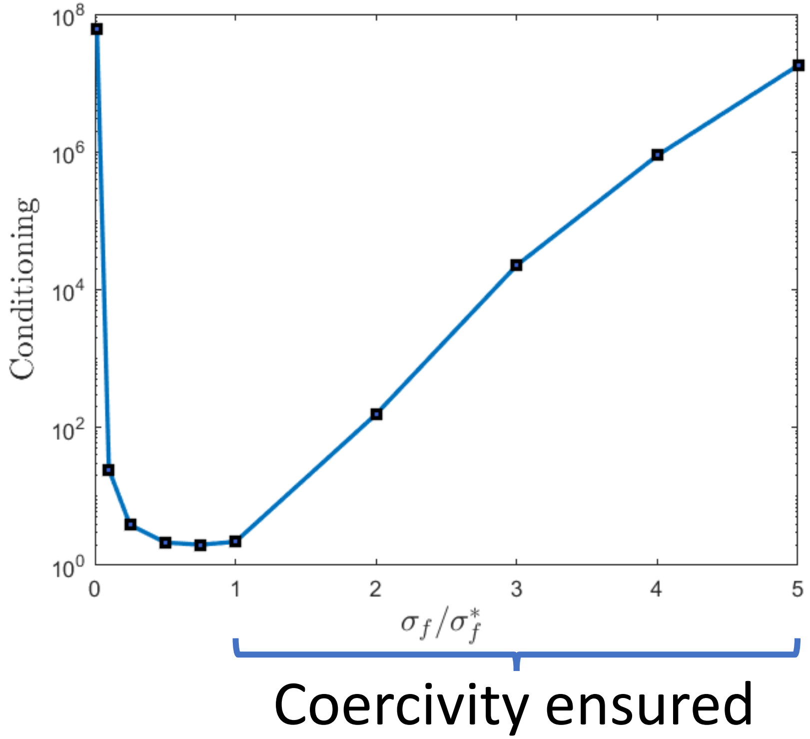


Trace inverse inequality

Sharp bound on inequality for finding optimal σ_f^*

$$\int_f v^2 dS \leq K \int_e v^2 dV, \forall v \in \mathcal{V}$$

$$K \sim C \frac{p^2}{h}$$

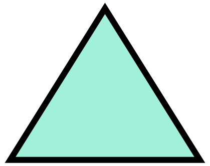
Trace inverse inequality

Sharp bound on inequality for finding optimal σ_f^*

$$\int_f v^2 dS = K \int_e v^2 dV, \forall v \in \mathcal{V}$$

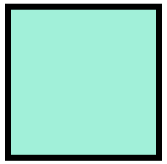
$$K \sim C\frac{p^2}{h} \longrightarrow K = \frac{C(p)A(f)}{V(e)}$$

[Warburton, 2003]
[Hillewaert, 2013]



Pascal
functional space

$$C(p) = \frac{(p+1)(p+d)}{d}$$



Pascal & Tensor product
functional space

$$C(p) = (p+1)^2$$

Trace inverse inequality

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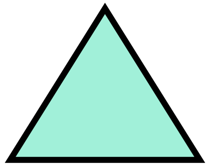

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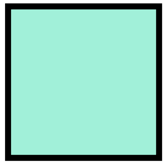

Pascal & Tensor product
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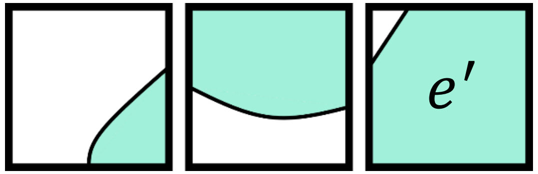
What about cut elements?



For 2D elements

$$K = \eta_0 \frac{p^2 A(\partial e')}{V(e')}$$

[Kummer, 2016]

with $\eta_0 = 4$ (empirical)

Trace inverse inequality

Sharp bound on inequality for finding optimal σ_f^*

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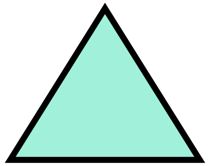

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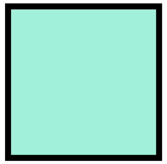

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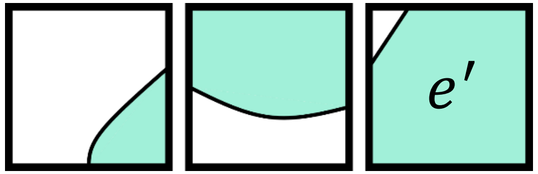

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$$K = \eta_0 \frac{p^2 A(\partial e')}{V(e')}$$

[Kummer, 2016]

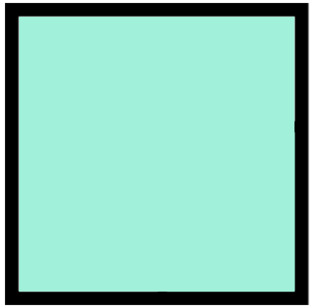
with $\eta_0 = 4$ (empirical)

Does it always ensure coercivity?
Is it a sharp bound?
Can we find one if not?

Trace inverse inequality for straight cut

Simplify the set-up

2D quadrangle equipped with Tensor product functional space of order p



Trace inverse inequality for straight cut

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2D quadrangle equipped with Tensor product functional space of order p

Straight cut from level-set

Trace inverse inequality for straight cut

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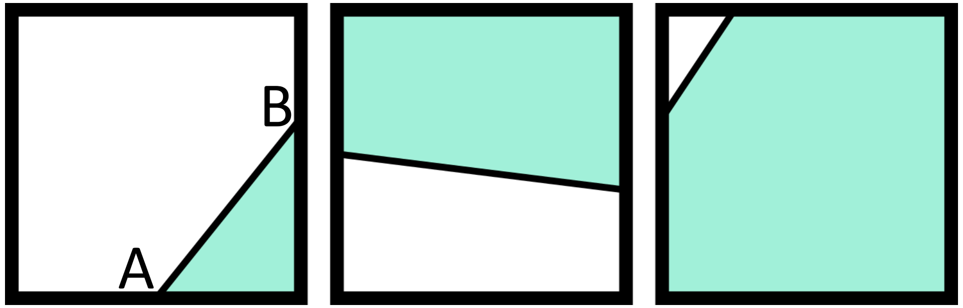


Type I Type II Type III

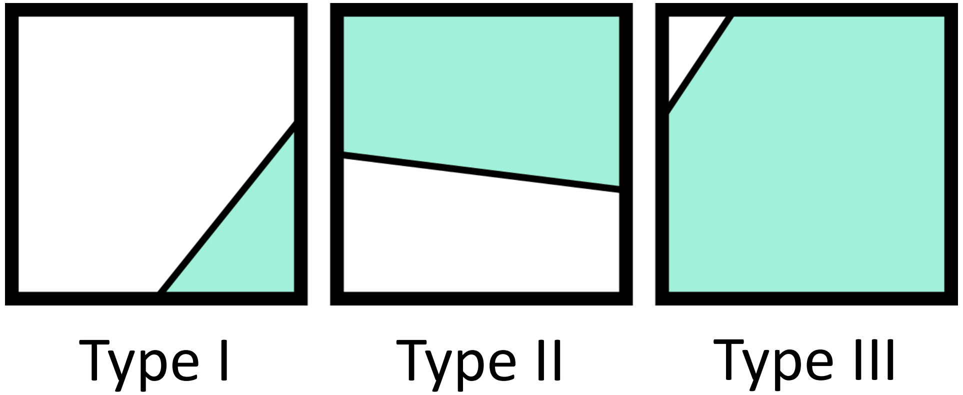
Search for $C(p)$ and h

Trace inverse inequality for straight cut

Simplify the set-up

2D quadrangle equipped with Tensor product functional space of order p

Straight cut from level-set



Search for $C(p)$ and h, and verify with Owen's numerical approach:

$$\int_f v^2 dS \leq K \int_e v^2 dV \,, \forall v \in \mathcal{V} \rightarrow K = \max \lambda : \left(\boldsymbol{M_f} - \lambda \boldsymbol{M_e}\right)\boldsymbol{x} = 0$$

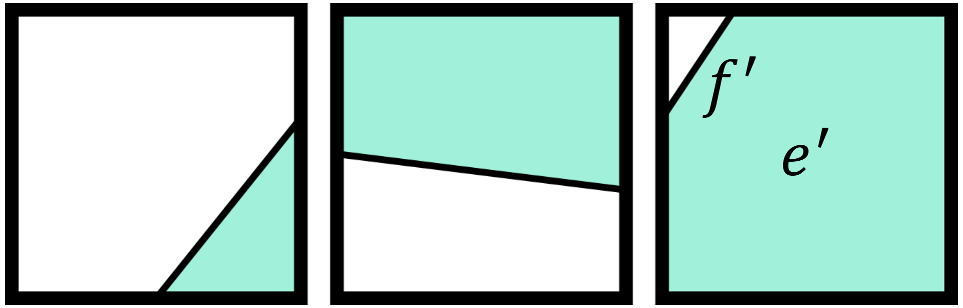
[Owens, 2017]

Trace inverse inequality for straight cut

Simplify the set-up

2D quadrangle equipped with Tensor product functional space of order p

Straight cut from level-set



Type I Type II Type III

Search for $C(p)$ and h, and verify with Owen's numerical approach:

$$\int_{f'} v^2 dS \leq K \int_{e'} v^2 dV, \forall v \in \mathcal{V} \rightarrow K = \max \lambda : \left(\boldsymbol{M}_{\boldsymbol{f}'} - \lambda \boldsymbol{M}_{\boldsymbol{e}'}\right)\boldsymbol{x} = 0$$

[Owens, 2017]

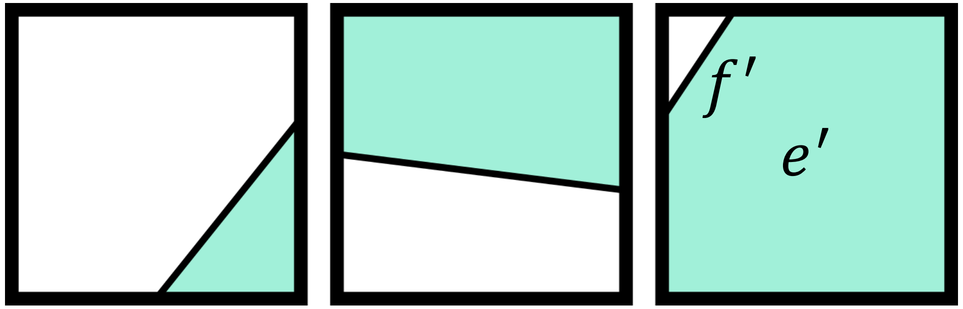
Still need to separate geometric contribution to functional space

Trace inverse inequality for straight cut

Simplify the set-up

2D quadrangle equipped with Tensor product functional space of order p

Straight cut from level-set



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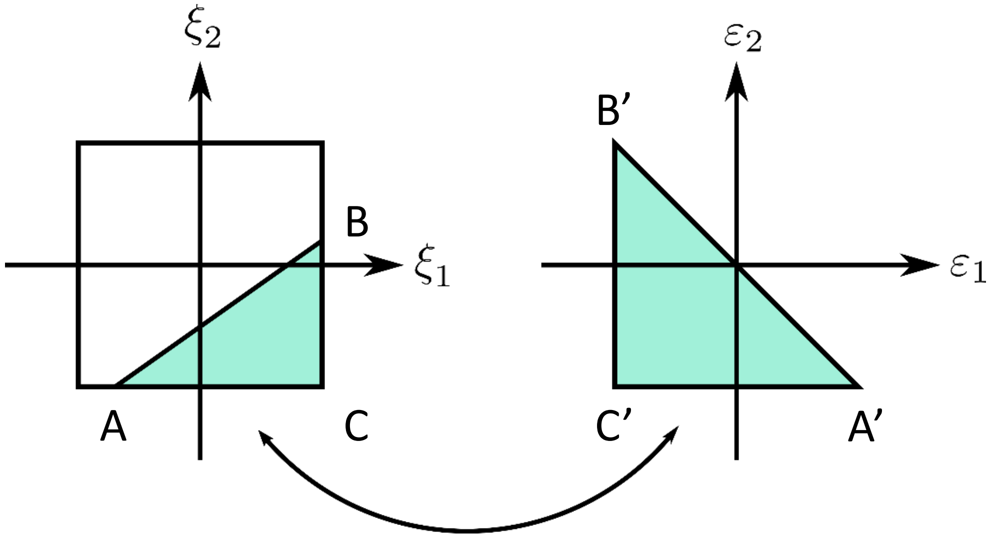
[Owens, 2017]

Why not using this approach "on the fly" for any cut? ▶ Computationally more expensive
Might be ill-conditioned

Type I: Sub-triangle



$$\xi_1 = \frac{1}{2}(\xi_1^A - 1)\varepsilon_1 + \frac{1}{2}(\xi_1^A + 1)$$

$$\xi_2 = \frac{1}{2}(\xi_2^B + 1)\varepsilon_2 + \frac{1}{2}(\xi_2^B - 1)$$

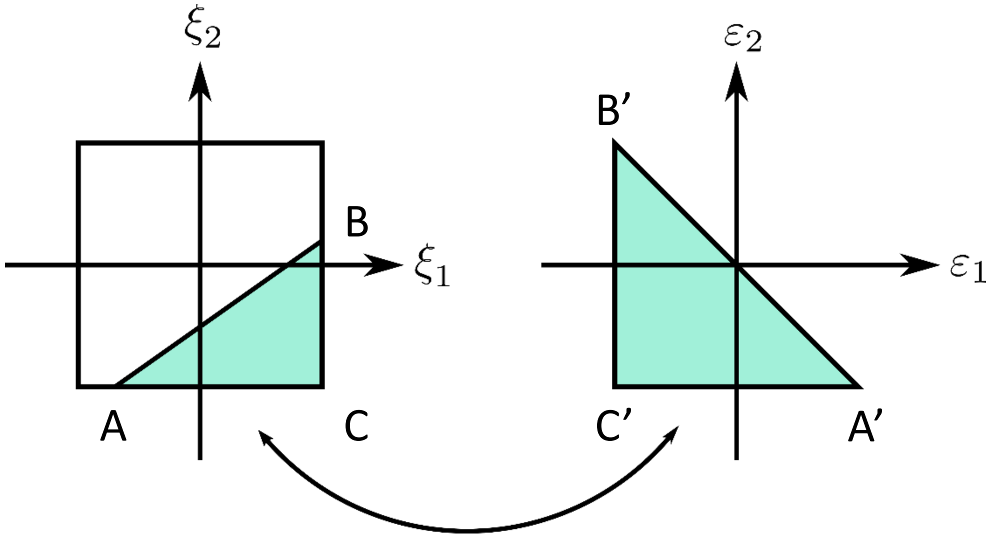
Pascal *p* functional space

$$1$$

$$\xi_1 \qquad \xi_2$$

$$\xi_1^2 \qquad \xi_1 \cdot \xi_2 \qquad \xi_2^2$$

Type I: Sub-triangle



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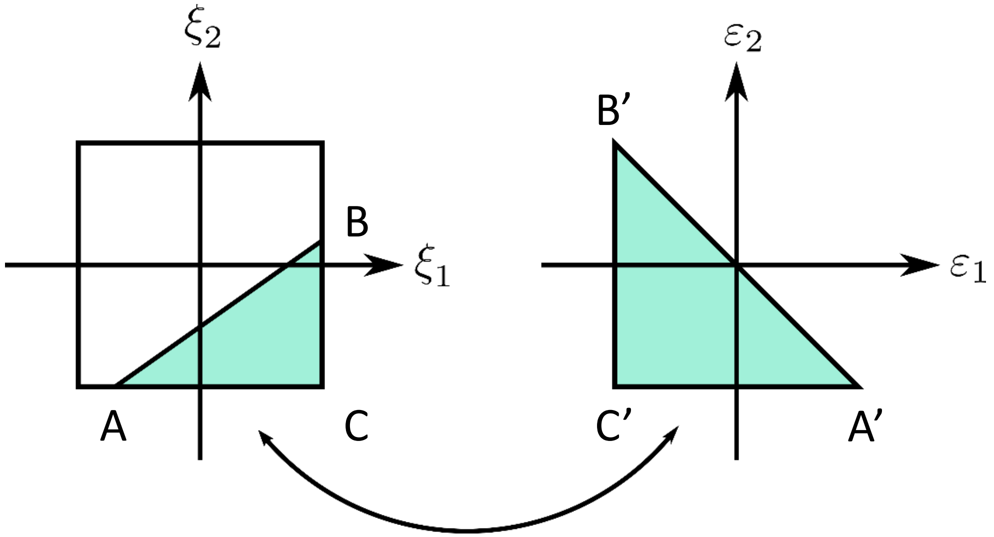
Pascal *p* functional space

$$1$$
$$(a\varepsilon_1 + b) \qquad (c\varepsilon_2 + d)$$
$$(a\varepsilon_1 + b)^2 \qquad (a\varepsilon_1 + b)\cdot(c\varepsilon_2 + d) \qquad (c\varepsilon_2 + d)^2$$

Type I: Sub-triangle



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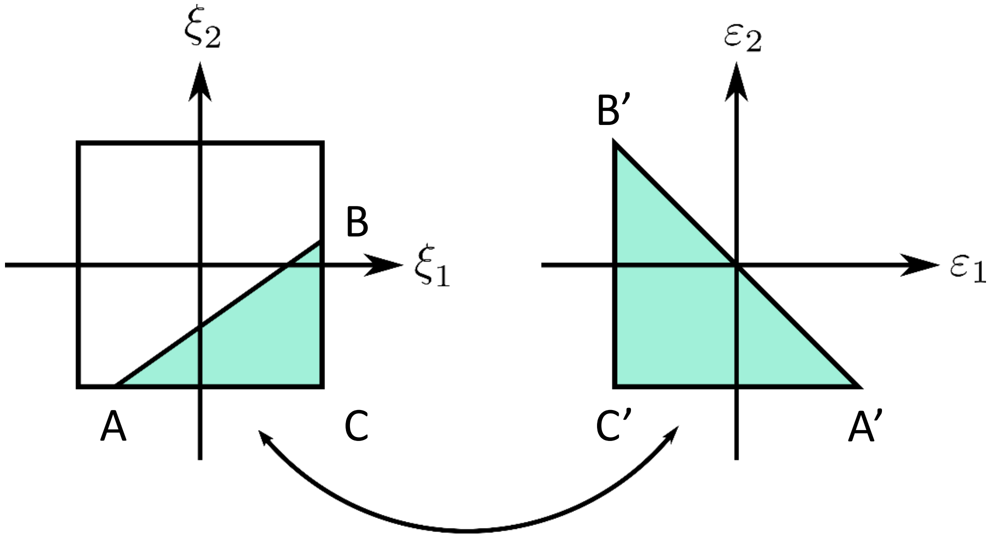
Pascal *p* functional space

$$1$$
$$\varepsilon_1 \qquad \varepsilon_2$$
$$\varepsilon_1^2 \qquad \varepsilon_1 \cdot \varepsilon_2 \qquad \varepsilon_2^2$$

Type I: Sub-triangle



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$$\xi_2 = \frac{1}{2}(\xi_2^B + 1)\varepsilon_2 + \frac{1}{2}(\xi_2^B - 1)$$

Pascal p functional space

$$\begin{array}{ccccc} & & 1 & & \\ & \xi_1 & & \xi_2 & \\ \xi_1^2 & & \xi_1 \cdot \xi_2 & & \xi_2^2 \end{array}$$

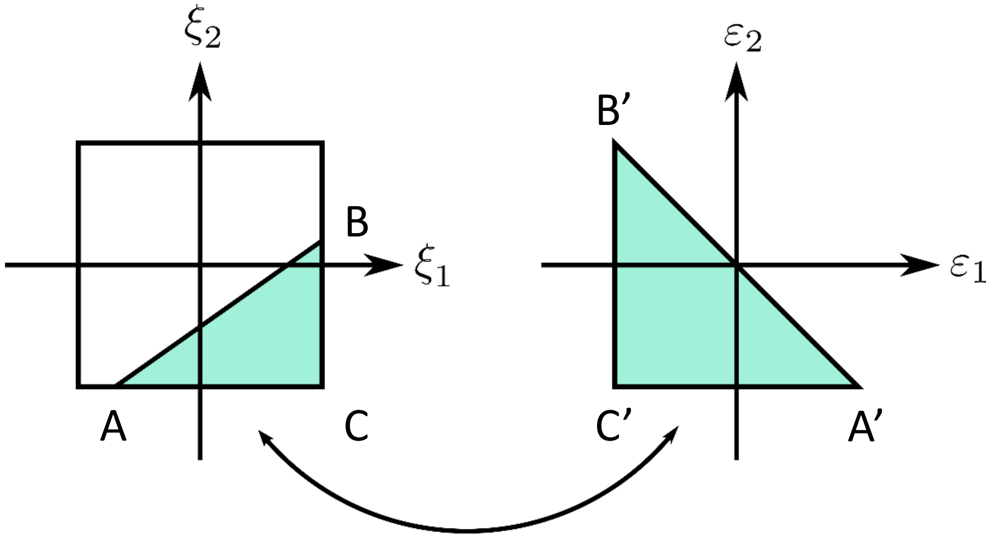
$$C(p) = \frac{(p+1)(p+d)}{d}$$

Integrand remains in same functional space

Geometric factor appears naturally with mapping

- $h = \frac{V(ABC)}{A(AB)}$ and $C(p)$ of reference element

Type I: Sub-triangle



$$\xi_1 = \frac{1}{2}(\xi_1^A - 1)\varepsilon_1 + \frac{1}{2}(\xi_1^A + 1)$$
$$\xi_2 = \frac{1}{2}(\xi_2^B + 1)\varepsilon_2 + \frac{1}{2}(\xi_2^B - 1)$$

Tensor p functional space

$$1$$
$$\xi_1 \quad \xi_2$$
$$\xi_1^2 \quad \xi_1 \cdot \xi_2 \quad \xi_2^2$$
$$\xi_1^2 \cdot \xi_2 \quad \xi_1 \cdot \xi_2^2$$
$$\xi_1^2 \cdot \xi_2^2$$

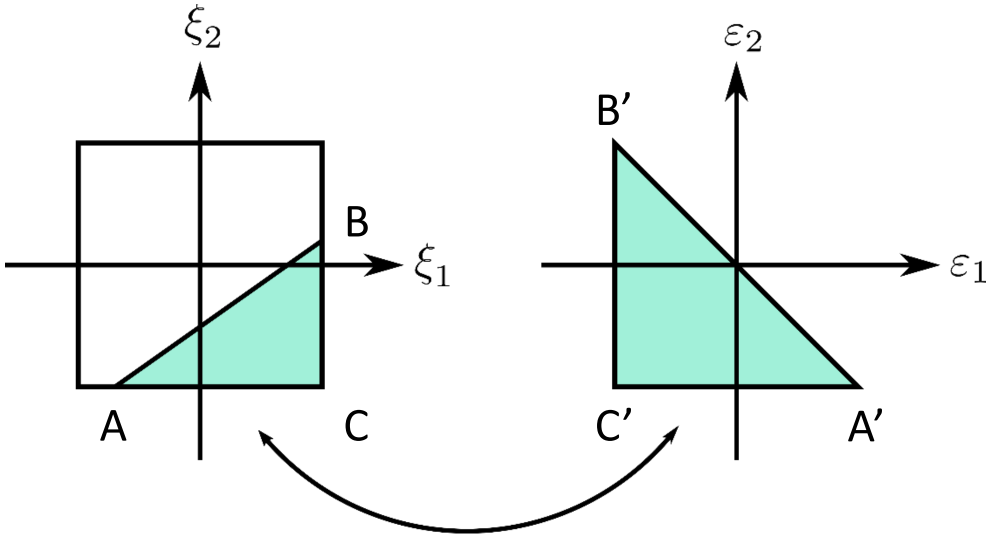
$$C(p) = ?$$

Integrand remains in same functional space

Geometric factor appears naturally with mapping

$h = \frac{V(ABC)}{A(AB)}$ and $C(p)$ of reference element

Type I: Sub-triangle



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$$\xi_2 = \frac{1}{2}(\xi_2^B + 1)\varepsilon_2 + \frac{1}{2}(\xi_2^B - 1)$$

Pascal $2p$ functional space

$$1$$
$$\xi_1 \quad \xi_2$$
$$\xi_1^2 \quad \xi_1 \cdot \xi_2 \quad \xi_2^2$$
$$\xi_1^3 \quad \xi_1^2 \cdot \xi_2 \quad \xi_1 \cdot \xi_2^2 \quad \xi_2^3$$
$$\xi_1^4 \quad \xi_1^3 \cdot \xi_2 \quad \xi_1^2 \cdot \xi_2^2 \quad \xi_1 \cdot \xi_2^3 \quad \xi_2^4$$

$$C(p) = \frac{(2p+1)(2p+2)}{2} = (2p+1)(p+1)$$

Integrand remains in same functional space

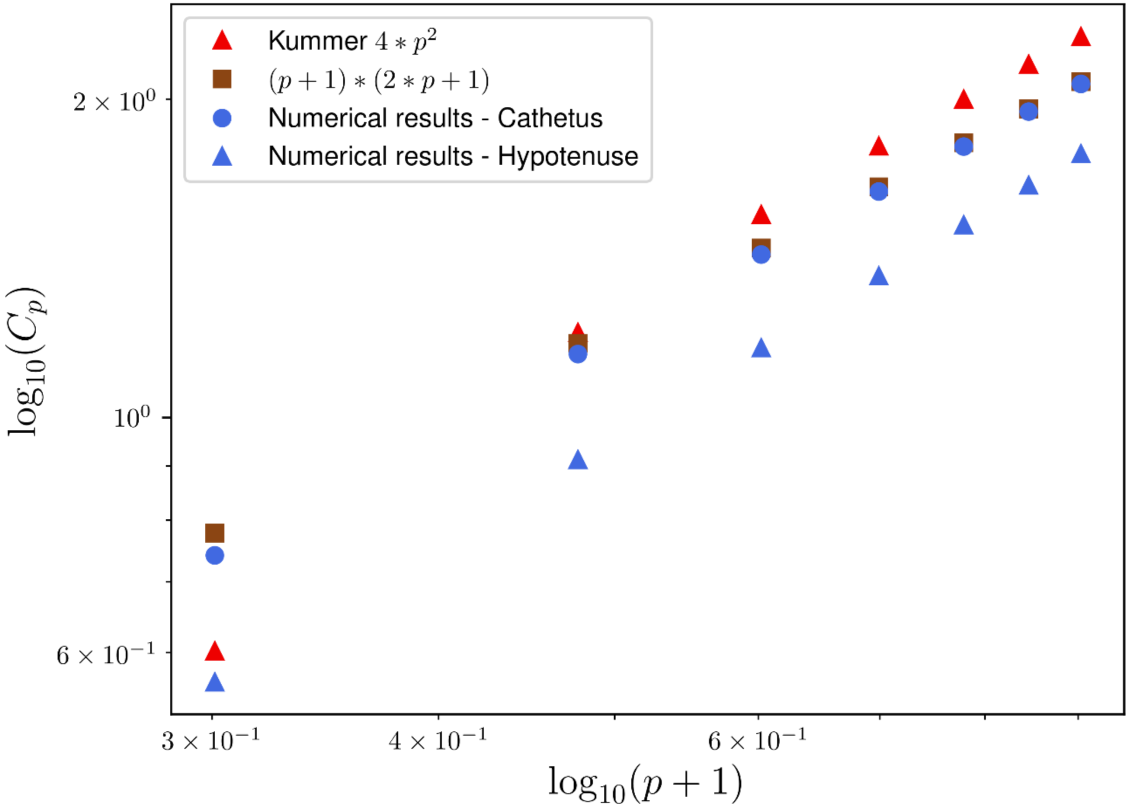
Geometric factor appears naturally with mapping

$h = \dfrac{V(ABC)}{A(AB)}$ and $C(p)$ of reference element

It ensures coercivity but is this value not too large?

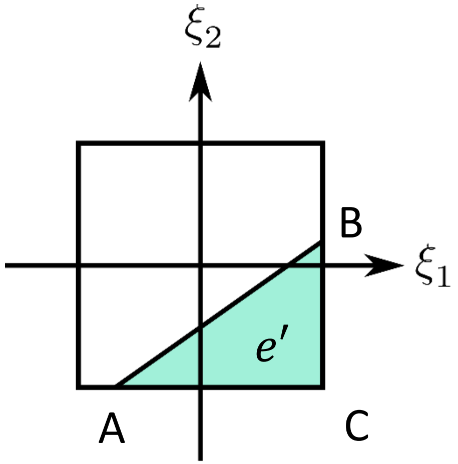
Type I: Sub-triangle verification

On reference triangle



Numerical results limited to $p = 7$ because of available quadrature rule
Closer to experiments than Kummer

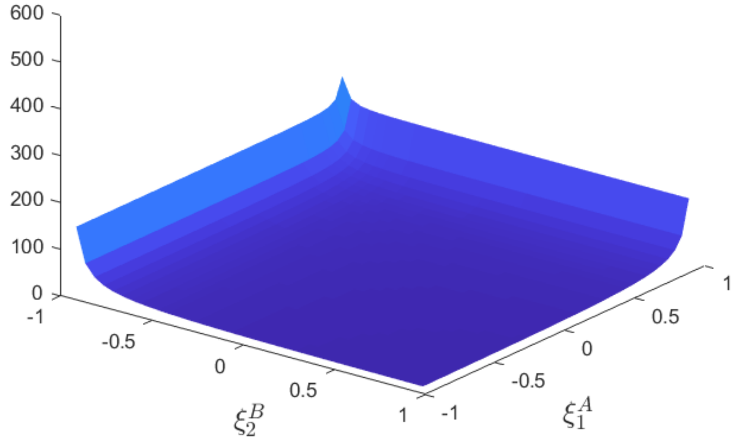
Type I: Sub-triangle verification

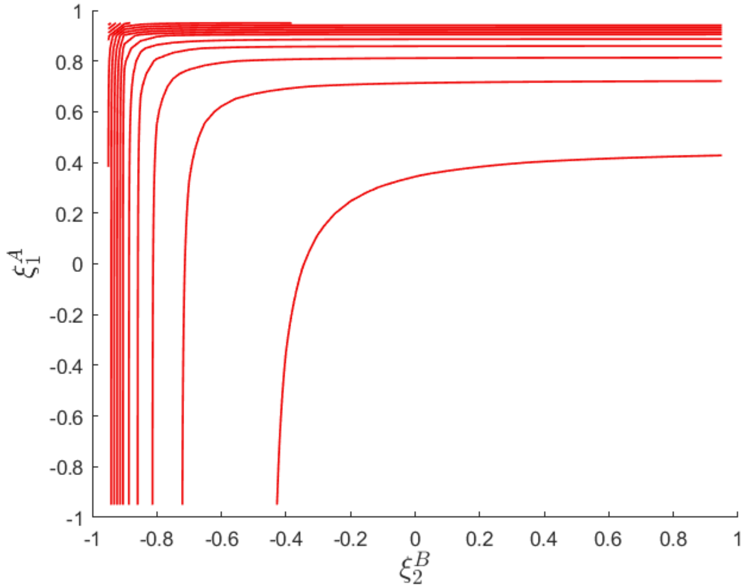


Trace inverse inequality on hypothenuse

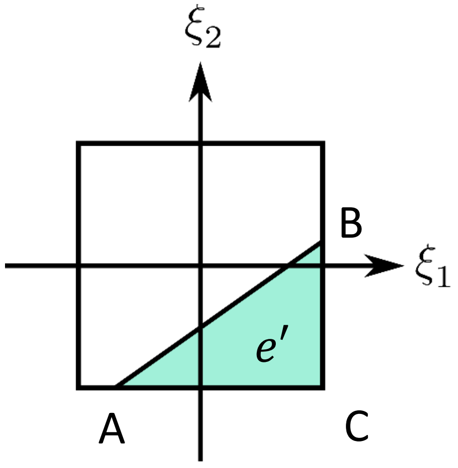
Order $p = 1$

Numerical results





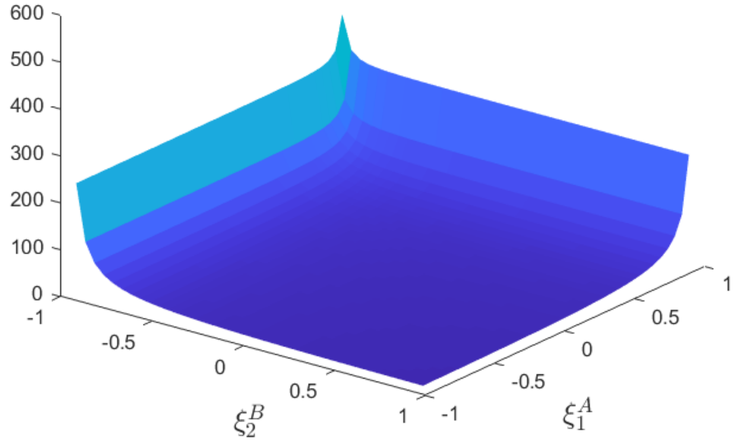
Type I: Sub-triangle verification

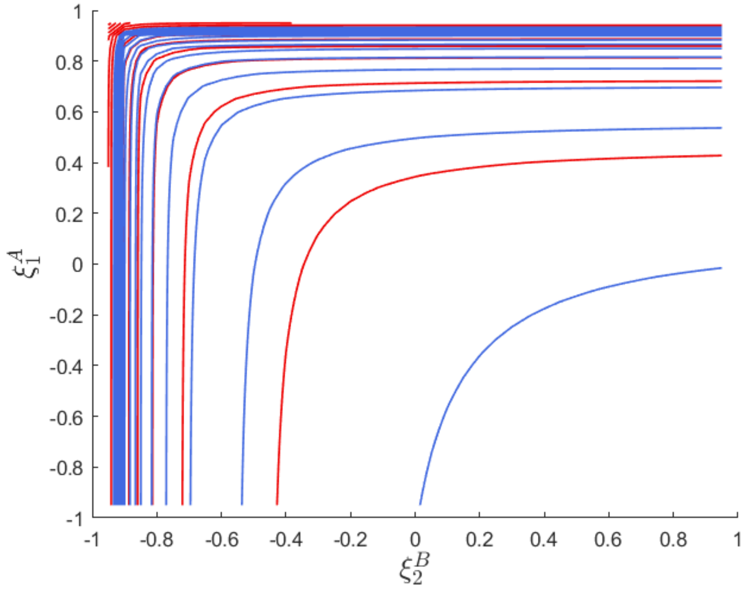


Trace inverse inequality on hypothenuse

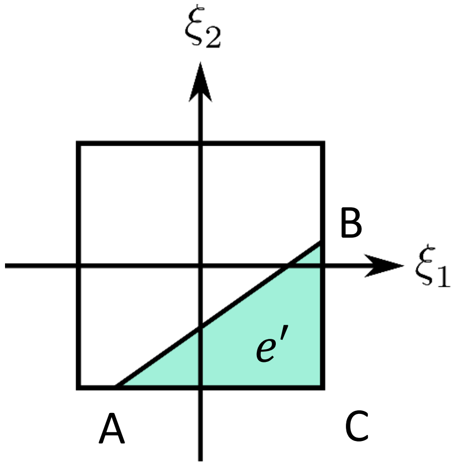
Order $p = 1$

$$K = (2p+1)(p+1)\frac{A(AB)}{V(ABC)}$$



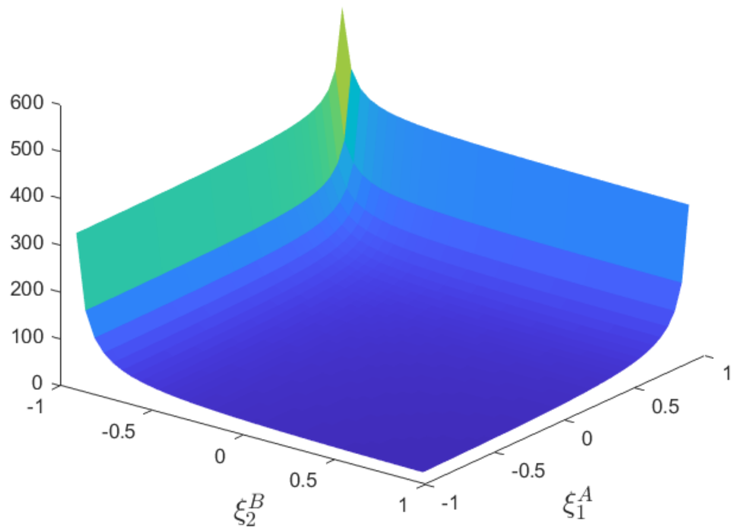


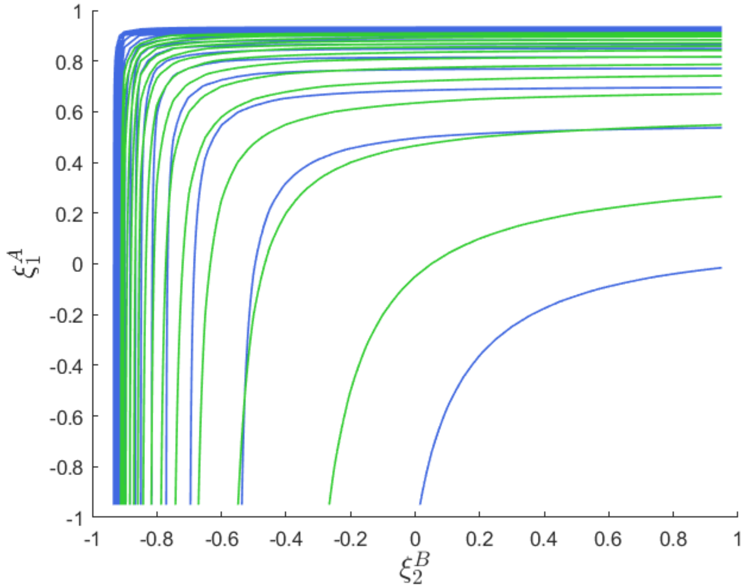
Type I: Sub-triangle verification



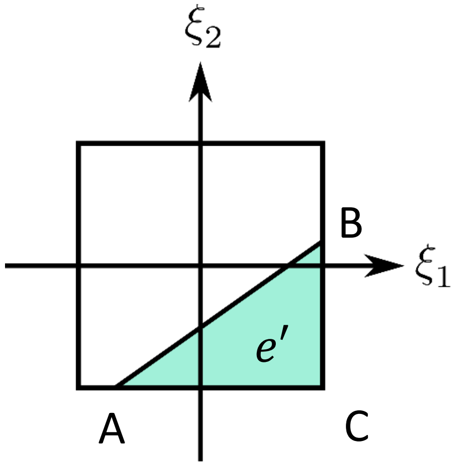
Trace inverse inequality on hypothenuse
Order $p = 1$

$$K = 4p^2 \frac{A(\partial e')}{V(e')}$$





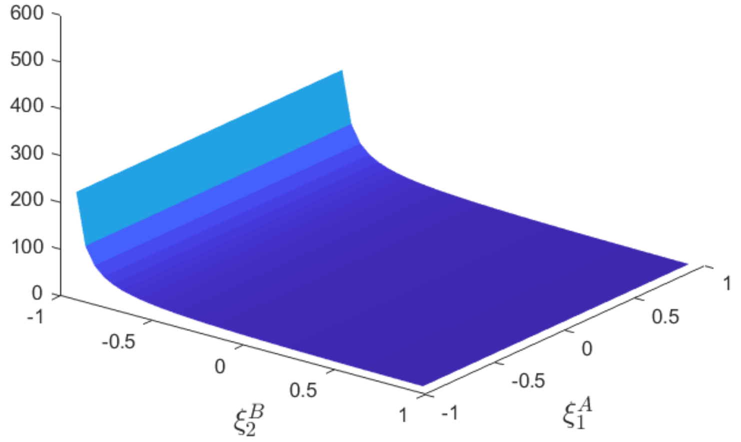
Type I: Sub-triangle verification

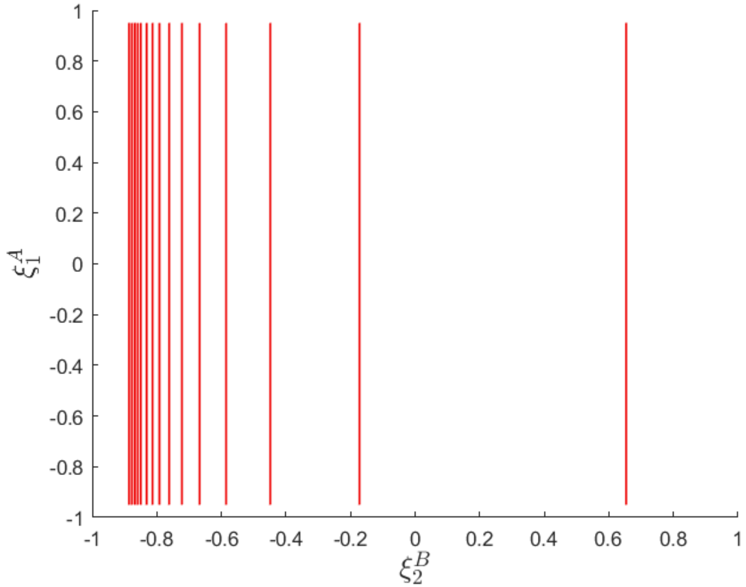


Trace inverse inequality on cathetus

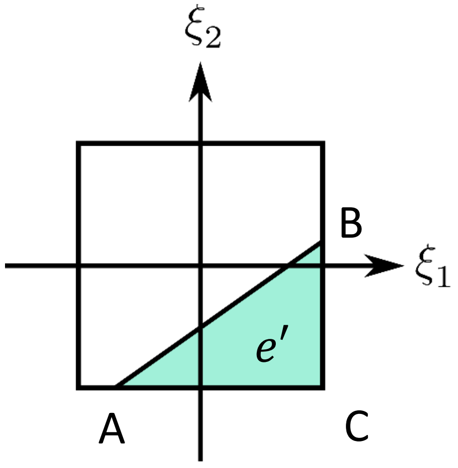
Order $p = 1$

Numerical results





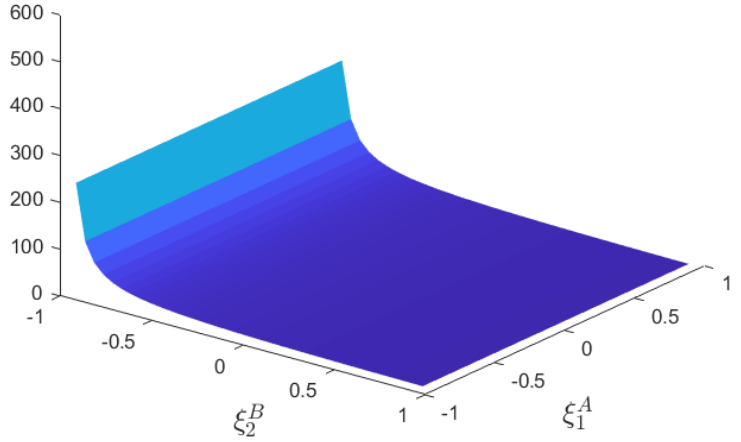
Type I: Sub-triangle verification

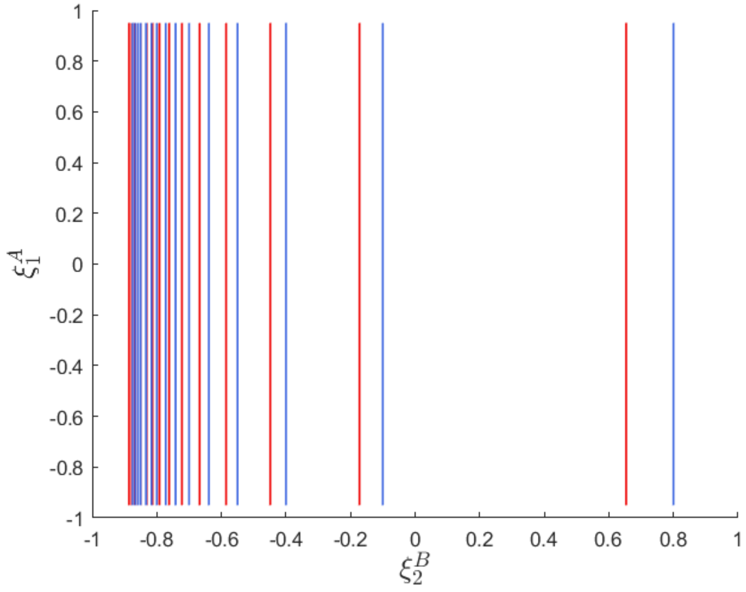


Trace inverse inequality on cathetus

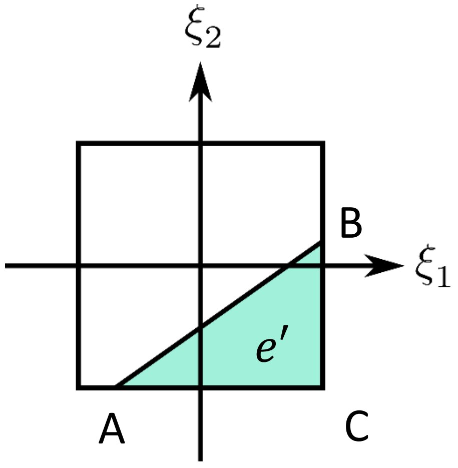
Order $p = 1$

$$K = (2p+1)(p+1)\frac{A(BC)}{V(ABC)}$$





Type I: Sub-triangle verification



Trace inverse inequality on cathetus

Order $p = 1$

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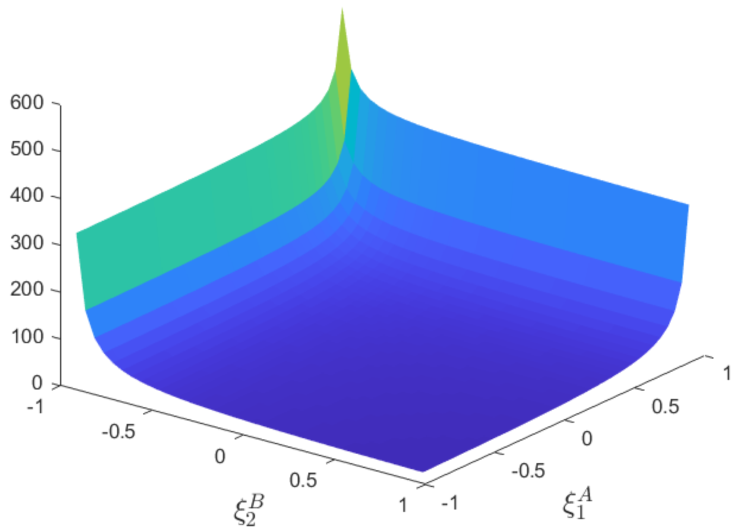
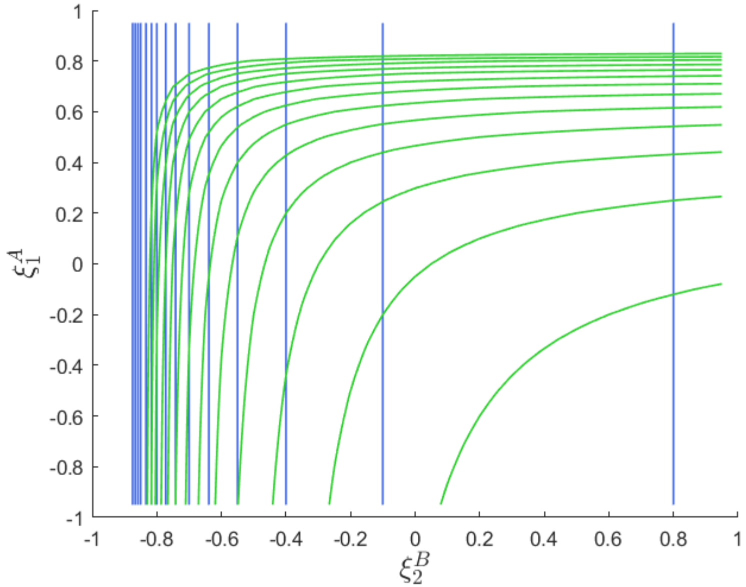
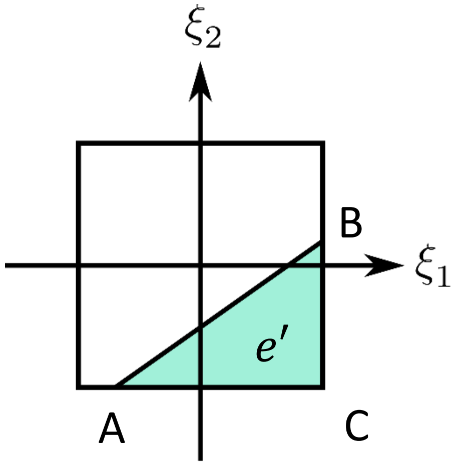




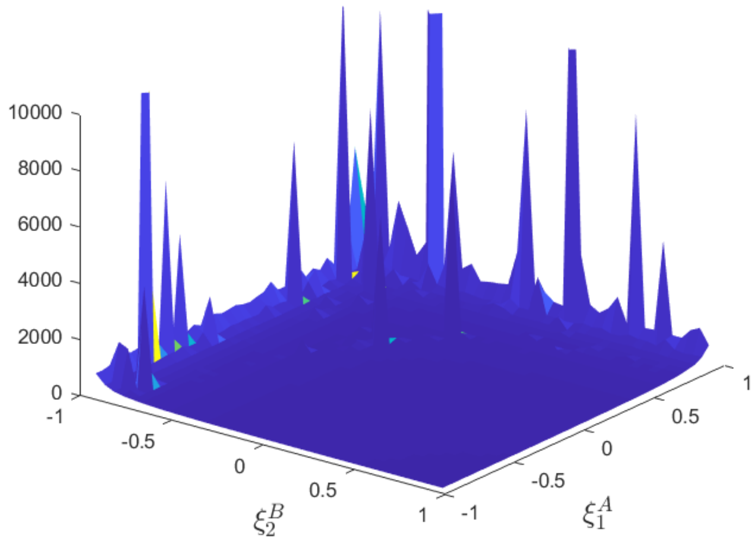
Type I: Sub-triangle verification

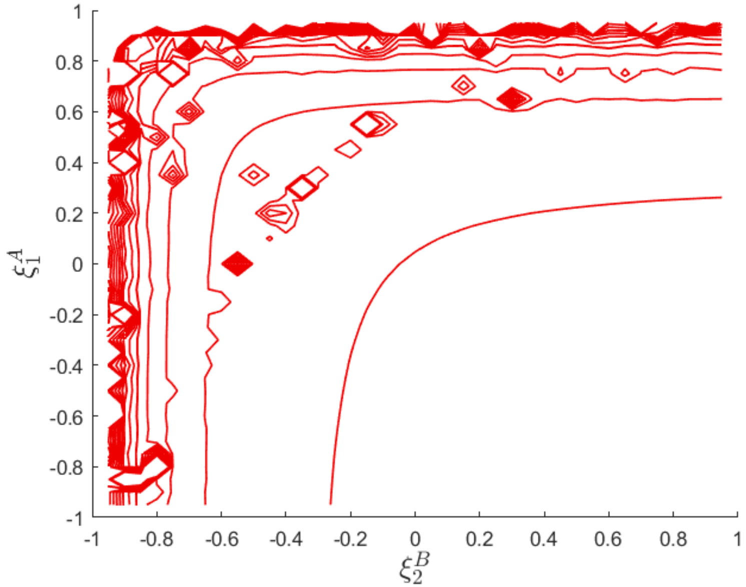


Trace inverse inequality on hypothenuse

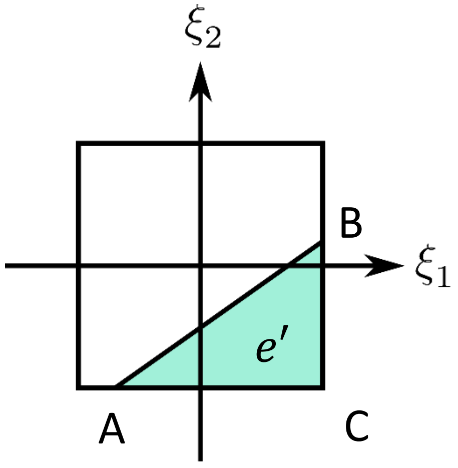
Order $p = 4$

Numerical results





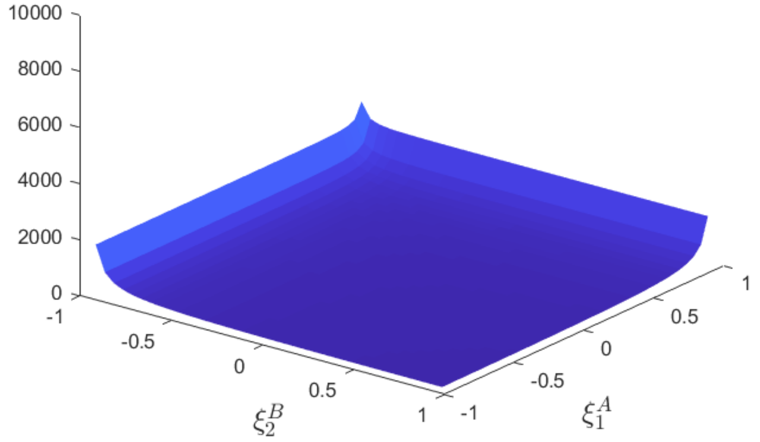
Type I: Sub-triangle verification

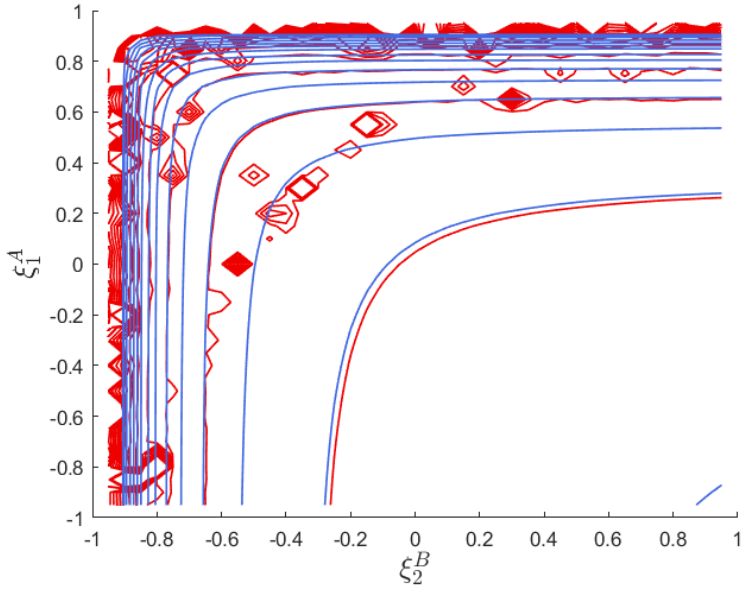


Trace inverse inequality on hypothenuse

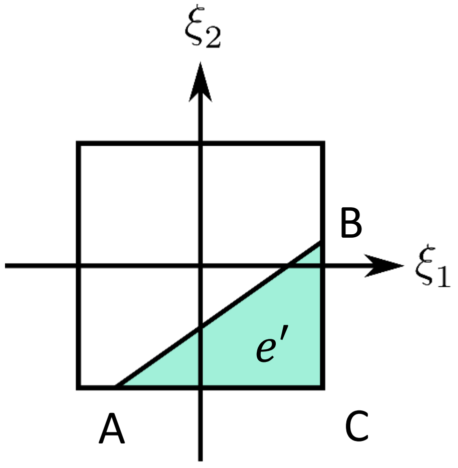
Order $p = 4$

$$K = (2p+1)(p+1)\frac{A(AB)}{V(ABC)}$$





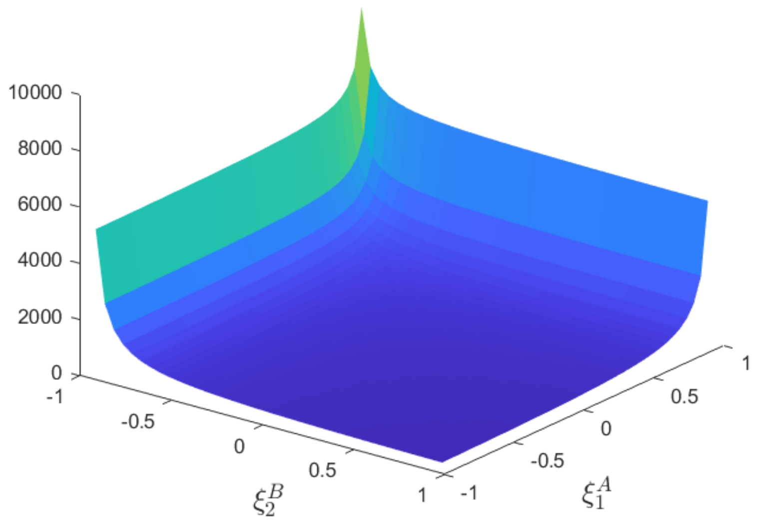
Type I: Sub-triangle verification

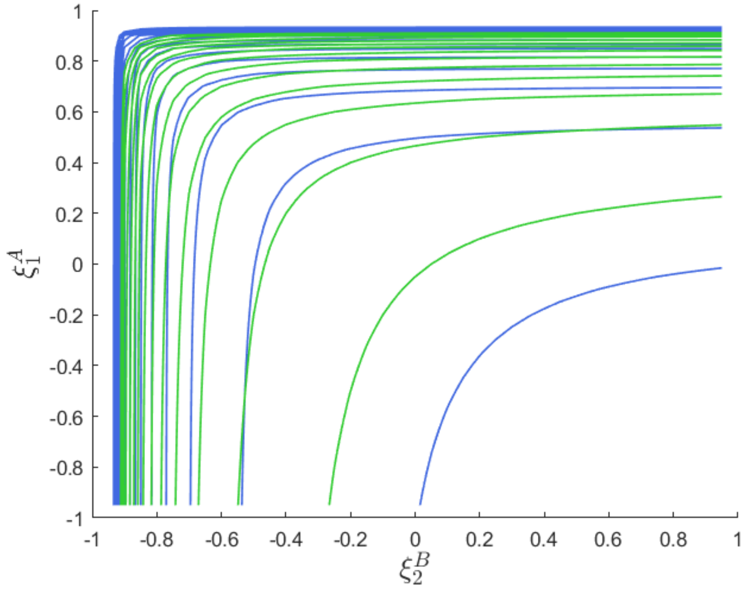


Trace inverse inequality on hypothenuse

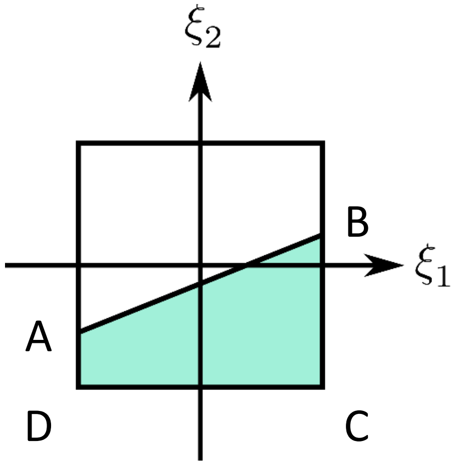
Order $p = 4$

$$K = 4p^2 \frac{A(\partial e')}{V(e')}$$

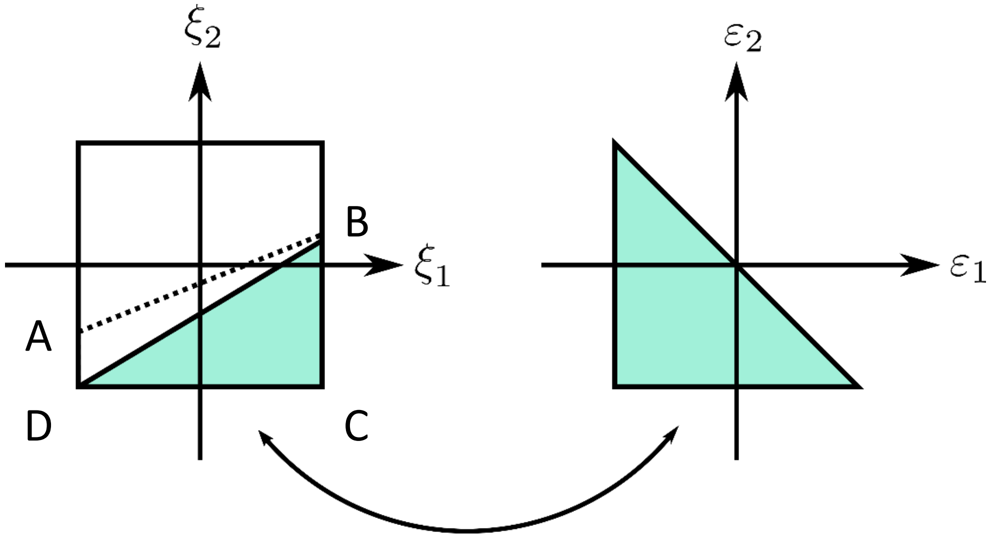




Type II: Sub-quadrangle



Type II: Sub-quadrangle

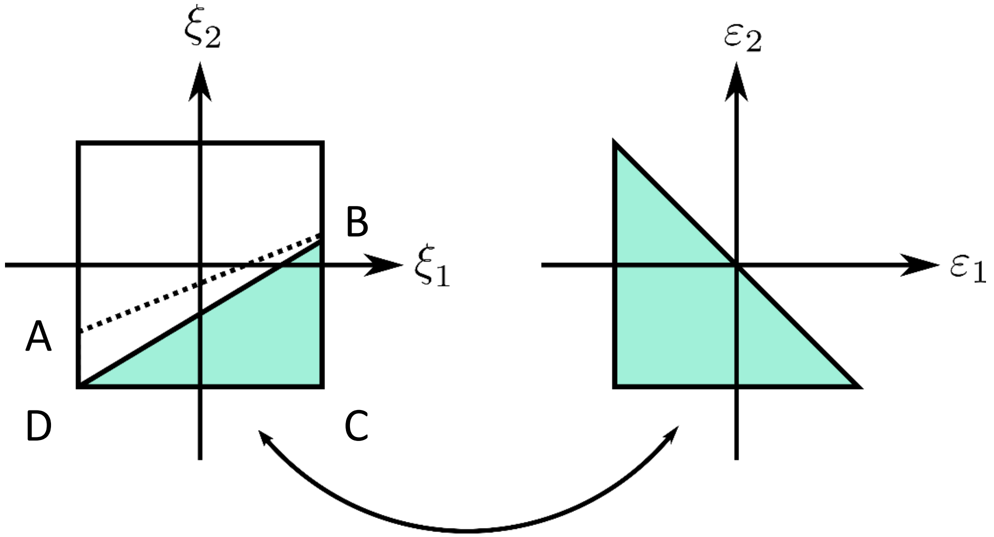


Mapping without crossed-terms

1st approach: largest triangle in sub-quadrangle

$$\int_{BC} v^2 dS \leq (p+1)(2p+1)\frac{A(BC)}{V(DBC)}\int_{DBC} v^2 dV$$

Type II: Sub-quadrangle



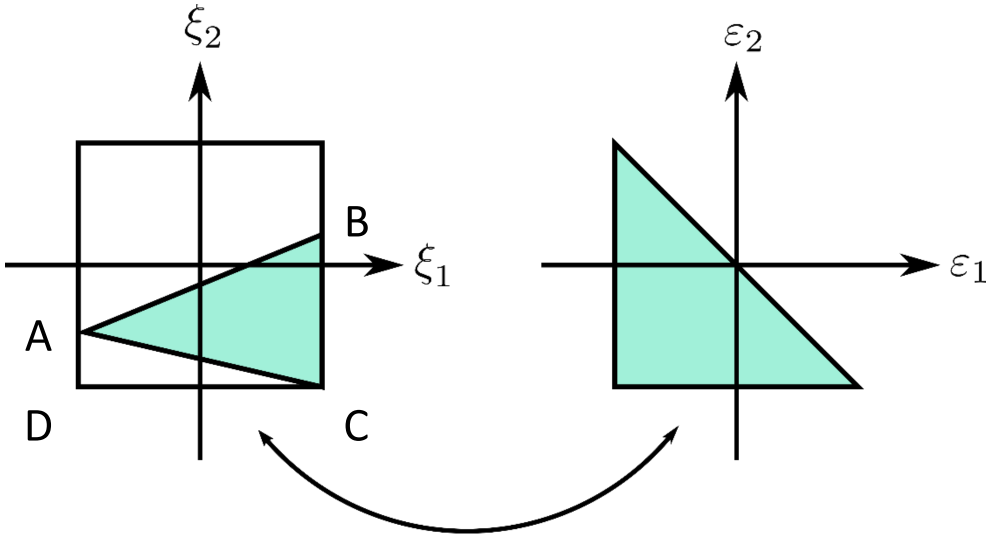
Mapping without crossed-terms

1st approach: largest triangle in sub-quadrangle

$$\int_{BC} v^2 dS \le (p+1)(2p+1)\frac{A(BC)}{V(DBC)}\underbrace{\int_{DBC} v^2 dV}$$

$$\le \int_{ABCD} v^2 dV$$

Type II: Sub-quadrangle



Mapping with crossed-terms

1st approach: largest triangle in sub-quadrangle

$$\int_{AB} v^2 dS \leq (p+1)(2p+1)\frac{A(AB)}{V(ABC)}\int_{ABC} v^2 dV$$

$$\leq \int_{ABCD} v^2 dV$$

$$\xi_1 = a\varepsilon_1 + b\varepsilon_2 + c$$
$$\xi_2 = d\varepsilon_1 + e\varepsilon_2 + f$$

1

ξ_1 ξ_2

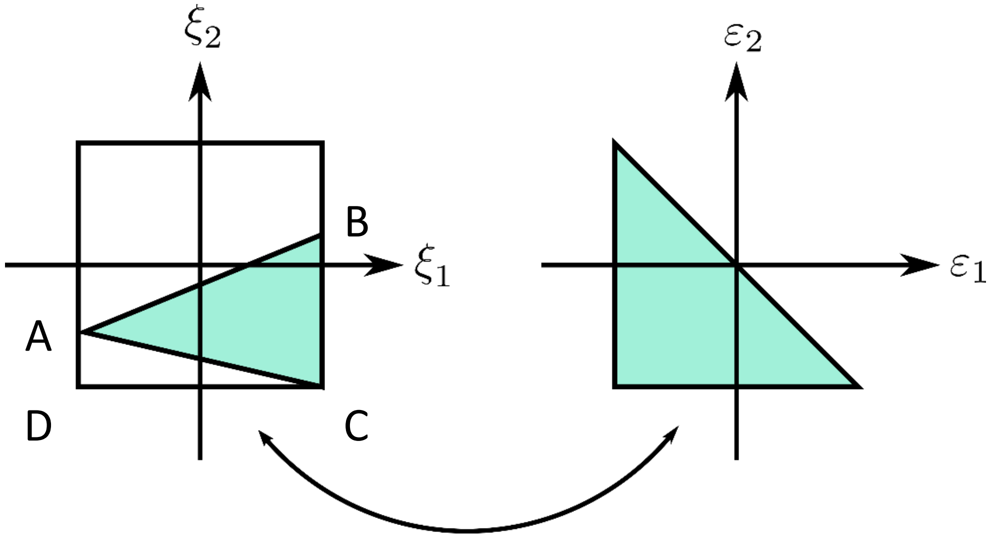
ξ_1^2 $\xi_1 \cdot \xi_2$ ξ_2^2

$\xi_1^2 \cdot \xi_2$ $\xi_1 \cdot \xi_2^2$

$\xi_1^2 \cdot \xi_2^2$

$$(a\varepsilon_1 + b\varepsilon_2 + c)^2(d\varepsilon_1 + e\varepsilon_2 + f)^2$$

Type II: Sub-quadrangle



Mapping with crossed-terms

1st approach: largest triangle in sub-quadrangle

$$\int_{AB} v^2 dS \leq (p+1)(2p+1)\frac{A(AB)}{V(ABC)}\int_{ABC} v^2 dV$$

$$\leq \int_{ABCD} v^2 dV$$

$$\int_{AB} v^2 dS \leq (p+1)(2p+1)\frac{A(AB)}{V(ABC)}\int_{ABC} v^2 dV$$

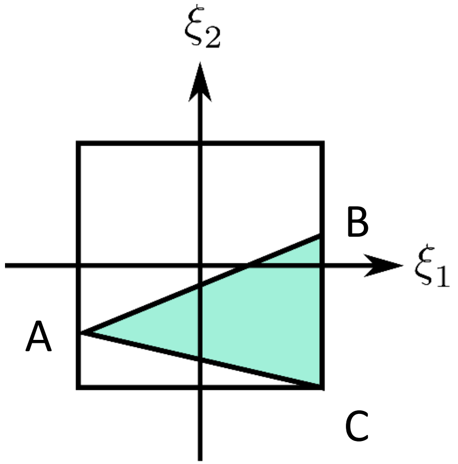
$$1$$
$$\xi_1 \qquad \xi_2$$
$$\xi_1^2 \qquad \xi_1 \cdot \xi_2 \qquad \xi_2^2$$
$$\xi_1^2 \cdot \xi_2 \qquad \xi_1 \cdot \xi_2^2$$
$$\boxed{\xi_1^2 \cdot \xi_2^2}$$

$$(a^2 d^2 \varepsilon_1^4 + a^2 e^2 \varepsilon_1^2 \varepsilon_2^2 + a^2 f^2 \varepsilon_1^2 + \ldots)$$

After mapping, integrand is no longer in the same functional space

$$C(p) = \frac{(4p+1)(4p+2)}{2}$$

Type II: Sub-quadrangle verification

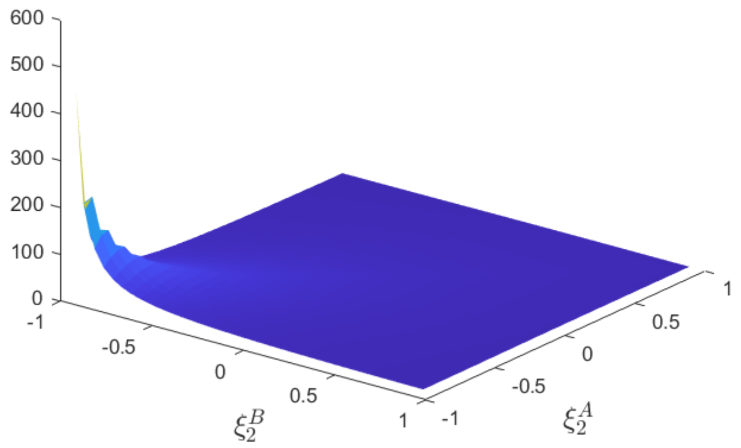


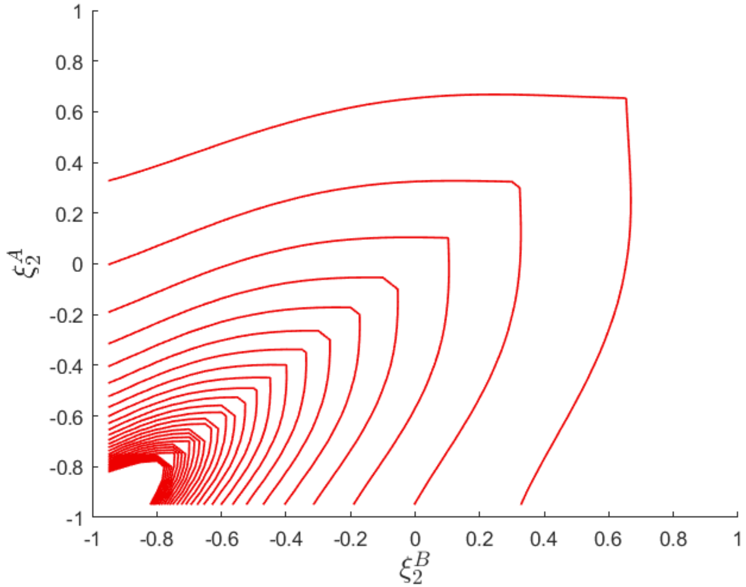
Does $h = \frac{V(ABC)}{A(AB)}$ a good definition for geometric factor?

Trace inverse inequality on AB

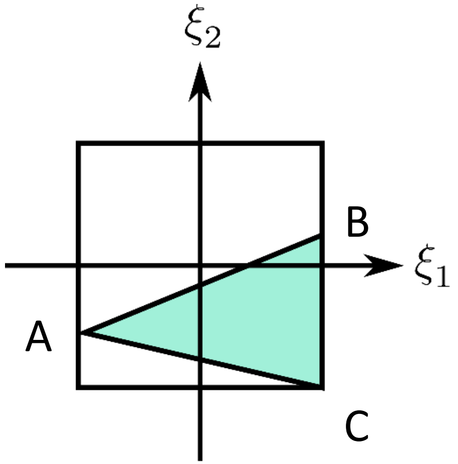
Order $p = 1$

Numerical results





Type II: Sub-quadrangle verification



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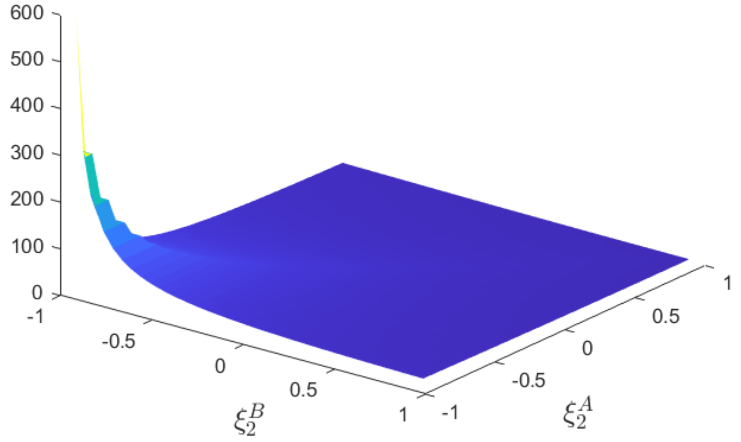
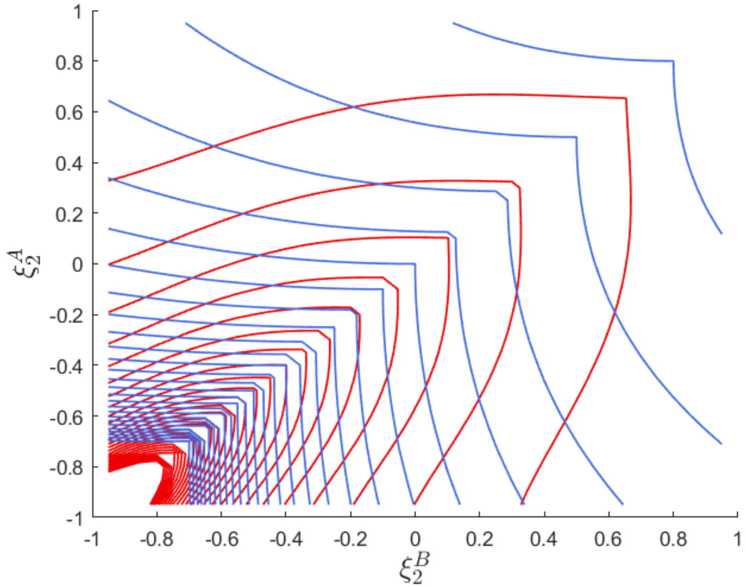
Trace inverse inequality on AB

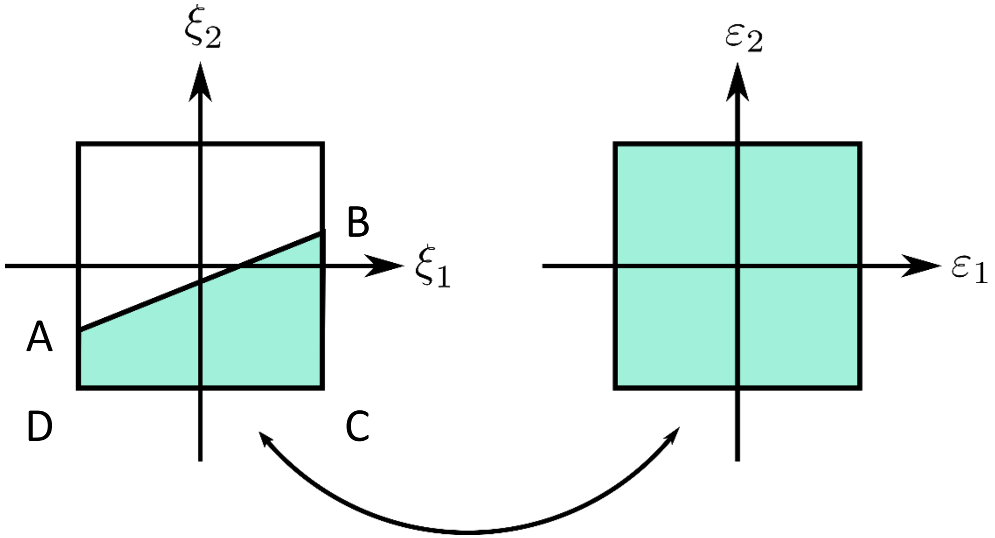
Order $p = 1$

$$K = (4p+1)(2p+1)\frac{A(AB)}{V(ABC)}$$





Type II: Sub-quadrangle verification



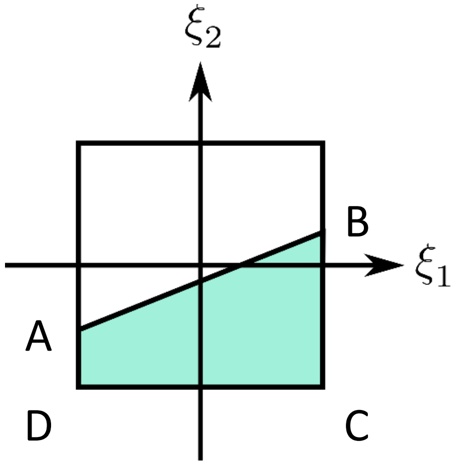
2nd approach: entire sub-quadrangle

$$\int_{AB} v^2 dS \leq \frac{C(p)}{h} \int_{ABCD} v^2 dV$$

Mapping with crossed-terms $C(p) = (2p+1)^2 < \frac{(4p+1)(4p+2)}{2}$

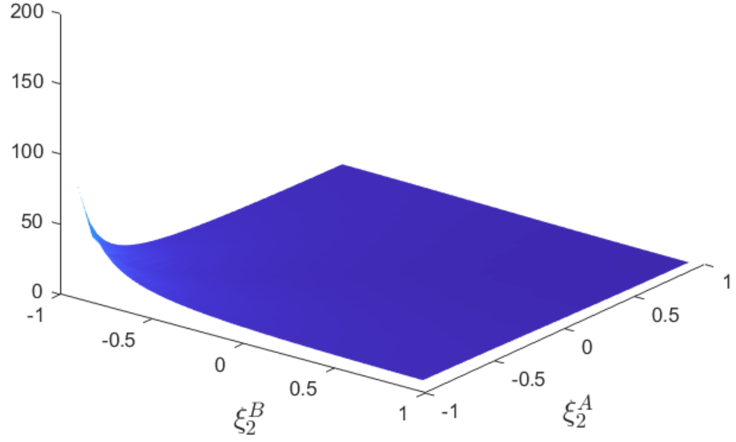
Is $h = \frac{V(ABCD)}{A(AB)}$ a good definition for geometric factor?

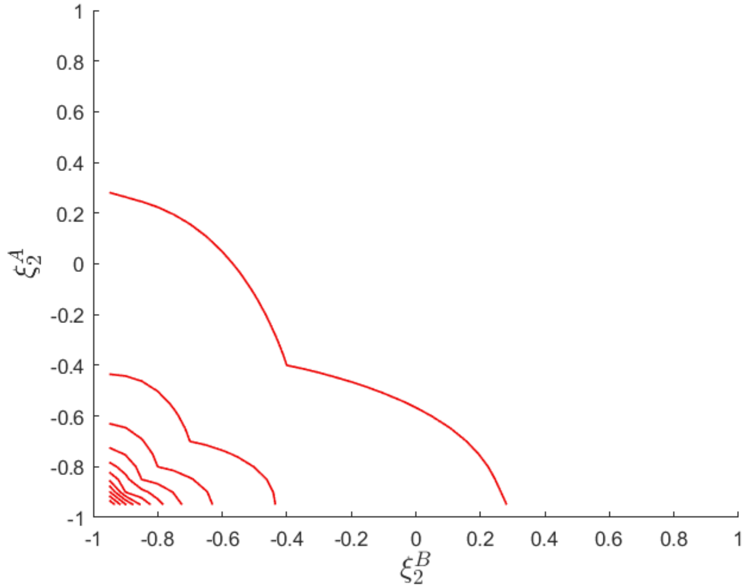
Type II: Sub-quadrangle verification



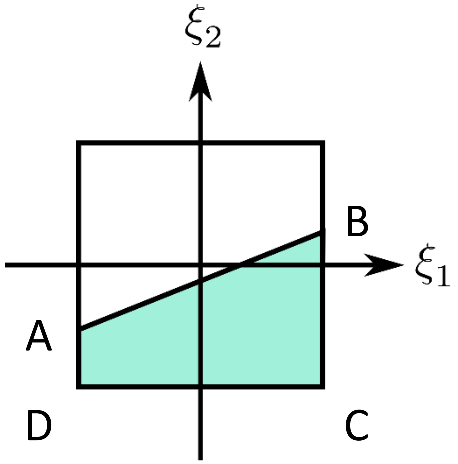
Trace inverse inequality on AB
Order $p = 1$

Numerical results



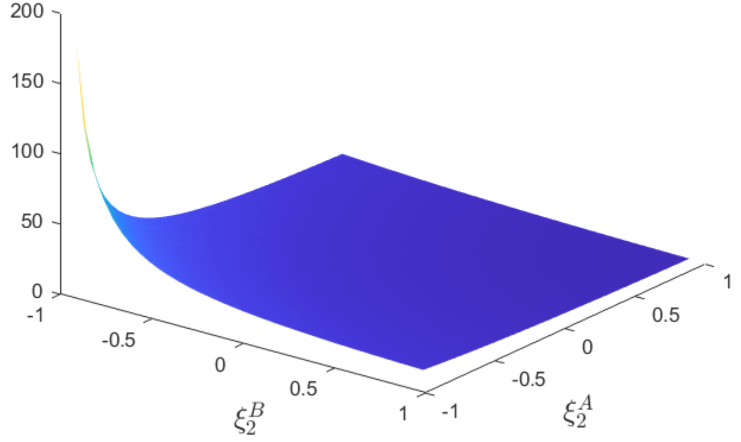


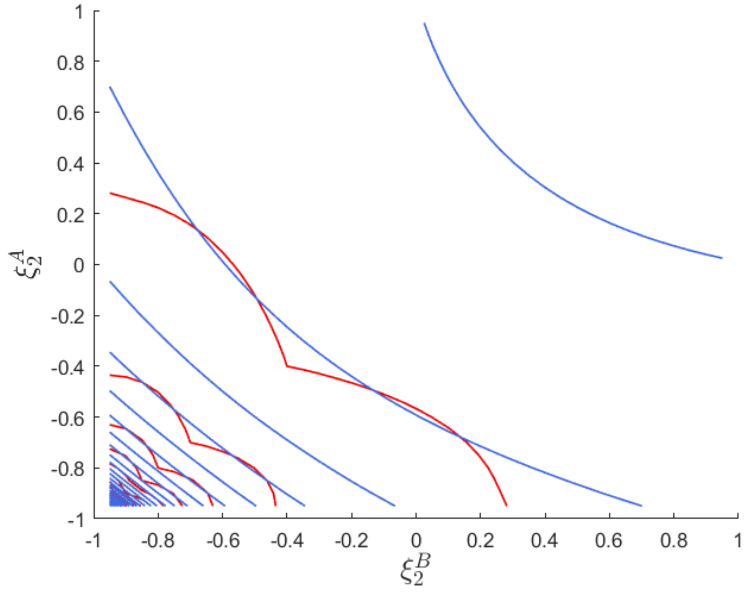
Type II: Sub-quadrangle verification



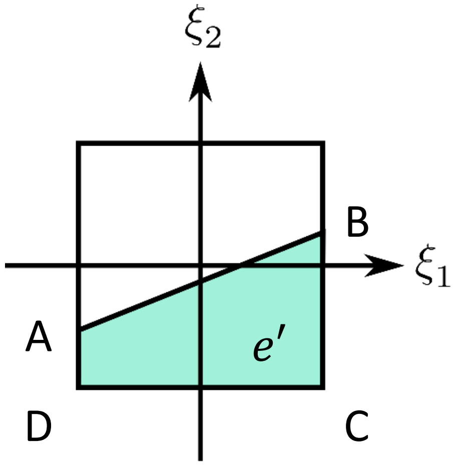
Trace inverse inequality on AB
Order $p = 1$

$$K = (2p+1)^2 \frac{A(AB)}{V(ABCD)}$$





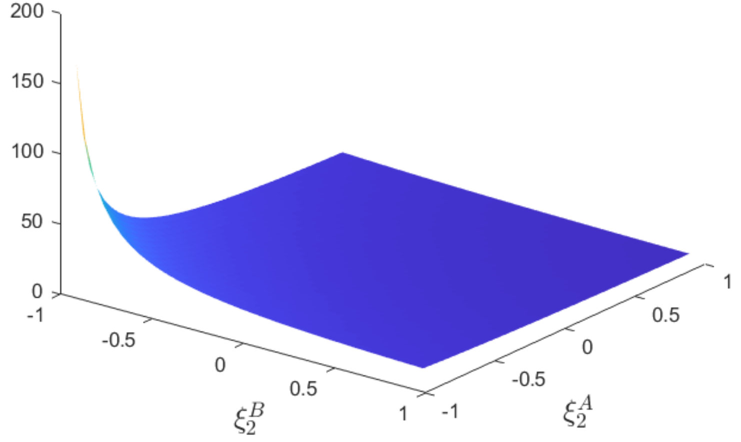
Type II: Sub-quadrangle verification

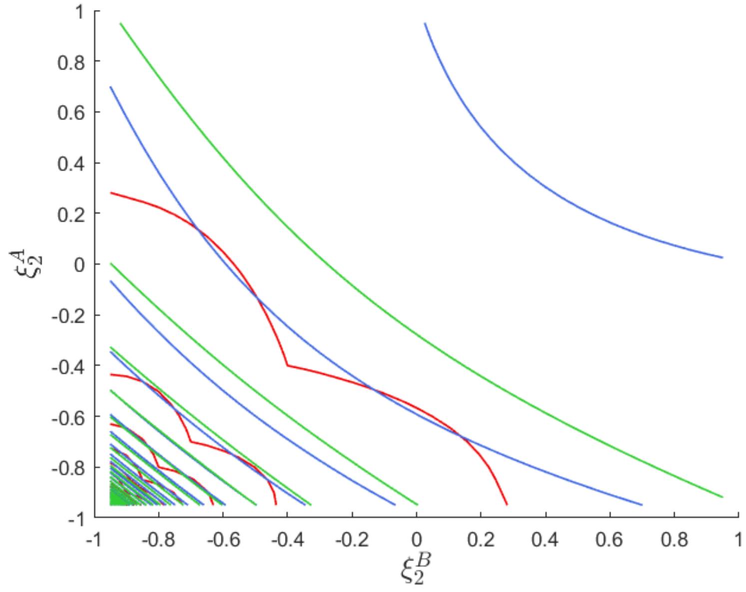


Trace inverse inequality on AB

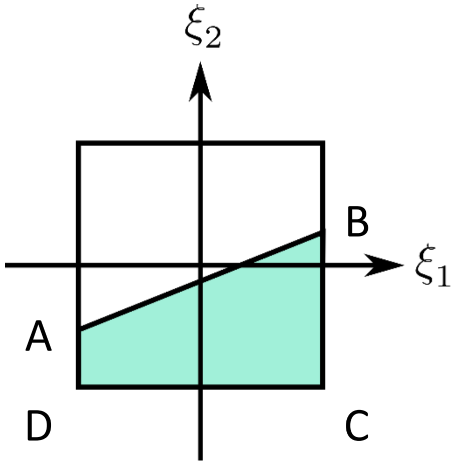
Order $p = 1$

$$K = 4p^2 \frac{A(\partial e')}{V(e')}$$





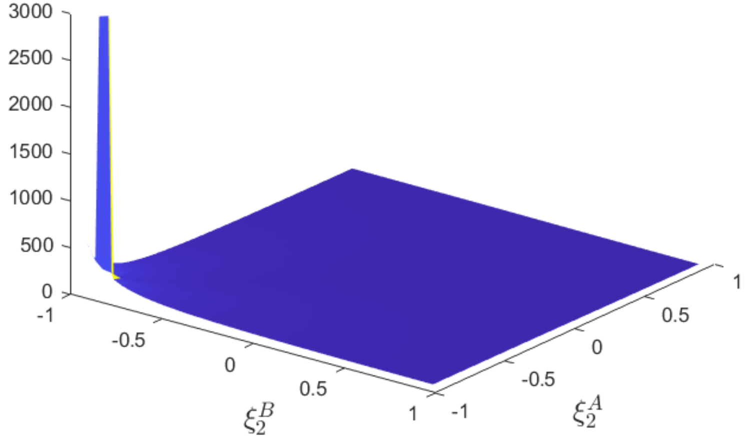
Type II: Sub-quadrangle verification

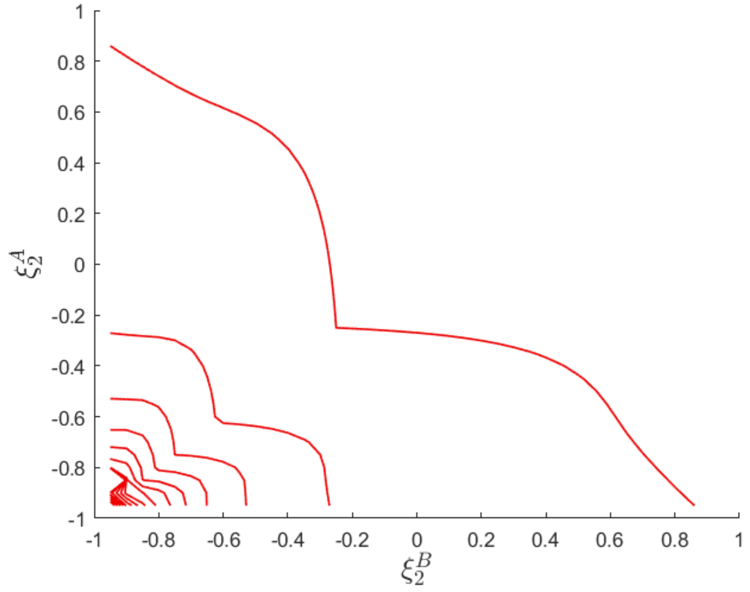


Trace inverse inequality on AB

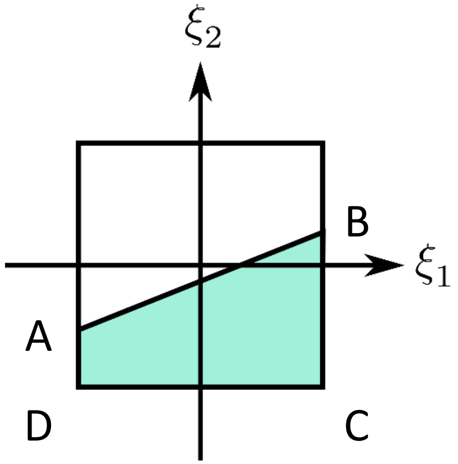
Order $p = 4$

Numerical results





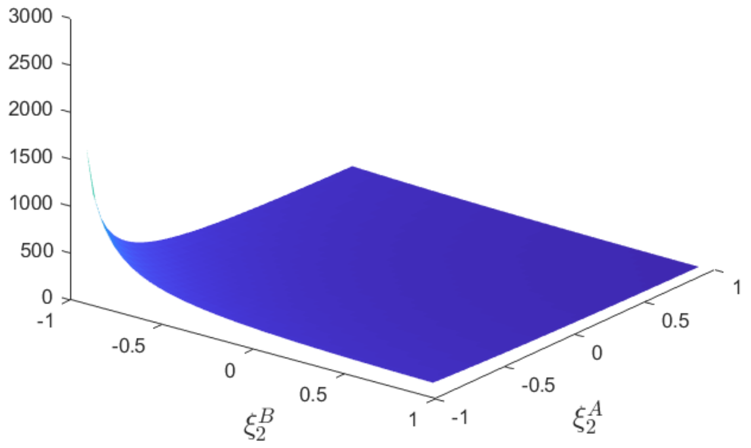
Type II: Sub-quadrangle verification

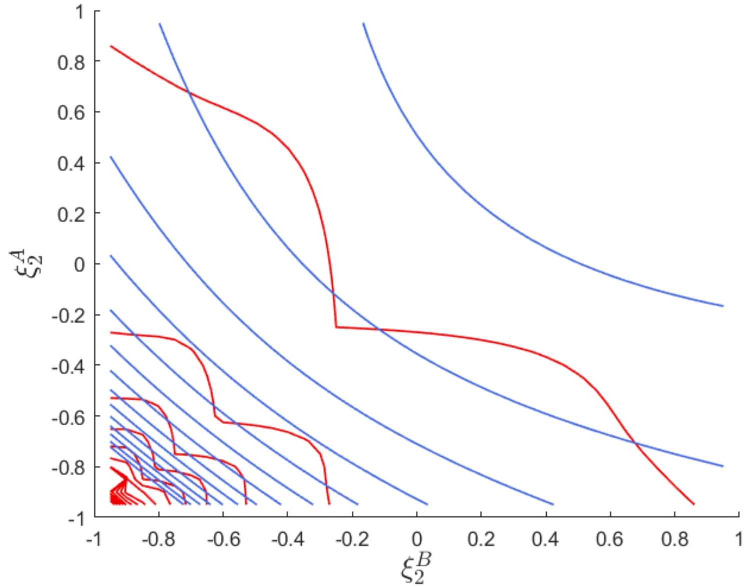


Trace inverse inequality on AB

Order $p = 4$

$$K = (2p+1)^2 \frac{A(AB)}{V(ABCD)}$$





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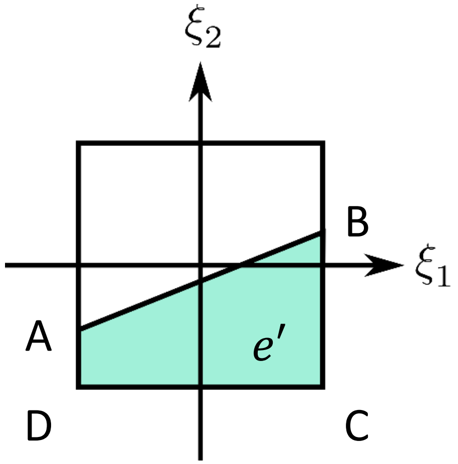

Trace inverse inequality on AB

Order $p = 4$

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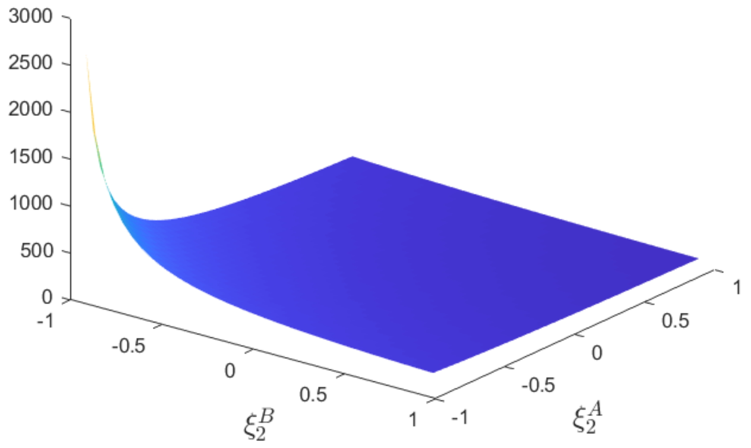
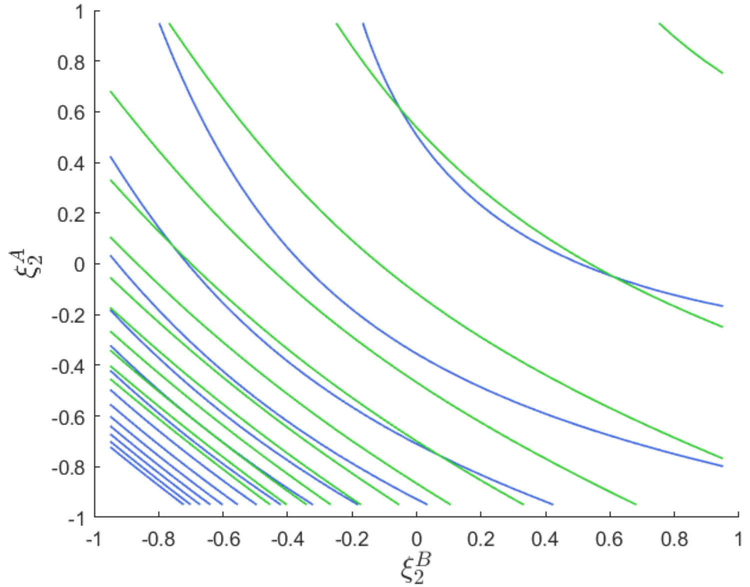
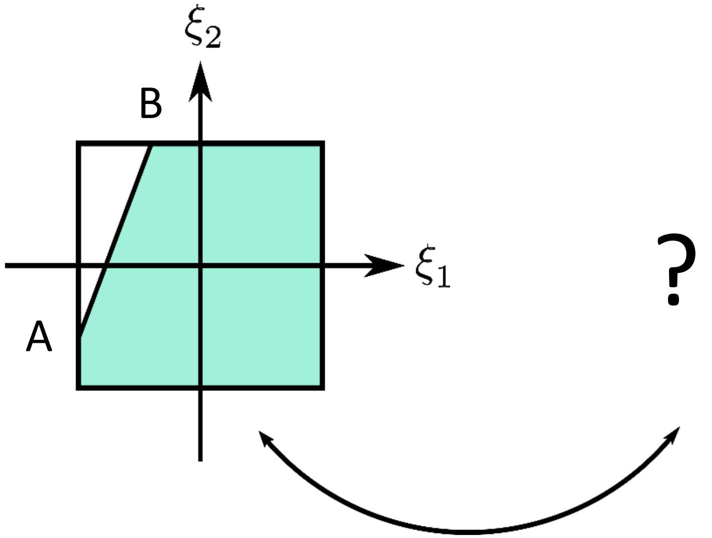


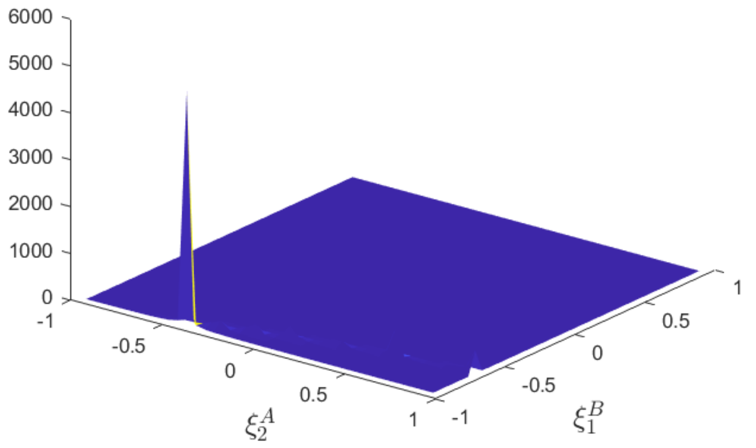


Type III: Sub-pentagon

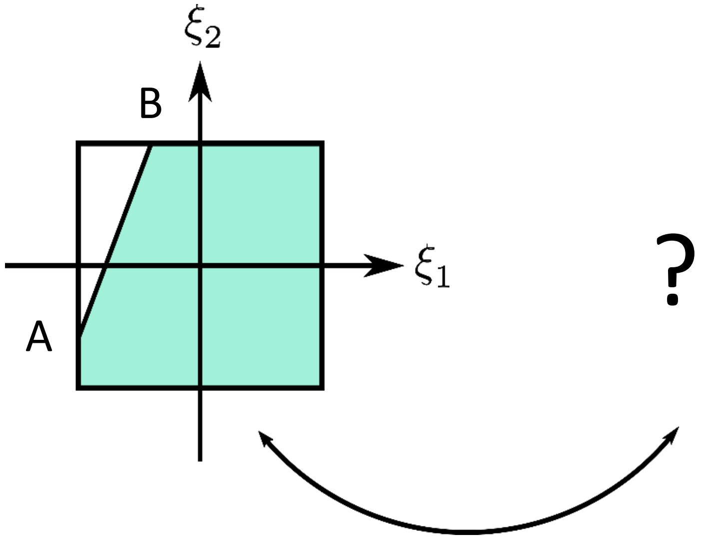


No sharp estimation, for both $C(p)$ and h
Trace inverse inequality on AB
Order $p = 1$

Numerical results



Type III: Sub-pentagon

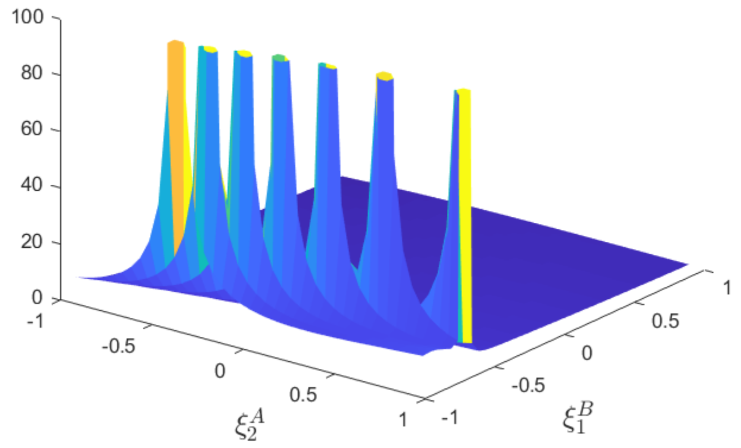


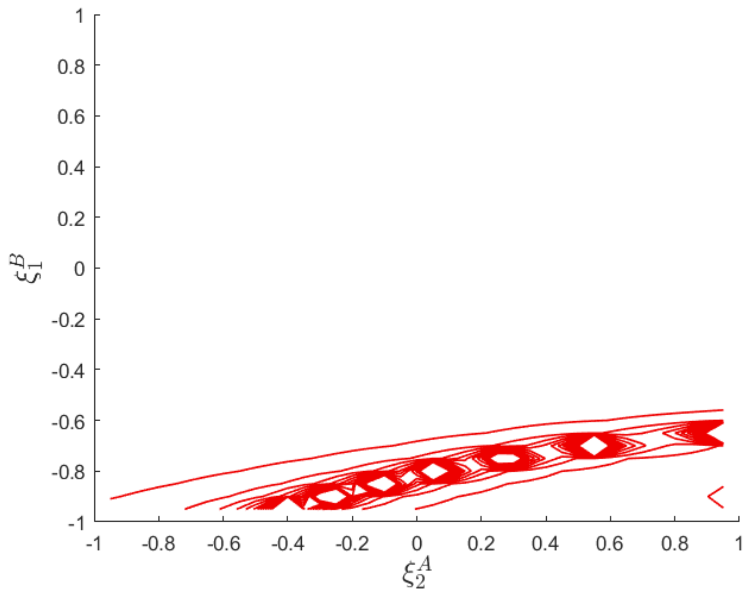
No sharp estimation, for both $C(p)$ and h

Trace inverse inequality on AB

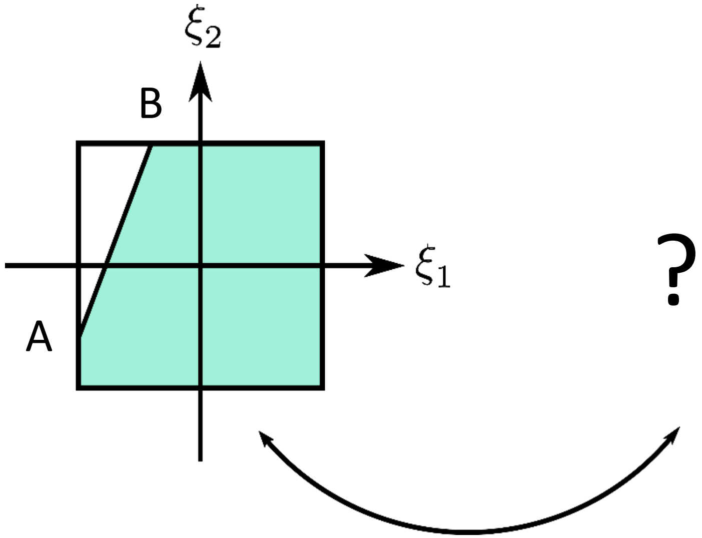
Order $p = 1$

Numerical results





Type III: Sub-pentagon

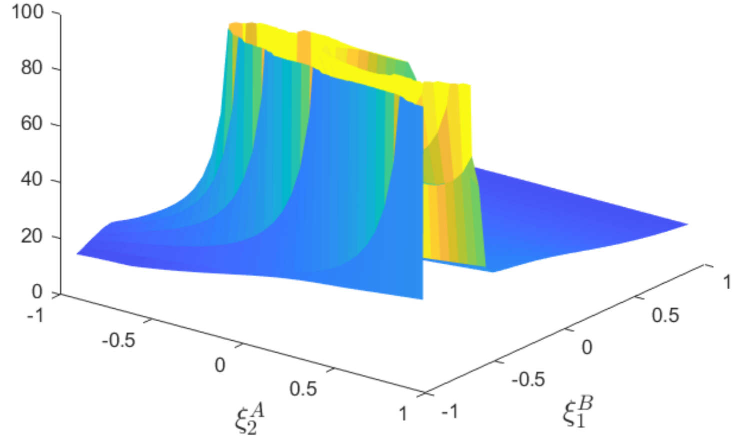


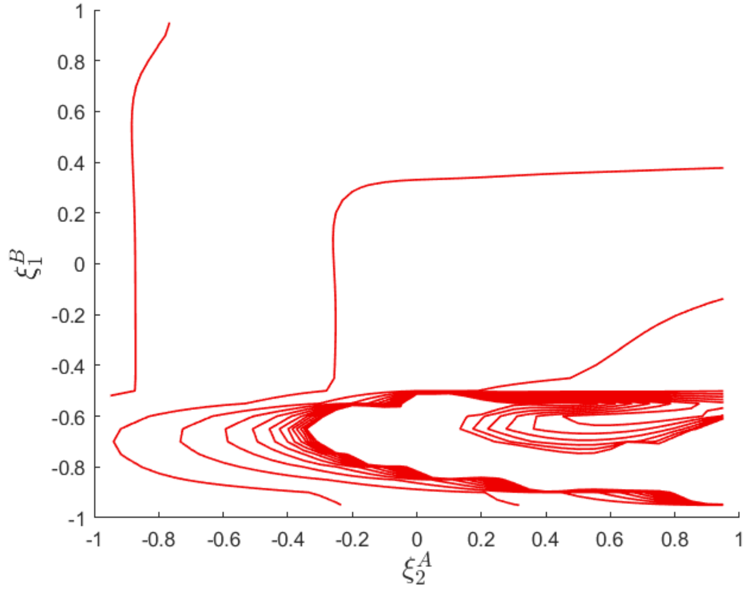
No sharp estimation, for both $C(p)$ and h

Trace inverse inequality on AB

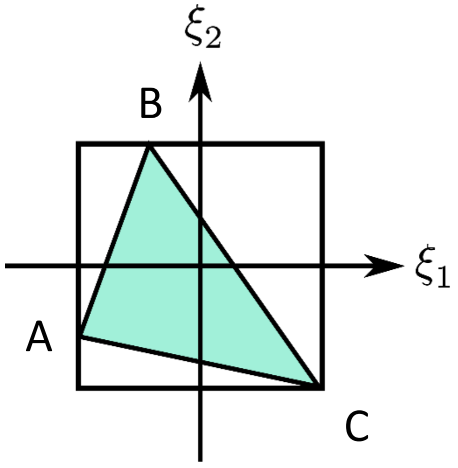
Order $p = 4$

Numerical results





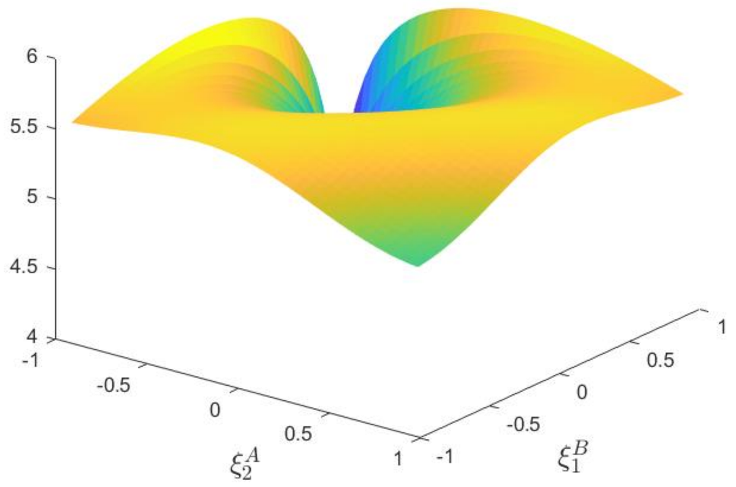
Type III: Sub-pentagon

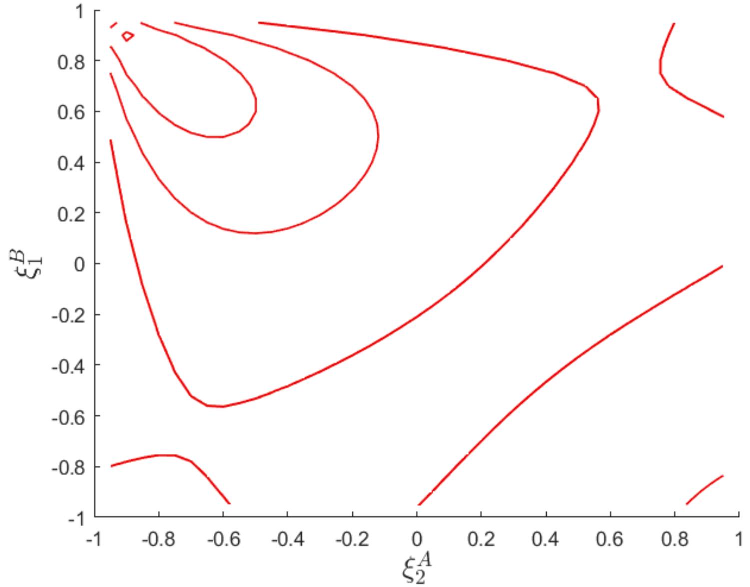


[Cangiani, 2014]

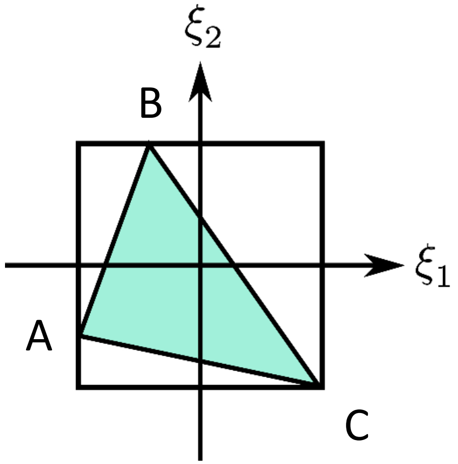
No sharp estimation, for both $C(p)$ and h
Trace inverse inequality on AB
Order $p = 1$

Numerical results





Type III: Sub-pentagon

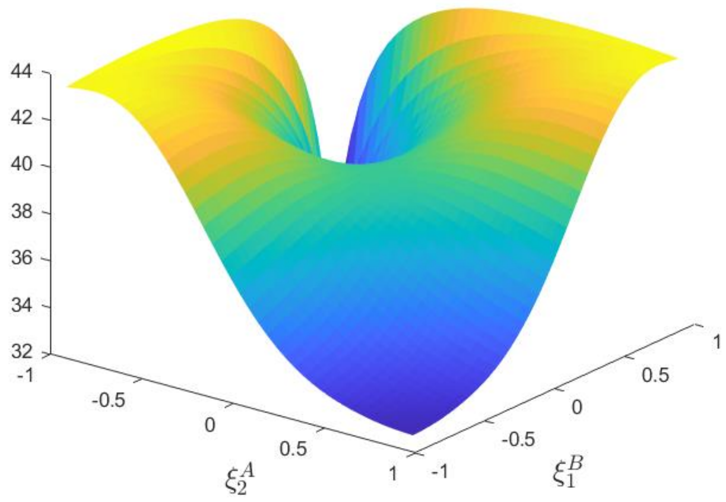


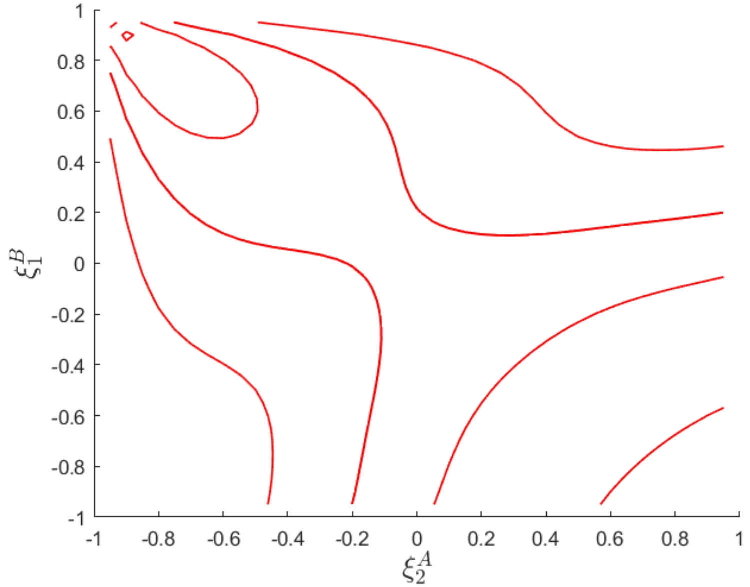
No sharp estimation, for both $C(p)$ and h

Trace inverse inequality on AB

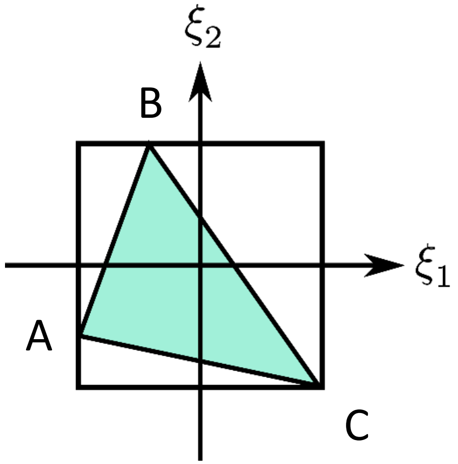
Order $p = 4$

Numerical results





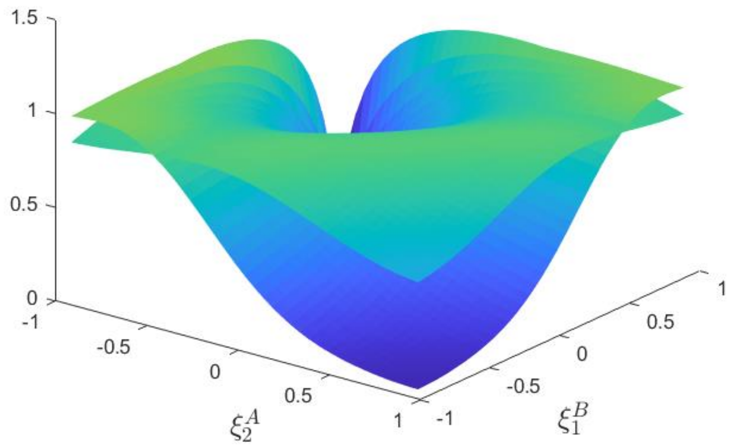
Type III: Sub-pentagon



No sharp estimation, for both $C(p)$ and h

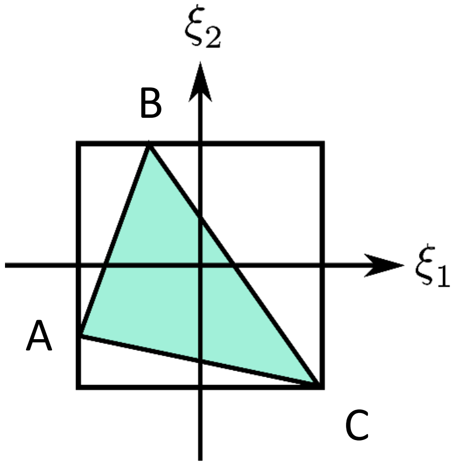
Trace inverse inequality on AB

Numerical results, scaled



Geometric factor depends on order

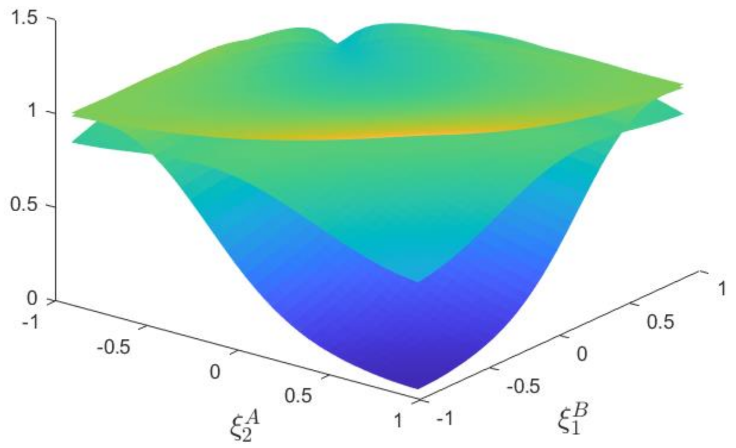
Type III: Sub-pentagon



No sharp estimation, for both $C(p)$ and h

Trace inverse inequality on AB

Numerical results, scaled

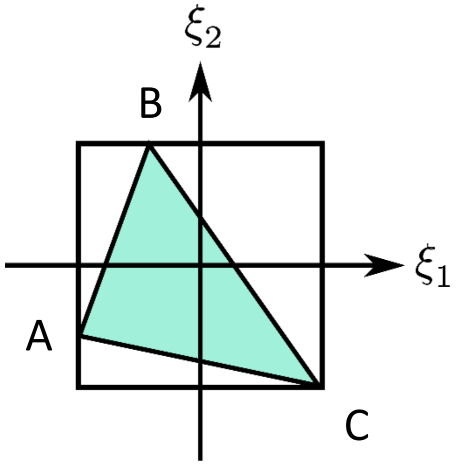


Bounding by above with

$$h = \frac{V(ABC)}{A(AB)}$$

Geometric factor depends on order

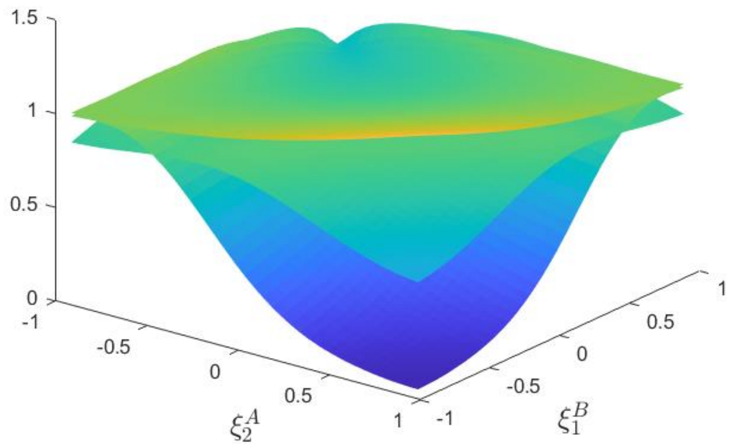
Type III: Sub-pentagon



No sharp estimation, for both $C(p)$ and h

Trace inverse inequality on AB

Numerical results, scaled



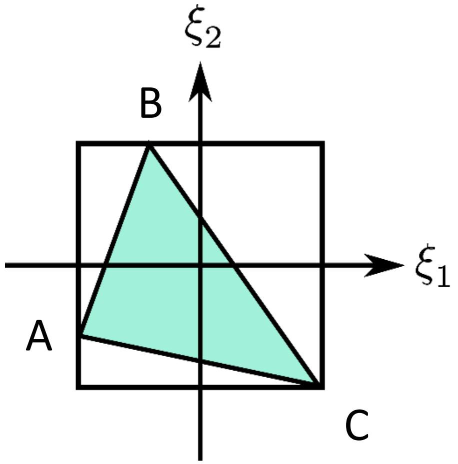
Scale back with

$$C(p) = (p + 1)(2p + 1)$$

to bound by above experimental results

Geometric factor depends on order

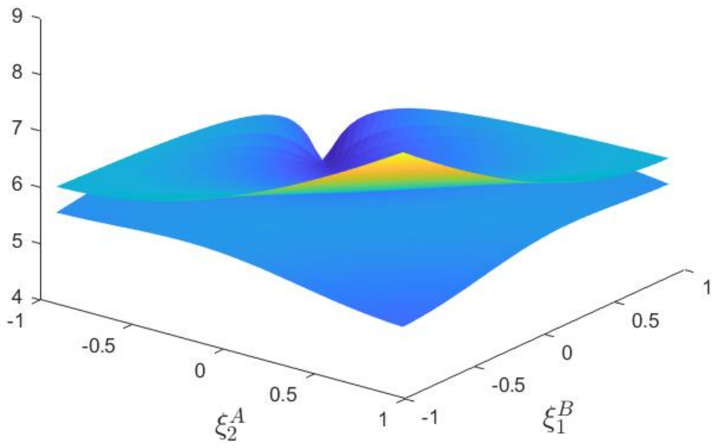
Type III: Sub-pentagon verification

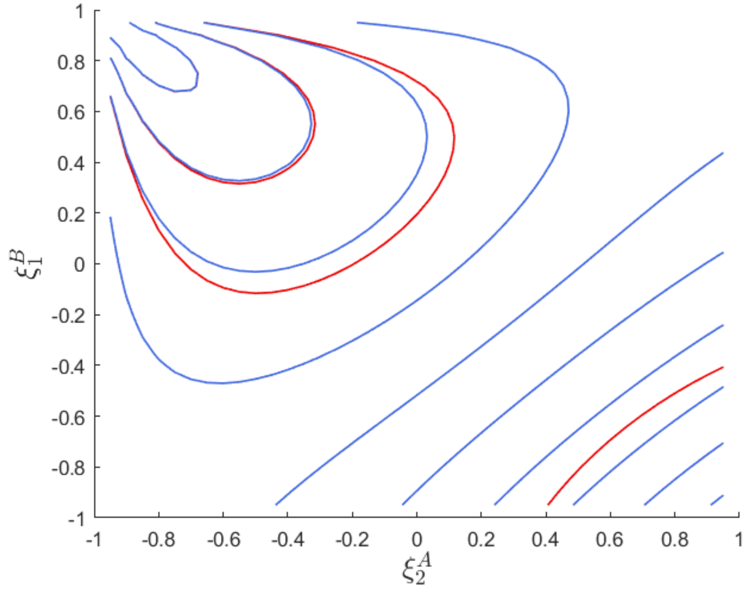


Trace inverse inequality on AB

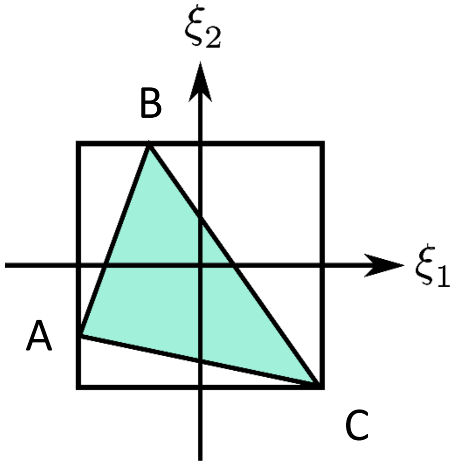
Order $p = 1$

$$K = (p+1)(2p+1)\frac{A(AB)}{V(ABC)}$$





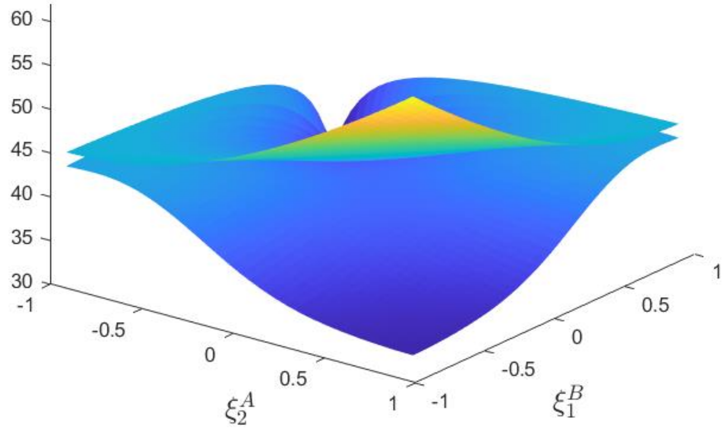
Type III: Sub-pentagon verification

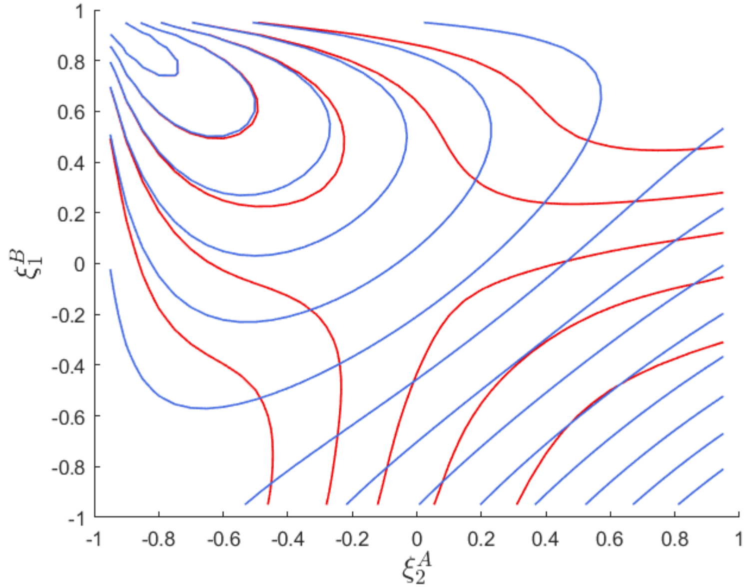


Trace inverse inequality on AB

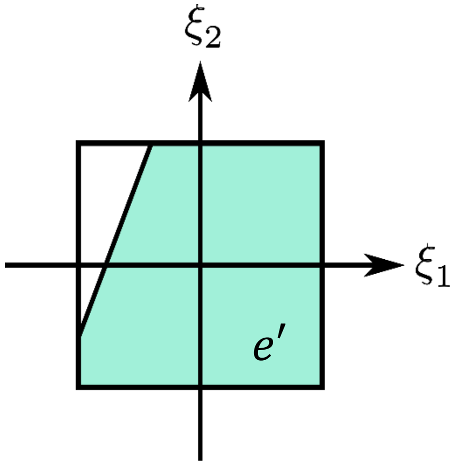
Order $p = 4$

$$K = (p+1)(2p+1)\frac{A(AB)}{V(ABC)}$$



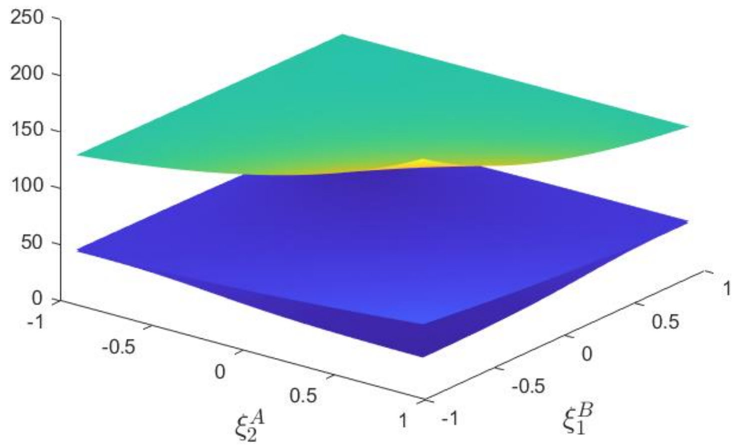


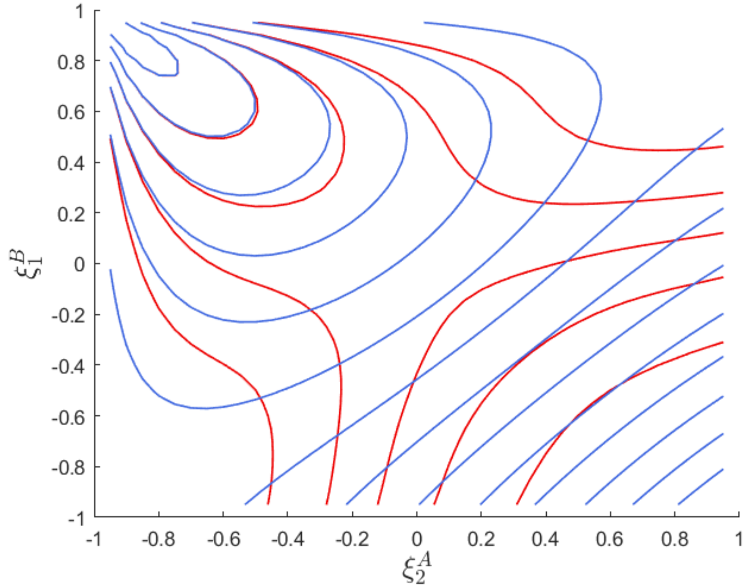
Type III: Sub-pentagon verification



Trace inverse inequality on AB
Order $p = 4$

$$K = 4p^2 \frac{A(\partial e')}{V(e')}$$



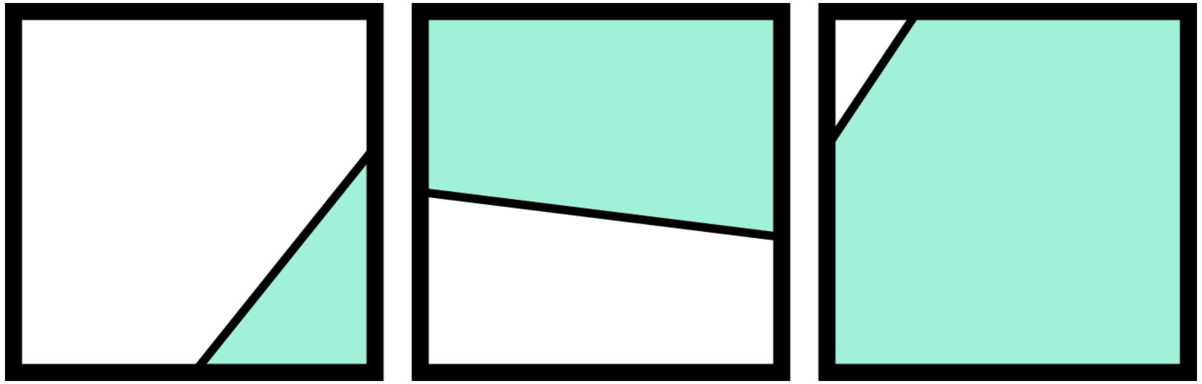


Conclusion & perspectives

To guarantee coercivity of interior penalty method & minimize conditioning

$$\int_{f'} v^2 dS \leq K \int_{e'} v^2 dV \,, \forall v \in \mathcal{V}$$

for 2D tensor product elements of order p, implicitly cut by a level-set

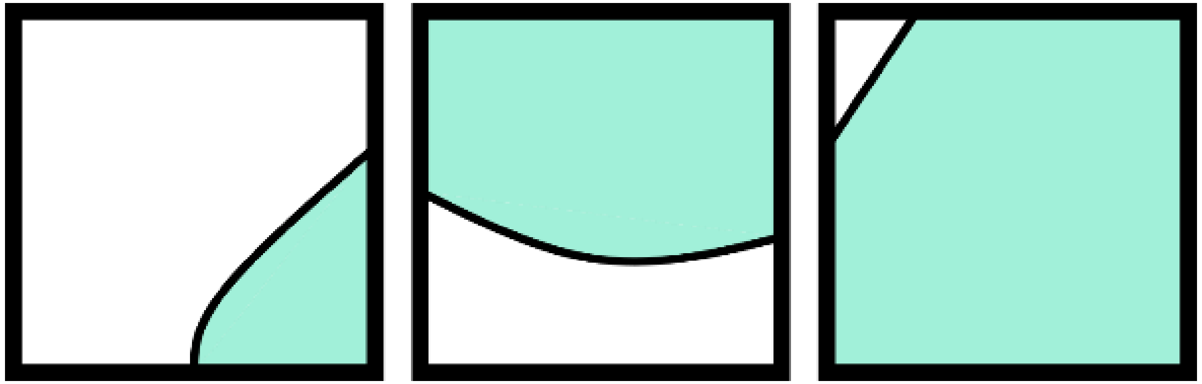


- Numerical results using Owen's approach
- Kummer's definition for K: though simple, too safe → higher conditioning
- Proposed new $C(p)$ and h expressions, case-dependent & better calibrated

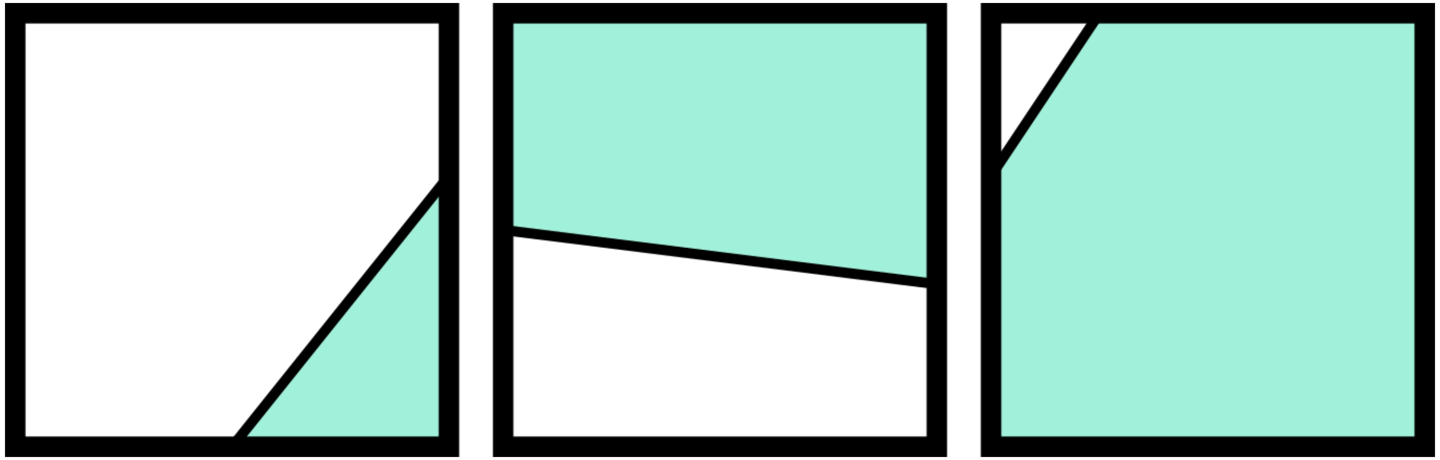
Conclusion & perspectives

What's next

- Consider curved cut with parameterized level-set
- Going to 3D tensor product elements



Interior Penalty Method for a Cartesian discontinuous Galerkin solver with immersed boundaries



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References

[Alexandersen, 2016] Alexandersen, J., Sigmund, O., & Aage, N. (2016). Large scale three-dimensional topology optimisation of heat sinks cooled by natural convection. *Int. J. Heat Mass Tran.*, **100**, 876-891.

[Thomas, 2019] Thomas, D., Cerquaglia, M. L., Boman, R., Economon, T. D., Alonso, J. J., Dimitriadis, G., & Terrapon, V. E. (2019). CUPyDO-An integrated Python environment for coupled fluid-structure simulations. *Adv. Eng. Softw.*, **128**, 69-85.

[Frère, 2015] Frere, A., Hillewaert, K., Chivaee, H. S., Mikkelsen, R. F., & Chatelain, P. (2015). Cross-validation of numerical and experimental studies of transitional airfoil performance. In *33rd Wind Energy Symposium* (p. 0499).

[Das, 2018] Das, S., Panda, A., Deen, N. G., & Kuipers, J. A. M. (2018). A sharp-interface immersed boundary method to simulate convective and conjugate heat transfer through highly complex periodic porous structures. *Chem. Eng. Sci.*, **191**, 1-18.

[Warburton, 2003] Warburton T, Hesthaven JS, (2003). On the constants in hp-finite element trace inverse inequalities, Comput. Methods Appl. Mech. Eng., **192**(25),2765–2773.

[Hillewaert, 2013] Hillewaert K., Development of the discontinuous Galerkin method for high-resolution, large scale CFD and acoustics in industrial geometries, PhD thesis, Université catholique de Louvain, Louvain-la-Neuve, Belgium.

[Kummer, 2016] Kummer, F. (2016) Extended discontinuous Galerkin methods for two-phase flows: the spatial discretization, Int. J. Numer. Meth. Engng, **109**.2:259-289.

References

[Owens, 2017] Owens, A. R., Kópházi, J., & Eaton, M. D. (2017). Optimal trace inequality constants for interior penalty discontinuous Galerkin discretisations of elliptic operators using arbitrary elements with non-constant Jacobians. *J. Comput. Phys.*, **350**, 847-870.

[Cangiani, 2014] Cangiani, A., Georgoulis, E. H., & Houston, P. (2014). hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, **24**(10), 2009-2041.