Fundamental Groups and Dendricity

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Dendric shifts

A *shift space* X is a closed subset of $A^{\mathbb{Z}}$ stable under the shift action of $\mathbb{Z}$.

The *extension graph* $\mathbf{E}_X(u)$, or $\mathbf{E}(u)$, of a factor u of X is the bipartite graph for which the left (resp. right) vertices are the letters a such that au (resp. ua) is a factor of X and (a,b) is an edge if aub is a factor of X.

X is *dendric* if for any factor u, $\mathbf{E}_X(u)$ is a tree.

The *return words of u* are the nonempty factors v of X such that uv is a factor in which u occurs exactly twice, once as a prefix and once as a suffix.

1. Example

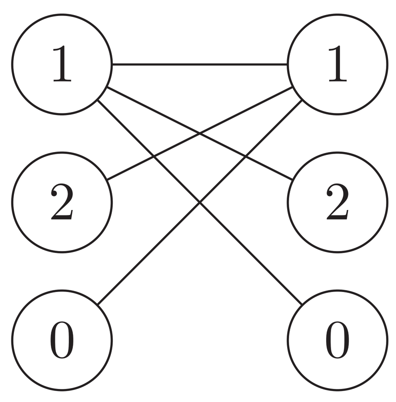
The *Tribonacci shift*

$$\ldots 0102010010201010201001020 1020\ldots$$

is the fixed point of the morphism

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0.$$

Its extension graph $\mathbf{E}(0)$ is given by



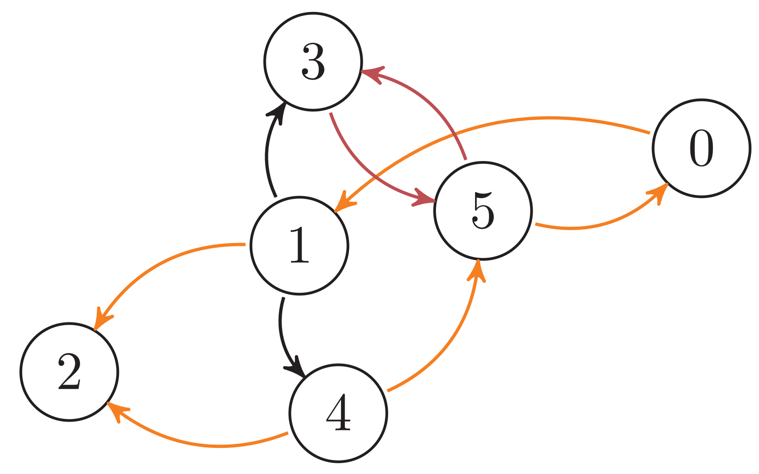
The set of return words of 01 in the Tribonacci shift is $\{0102, 010, 01\}$.

2. Eventually dendric shifts

A shift is *eventually dendric* if there is a threshold N such that $\mathbf{E}(u)$ is a tree if $|u| \geq N$. These are stable under dynamical isomorphisms which corrects a major drawback of dendric shifts.

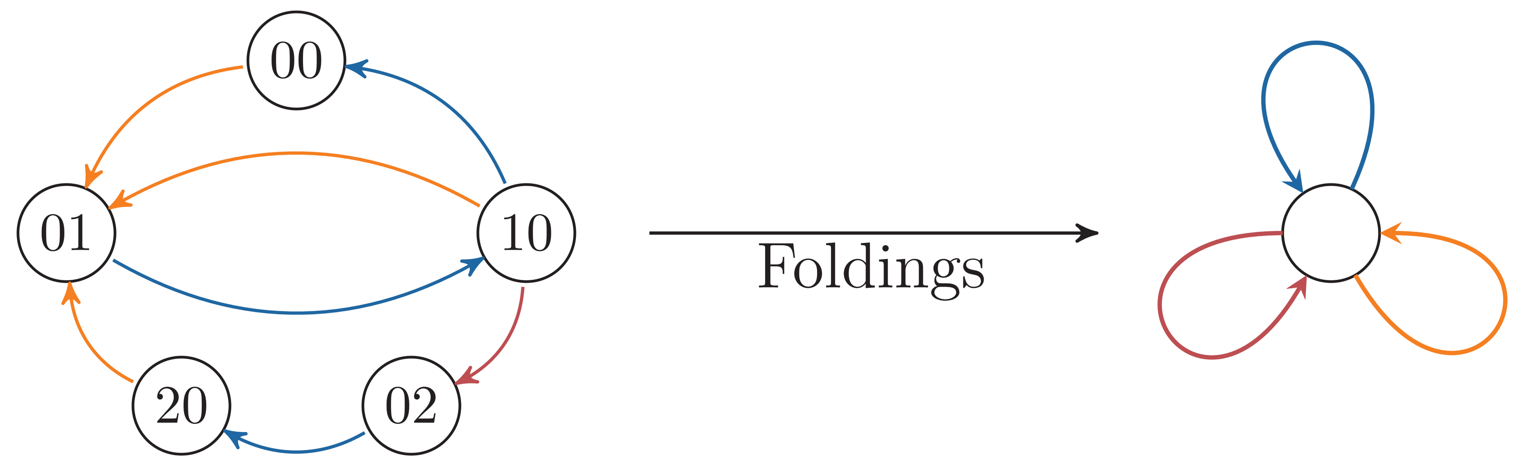
Fundamental groups

A directed graph G coupled with one of its vertices v define a *fundamental group* as its set of loops (where directed edges can be used in both directions) endowed with concatenation.



1. Application to dendric shifts

The *order-m Rauzy graph* of a language is the di-graph for which the vertices are the length m factors and the b-labelled edges are between au and ub if aub is a factor. On the Tribonacci shift, we have G_2 and its folding given bellow:



The *Rauzy group of u* for u in the language is projection of the fundamental group of $G_{|u|}$ with base point u on the free group F_A. An example of such a projection is given bellow.

2. Return theorem

Let X be a minimal dendric shift. For any factor u, the group generated by the return words of u (called the *return group of u*) is free of rank $|A|$.

Sketch of the proof

- Prove that the return words of u form a basis of the loops of the Rauzy group of u. Hence, our study of return groups can be reduced to a study of Rauzy groups.
- Using the combinatorial restrictions of dendric shifts, prove that we can find a trivial path between two vertices having the same proper suffixes.
- Show that the previous item allows us to iteratively reduce the order of the Rauzy graph. Once the order is 1, the graph is a bouquet of circles and we know its fundamental group.

Return theorem for eventually dendric shifts

Let X be a minimal eventually dendric shift with threshold N. The return groups of all factors u such that $|u| \geq N$ are conjugate of rank lower or equal to

$$p_X(N) - \sum_{v \in \mathcal{L}_{N-1}(X)} C(v) + 1,$$

where $C(v)$ is the number of connected components of $\mathbf{E}_X(v)$.

Sketch of the proof

- We apply the conjugacy $\varphi : u_1 \ldots u_N \mapsto (u_1, \ldots, u_N)^{\mathrm{T}}$ to X. This provides an eventually dendric shift $X^{(N)}$ (called the N^{th} higher block code) of threshold 1.
- In [4], tools are provided to study the return groups of such shifts. For instance, these groups are free of rank $S - C + 1$, where S is the size of the alphabet of $X^{(N)}$ and C the number of connected components of $\mathbf{E}_{X^{(N)}}(\varepsilon)$.
- Exploiting the equivalence between Rauzy groups and return groups, we see that any folding occurring in a Rauzy graph of $X^{(N)}$ will also occur in the corresponding Rauzy graph of X. Thus, the rank in X is bounded by the announced value.

Example

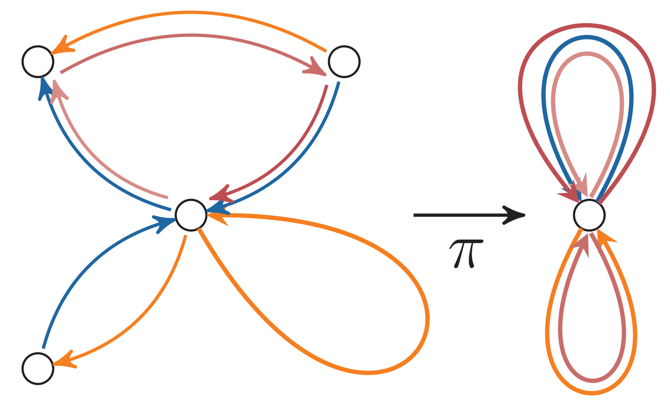
We consider the eventually dendric shift X of threshold 2 obtained by applying the morphism $0 \mapsto 0, 1 \mapsto 012, 2 \mapsto 022$ and $3 \mapsto 02$ to the 4-bonacci shift:

$$\ldots 0102010301020100102010301020101020103010\ldots$$

The shift space X has the property that $X^{(2)}$ has return groups of rank 4, whereas X has return groups for words of length bigger than 2 of rank 3. We can conclude that the rank of the return groups of eventually dendric shifts is not a conjugacy invariant.

Projection of a fundamental group

The purple loop in the fundamental group of (G, v) is sent by the projection on F_2 to the purple loop on the bouquet of two circles.



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