Return Groups of Eventually Dendric Shifts

Pierre Stas

March 2023

Joint work with F. Gheeraert and J. Leroy.

Definition I

Definition

A *shift space* X is a closed subset of $A^{\mathbb{Z}}$ stable under the shift action of $\mathbb{Z}$.

Notations

- $\mathcal{L}(X) = \{w \in A^* | \exists x \in X : w \in \mathrm{Fact}(x)\}$
- $\mathbf{L}(w) = \{a \in A | aw \in \mathcal{L}(X)\}$
- $\mathbf{R}(w) = \{b \in A | wb \in \mathcal{L}(X)\}$
- $\mathbf{E}(w) = \{(a, b) \in \mathbf{L}(w) \times \mathbf{R}(w) | awb \in \mathcal{L}(X)\}$

Definition ([BDFD+15])

The shift space X is *dendric* if $\mathbf{E}(w)$ is a tree for all w.

Definition

Examples

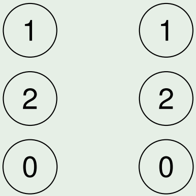
- *Sturmian* shifts, *Arnoux-Rauzy* shifts or RIE.
- The *Tribonacci shift*

$$\ldots 0102010010201010201001020102 0\ldots$$

is the shift generated by the fixed points of the morphism

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0.$$

Its extension graph $\mathbf{E}(0)$ is given by



Definition

Examples

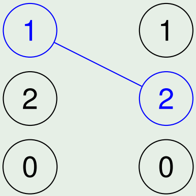
- *Sturmian* shifts, *Arnoux-Rauzy* shifts or RIE.
- The *Tribonacci shift*

$$\dots 0102010010201010201001020102 0\dots$$

is the shift generated by the fixed points of the morphism

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0.$$

Its extension graph $\mathbf{E}(0)$ is given by



Definition

Examples

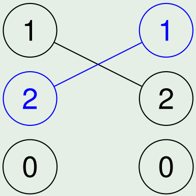
- *Sturmian* shifts, *Arnoux-Rauzy* shifts or RIE.
- The *Tribonacci shift*

$$\ldots 010201001020101020100102010 20 \ldots$$

is the shift generated by the fixed points of the morphism

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0.$$

Its extension graph $\mathbf{E}(0)$ is given by



Definition

Examples

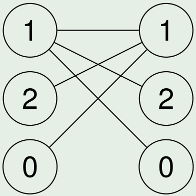
- *Sturmian* shifts, *Arnoux-Rauzy* shifts or RIE.
- The *Tribonacci shift*

$$\ldots 01020100102010102010010201020 \ldots$$

is the shift generated by the fixed points of the morphism

$$\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0.$$

Its extension graph $\mathbf{E}(0)$ is given by



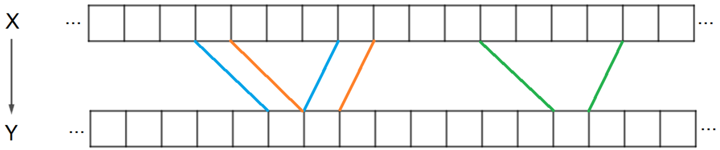
Dendric shifts and dynamical systems I

Definition

Let X, Y be shift spaces.
A map $\phi : X \to Y$ is a *morphism (or sliding block code)* if

- ϕ is continuous and
- $\phi \circ \sigma_A = \sigma_B \circ \phi$

A *conjugacy* is a bijective morphism.



Dendric shifts and dynamical systems II

Definition

The *k^{th} higher block code* is the map

$$(x_i)_{i\in\mathbb{Z}} \mapsto \left(\begin{pmatrix} x_i \\ x_{i+1} \\ \dots \\ x_{i+k-1} \end{pmatrix} \right)_{i\in\mathbb{Z}} .$$

Let $X^{(k)}$ denote the image by the k^{th} higher block code of X.

Dendric shifts and dynamical systems III

Example

For the Tribonnacci shift,

$$\ldots 0102010010201010201001020102 0\ldots$$

the fifth higher block code is

$$\ldots \begin{pmatrix}0\\1\\0\\2\\0\end{pmatrix} \begin{pmatrix}1\\0\\2\\0\\1\end{pmatrix} \begin{pmatrix}0\\2\\0\\1\\0\end{pmatrix} \begin{pmatrix}2\\0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\\1\end{pmatrix} \begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\\2\end{pmatrix} \begin{pmatrix}0\\1\\0\\2\\0\end{pmatrix} \ldots$$

Dendric shifts and dynamical systems IV

Proposition

Dendric shifts are not closed under conjugacy.

Proof.

If X is the Tribonacci shift, then $X^{(2)}$ is not dendric. This is seen by counting the number of connected components of $\mathbf{E}(\varepsilon)$. Indeed, in the extension graph, $\binom{0}{2}$ is connected to $\binom{2}{0}$ but cannot be connected to any other letter. □

Corollary

Being dendric is not a dynamical property.

Eventually dendric shifts

Definition ([DP21])

A shift space X is *eventually dendric* if $\exists n : |w| \geq n \Rightarrow \mathbf{E}(w)$ is a tree. The smallest n s.t. $|w| \geq n \Rightarrow \mathbf{E}(w)$ is a tree is called the threshold of X.

Remark

For any non-empty $w \in \mathcal{L}\left(X^{(n)}\right)$, $\mathbf{E}(w)$ is a tree.

Eventually dendric shifts

Definition ([DP21])

A shift space X is *eventually dendric* if $\exists n : |w| \geq n \Rightarrow \mathbf{E}(w)$ is a tree. The smallest n s.t. $|w| \geq n \Rightarrow \mathbf{E}(w)$ is a tree is called the threshold of X.

Remark

For any non-empty $w \in \mathcal{L}\left(X^{(n)}\right)$, $\mathbf{E}(w)$ is a tree.

Theorem ([DP21])

The class of eventually dendric shifts is closed under conjugacy.

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the purple factors.

$$\ldots 0102010010201010201001020102 0\ldots$$

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the coloured factors.

$$\ldots 0102010010201010201001020102 0\ldots$$

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the coloured factors.

$$\ldots 0102010010201010201001020102 0\ldots$$

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the coloured factors.

$$\ldots 0102010010201010 2010010201020 \ldots$$

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the coloured factors.

$$\ldots 0102010010201010201001020102 0\ldots$$

Return groups

Definition

The *return words of u* are the nonempty $v \in \mathcal{L}(X)$ such that $uv \in \mathcal{L}(X)$ in which u occurs exactly twice, once as a prefix and once as a suffix. The *return group of u* is the group generated by the return words of u.

Example

On the Tribonnacci shift, some return words of 010 are the coloured factors.

$$\ldots 0102010010201010201001020102 0\ldots$$

Return theorem

Theorem ([BDFD+15])

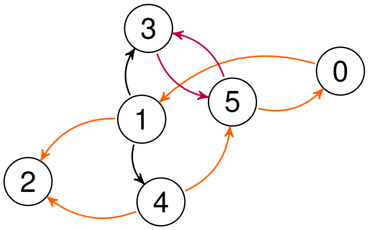
Let X be a minimal dendric shift. The return groups of X are equal to F_A.

Is there a version of this theorem that applies to eventually dendric shifts ?

Tools I

Definition

Let G be a di-graph and v one of its vertices.
The *fundamental group* $\pi_1(G, v)$ of the couple (G, v) is its set of loops endowed with concatenation.



Tools II

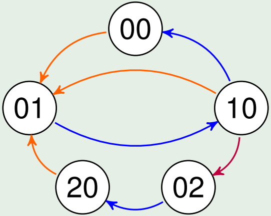
Definition

The *order-m Rauzy graph* G_m of X is the di-graph for which

- the vertices are the length-m factors and
- there is an edge labelled b between au and ub if aub is a factor.

Example

On the Tribonacci shift, G_2:

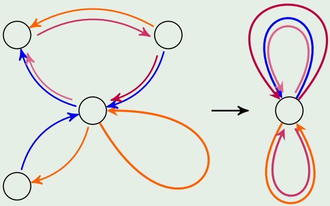


Tools III

Definition

Let u be a word. The *Rauzy group of u* is the projection of the fundamental group of $G_{|u|}$ with base point u on the free group F_A.

Example



Return theorem

Theorem ([BDFD$^+$15])

Let X be a minimal dendric shift.
The return groups of X are equal to F_A.

Sketch of the proof.

- Reduction of the problem to a study of Rauzy groups: because the return words of u form a basis of the Rauzy group of u.

Return theorem

Theorem ([BDFD$^+$15])

Let X be a minimal dendric shift.
The return groups of X are equal to F_A.

Sketch of the proof.

- Reduction of the problem to a study of Rauzy groups: because the return words of u form a basis of the Rauzy group of u.
- Combinatorial restrictions of dendric shifts $\Rightarrow \exists$ trivial path between two vertices having the same proper suffixes.

Return theorem

Theorem ([BDFD+15])

Let X be a minimal dendric shift.
The return groups of X are equal to F_A.

Sketch of the proof.

- Reduction of the problem to a study of Rauzy groups: because the return words of u form a basis of the Rauzy group of u.
- Combinatorial restrictions of dendric shifts $\Rightarrow \exists$ trivial path between two vertices having the same proper suffixes.
- Iterate the previous item to reduce the order of the Rauzy graph. Order 0: the graph is a bouquet of circles. □

The case of eventually dendric shifts

Theorem

Let X be a minimal eventually dendric shift with threshold N. The return groups of all factors u such that $|u| \geq N$ are conjugate of rank lower or equal to

$$p_X(N) - \sum_{v \in \mathcal{L}_{N-1}(X)} C(v) + 1,$$

where p_X is the factor complexity of X and $C(v)$ is the number of connected components of $\boldsymbol{E}_X(v)$.

The case of eventually dendric shifts

Sketch of the proof.

- The shift $X^{(N)}$ is eventually dendric shift of threshold 1.

The case of eventually dendric shifts

Sketch of the proof.

- The shift $X^{(N)}$ is eventually dendric shift of threshold 1.
- According to [GO22], these groups are free of rank $S - C + 1$, where S is the size of the alphabet of $X^{(N)}$ and C the number of connected components of $\mathbf{E}_{X^{(N)}}(\varepsilon)$.

The case of eventually dendric shifts

Sketch of the proof.

- The shift $X^{(N)}$ is eventually dendric shift of threshold 1.
- According to [GO22], these groups are free of rank $S - C + 1$, where S is the size of the alphabet of $X^{(N)}$ and C the number of connected components of $\mathbf{E}_{X^{(N)}}(\varepsilon)$.
- Clearly, $S = p_X(N)$. Less clearly,

$$C = \sum_{v \in \mathcal{L}_{N-1}(X)} C(v).$$

The case of eventually dendric shifts

Sketch of the proof.

- Since we can look at Rauzy groups or return groups equivalently, any folding occurring in a Rauzy graph of $X^{(N)}$ will occur in the Rauzy graph of X.

The case of eventually dendric shifts

Sketch of the proof.

- Since we can look at Rauzy groups or return groups equivalently, any folding occurring in a Rauzy graph of $X^{(N)}$ will occur in the Rauzy graph of X.
- The rank of the return groups of long enough factors of X is bounded by the announced value. □

Remark

The proof of the previous theorem holds for ks bigger than the threshold N. As such, the upper bound can be replaced by $p_X(k) - p_X(k-1) + 1$. This quantity is constant for eventually dendric shifts.

Let's compute some examples

Definition

The *Quadribonacci shift* is the shift defined by the morphism $\varphi : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 03$ and $3 \mapsto 0$.

$$\dots 0102010301020100102010301020101020103010 \dots$$

Example

If we apply the morphism $\psi : 0 \mapsto 0, 1 \mapsto 012, 2 \mapsto 022$ and $3 \mapsto 02$, we get

$$\dots 0012002200120020012002200120001200220012 \dots$$

- Rank of the return groups is 3.
- Rank of the return groups of the higher block codes is 4.

Let's compute some examples

Example

The shift space given by

$$\ldots 0012002200120020012002200120001200220012 \ldots$$

is such that the

- rank of the return groups is 3 and
- rank of the return groups of the higher block codes is 4.

Corollary

The rank of the return groups of eventually dendric shifts is not a dynamical invariant.

Questions

- Do the same techniques apply to eventually suffix-connected languages ? More generally, what restrictions of dendricity could be removed ?
- Can we say anything about the relative ranks of conjugate eventually dendric shifts ?
- How different are the two bound ?
- Can we find lower bounds to the ranks of return groups of eventually dendric shifts ?
- When fixing the $p_X(k+1) - p_X(k)$, are there any gaps in the ranks of the return groups that eventually dendric shifts can reach ?

Bibliography I

- *Office Hours with a Geometric Group Theorist*.
  Princeton University Press, 2017.
- Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique Perrin, Christophe Reutenauer, and Giuseppina Rindone.
  Acyclic, connected and tree sets.
  Monatshefte für Mathematik, 176(4), 2015.
- Francesco Dolce and Dominique Perrin.
  Eventually dendric shift spaces.
  Ergodic Theory Dyn. Syst., 41(7):2023–2048, 2021.
- Herman Goulet-Ouellet.
  Suffix-connected languages.
  Theoretical Computer Science, 2022.

Bibliography II

I. Kapovich and A. Myasnikov.
Stallings foldings and subgroups of free groups.
J. Algebra, 248(2):608–668, 2002.