Self-graviting systems in astrophysics: the equation of motion

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Galaxies are complex systems of a demanding dance between the action of gravity, rotation and collective phenomena. This qualitative approach offers a simple procedure in transforming the initial dynamical set of equations into purely kinematical equation of motion i.e. the evolution of galaxies are governed only by an initial setup.

Introduction

Galaxies are large systems of stars, gas and dust that are held together by gravity. They come in a wide variety of shapes and sizes, from small irregular dwarf galaxies to massive ellipticals and spirals. A widely used approach is to consider galaxies as an ideal collisionless fluid made of stars. Here, I derive a very general form of galactic equation of motion that describes the evolution of rotating galaxies derived within the framework of Newtonian fluid dynamics.

Equation of motion

We begin with the standard set of equations: the equation of continuity; the equation of motion and, and Poisson's equation for gravitational potential:

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}), \tag{1}$$

$$\dot{\mathbf{v}} = -\nabla \Phi - \frac{\nabla P}{\rho} - 2\Omega \times \mathbf{v} - \mathbf{r} \times \dot{\Omega} - \Omega \times (\Omega \times \mathbf{r}), \tag{2}$$

$$\nabla^2 \Phi = 4\pi G \rho, \tag{3}$$

where ρ is volume mass density, $\mathbf{r}$ is radius vector, $\mathbf{v} = \dot{\mathbf{r}}$ is velocity, Ω angular rotation, $\dot{\mathbf{v}}$ acceleration, $-2\Omega \times \mathbf{v}$ Coriolis acceleration, $-\Omega \times (\Omega \times \mathbf{r})$ centrifugal acceleration, Φ gravitational potential, P pressure and G gravitational constant. The integral quantities are total mass $\mathcal{M}$ and angular momentum J. The dot superscript indicates a total material derivative $\mathrm{d}/\mathrm{d}t$. At certain occasions, it is more convenient to write $\mathrm{d}/\mathrm{d}t$ when material derivative is applied to multiple functions. Vectors are denoted by bold characters.
As we can see, the starting set of equations belongs to the group of coupled nonlinear partial differential equations of higher order. Complexity of the system is self-consistency: large-scale mass distribution gives rise to a mean gravitational field that is coupled together with fictitious forces due to rotation. Thus the way galaxies rotate around their center departs significantly from the Keplerian rotation that is most intuitive from experience with the Solar system, planetary rings or accretion disks.

We transform the equation 1 in favor of a material derivative by eliminating the partial derivative after simple algebra and separating velocity and density terms; applying $\nabla\cdot$ to both sides of the equation 2 we eliminate the potential using the Poisson's equation. We obtain a system of two equations:

$$\frac{\dot{\rho}}{\rho} = \frac{\mathrm{d}}{\mathrm{dt}} \ln\rho = -\nabla\cdot\mathbf{v} \tag{4}$$

$$\nabla\cdot\left(\dot{\mathbf{v}} + 2\Omega\times\mathbf{v} + \mathbf{r}\times\dot{\Omega} + \Omega\times(\Omega\times\mathbf{r}) + \frac{\nabla P}{\rho}\right) = -4\pi G\rho. \tag{5}$$

Until this step, we have used no additional fluid nor geometry assumptions or simplifications. The previous two equations can be further simplified by assuming a barotropic fluid if velocity dispersion is significant. In the case of an ideal collisionless fluid interacting only through gravity, this term is often neglected[1]. We obtain the following system of equations which relates density with acceleration and velocity.

$$\nabla\cdot\mathbf{v} = -\frac{\mathrm{d}}{\mathrm{dt}} \ln\rho \tag{6}$$

$$\nabla\cdot\left(\dot{\mathbf{v}} + 2\Omega\times\mathbf{v} + \mathbf{r}\times\dot{\Omega} + \Omega\times(\Omega\times\mathbf{r})\right) = -4\pi G\rho. \tag{7}$$

The last system of two equations could be considered as a fundamental set describing the relation between the velocity, density, acceleration, and rotation. Moreover, it is possible to further transform the equations by eliminating density (we first apply ln to the equation 7 followed by d/dt), which yields the fundamental equation of motion (EoM) for collisionless, self-graviting systems supported by rotation:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\ln\left(\nabla\cdot\left(\dot{\mathbf{v}} + 2\Omega\times\mathbf{v} + \mathbf{r}\times\dot{\Omega} + \Omega\times(\Omega\times\mathbf{r})\right)\right)\right) + \nabla\cdot\mathbf{v} = 0. \tag{8}$$

Conclusion

Our initial assumptions are quite minimum: Newtonian fluid dynamics holds for systems not to large in terms of cosmological scales, but rather of regular galaxy sizes. Derived EoM supports the dynamics of rotating galaxies for given the initial and boundary conditions. The initial set of equations reduces counter-intuitively to a purely kinematical equation. An intriguing consequence is that besides the initial setup and boundary conditions, total mass itself therefore plays no obvious role in determining the evolution of the galaxy, however, the role of mass distribution is crucial. It seems that initial mass is just redistributed during the galaxy evolution influenced by rotation and some outer forces or perturbations.

[1] There is no collisional pressure and no equivalent equation of state for an ideal collisionless fluid that interacts via gravitational force only.