q-deformations of binomial coefficients of words

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Classical coefficients

The ***binomial coefficient*** $\binom{u}{v}$ of two words u, v counts the number of occurrences of v as a scattered subword of u.

Example:

$$\binom{abbab}{ab} = 4 \qquad abbab \quad abbab \quad abbab \quad abbab$$

The ***Gaussian binomial coefficient*** $\binom{m}{r}_q$ of two positive integers m, r is

$$\binom{m}{r}_q = \frac{(1-q^m)\cdots(1-q^{m-r+1})}{(1-q^r)\cdots(1-q)}.$$

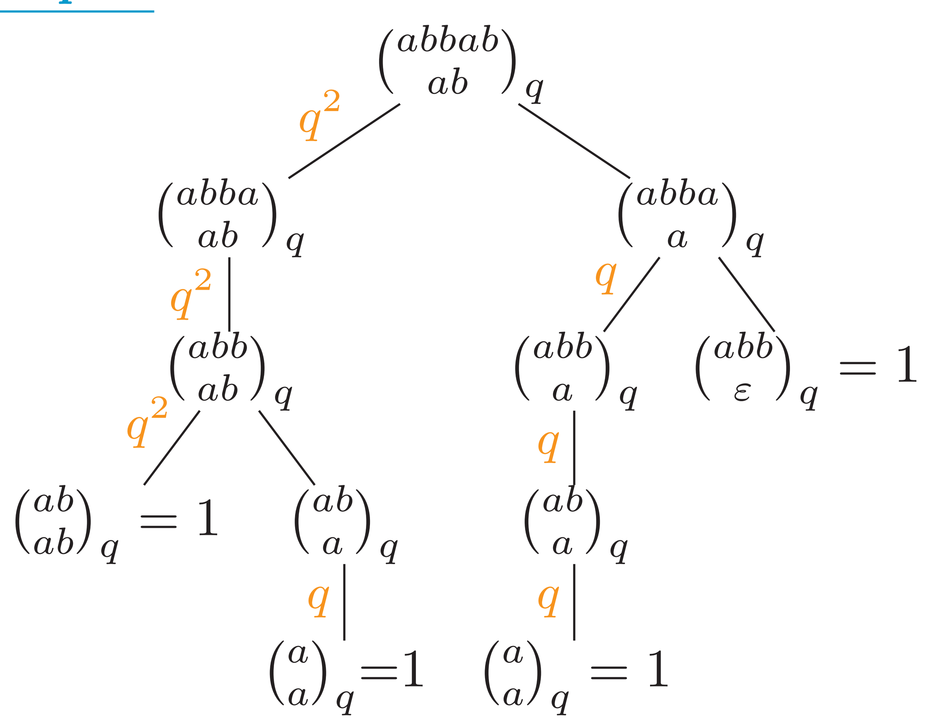
q-deformed coefficients of words

The ***q-deformation*** $\binom{\cdot}{\cdot}_q$ of binomial coefficients of words is a polynomial in $\mathbb{N}[q]$ defined as follows: for all $u, v \in A^*$ and $a, b \in A$,

$$\binom{u}{\varepsilon}_q = 1, \quad \binom{\varepsilon}{v}_q = 0 \text{ if } v \neq \varepsilon,$$

$$\binom{ua}{vb}_q = \binom{u}{vb}_q \cdot q^{|vb|} + \delta_{a,b}\binom{u}{v}_q.$$

Example:



$$\longrightarrow \binom{abbab}{ab}_q = q^6 + q^5 + q^3 + 1$$

Combinatorial interpretation

1. Main theorem

Let $u = u_n \cdots u_1$ and $v = v_k \cdots v_1$ be words.

$$\binom{u}{v}_q = \sum_{Y \in A_{n,k}} q^{s(Y) - \frac{k(k+1)}{2}} \quad \text{where} \quad A_{n,k} = \{n \geq y_k \geq \cdots \geq y_1 \geq 1 | u_{y_k} \cdots u_{y_1} = v\}, \; s(Y) = \textstyle\sum_{y \in Y} y.$$

Example: The occurrences of ab in $abbab$ give

Y	$(5,4)$	$(5,3)$	$(5,1)$	$(2,1)$
$s(Y)-3$	6	5	3	0

2. Alternative interpretation

Each occurrence of v in u contributes to $\binom{u}{v}_q$ with a term q^α, where α is the sum over all letters of v of the number of letters at the right of them and that are not part of this particular occurrence of v.

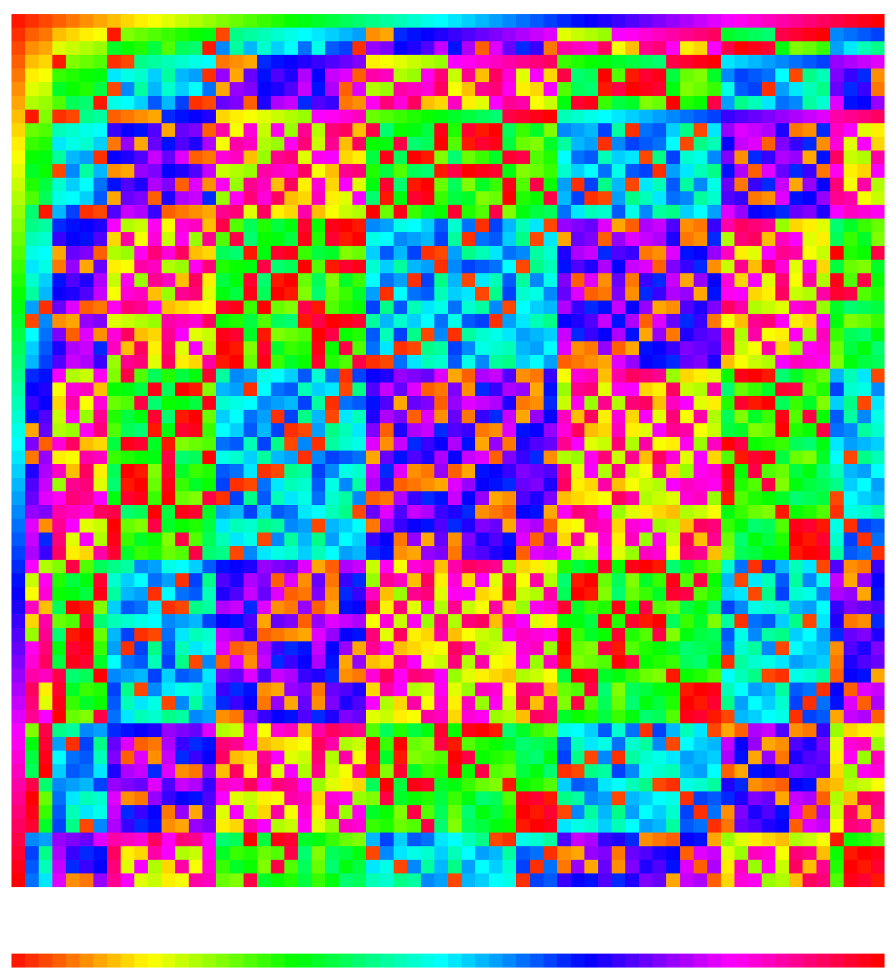
Example:

$$\underset{q^{3+3}}{abbab} + \underset{q^{2+3}}{abbab} + \underset{q^{0+3}}{abbab} + \underset{q^{0+0}}{abbab} = \binom{abbab}{ab}_q$$

3. Information within the coefficients

In particular, we have

$$\binom{u}{v}_q(0) = \begin{cases} 1, & \text{if } v \text{ is a suffix of } u, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \binom{u}{v}_q\left[q^{|v|(|u|-|v|)}\right] = \begin{cases} 1, & \text{if } v \text{ is a prefix of } u, \\ 0, & \text{otherwise.} \end{cases}$$



Multiplication table of $\{a,b\}^*/\equiv_{ab,q^2+1}$

Classical formulas revisited

Sums over all words of a fixed length

For $u \in A^*$ and $n \geq 1$,

$$\sum_{v \in A^n} \binom{u}{v}_q = \binom{|u|}{n}_q.$$

For $v \in A^*$ and $n \geq |v|$,

$$\sum_{u \in A^n} \binom{u}{v}_q = (\#A)^{n-|v|}\binom{n}{|v|}_q.$$

q-Vandermonde formula

For $u, x, y \in A^*$,

$$\binom{xy}{u}_q = \sum_{\substack{u = u_1 u_2 \\ u_1, u_2 \in A^*}} q^{|u_1|(|y|-|u_2|)} \binom{x}{u_1}_q \binom{y}{u_2}_q.$$

p-group languages

A language $L \subset A^*$ is ***regular*** if and only if there exist

a finite monoid M, a subset $S \subset M$ and a monoid morphism $\varphi : A^* \to M$,

such that $L = \varphi^{-1}(S)$. A language recognized by a p-group is a ***p-group language***.

Idea: Find a congruence $\equiv$ so that $A^*/\equiv$ is a finite monoid, and thus have a regular language.

Let p be a prime and $\mathfrak{M}$ be a polynomial in $\mathbb{F}_p[q]$. For $u \in A^*$, we define the ***equivalence relation***

$$w_1 \sim_{u,\mathfrak{M}} w_2 \Leftrightarrow \forall v \in \mathrm{Fac}(u) : \binom{w_1}{v}_q \equiv \binom{w_2}{v}_q \pmod{\mathfrak{M}}.$$

Problem: this is not always a congruence $\to$ refinement $\equiv_{u,\mathfrak{M}}$ defined as $\sim_{u,\mathfrak{M}} \cap \sim_{\mathrm{ord}(q)}$, where

$$w_1 \sim_{\mathrm{ord}(q)} w_2 \Leftrightarrow |w_1| \equiv |w_2| \pmod{\mathrm{ord}(q)}.$$

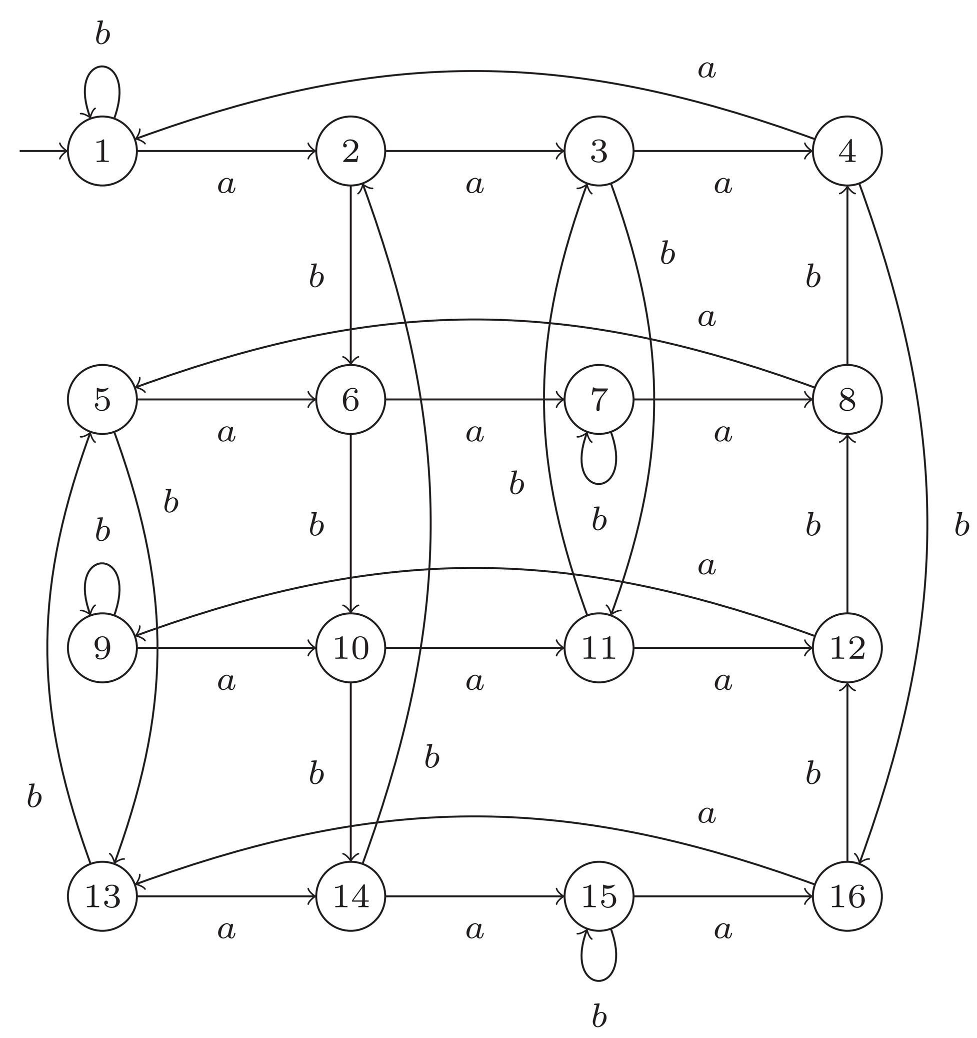
Two cases:

q is not a unit in $\mathbb{F}_p[q]/\langle\mathfrak{M}\rangle$	q is a unit in $\mathbb{F}_p[q]/\langle\mathfrak{M}\rangle$
$\to$ $A^*/\equiv_{u,\mathfrak{M}}$ is not a group	$\to$ $A^*/\equiv_{u,\mathfrak{M}}$ is a group
$\to$ no element is a unit (except for $[\varepsilon]$)	$\to$ its order divides $\mathrm{ord}(q) \cdot p^{\lvert u\rvert}$

Thanks to this, we get a new version of Eilenberg's theorem characterizing p-group languages.

Theorem. A language is a p-group if and only if it is a Boolean combination of languages of the form

$$L_{v,\mathfrak{R},p} = \{u \in A^* : \binom{u}{v}_q \equiv \mathfrak{R} \pmod{q^p - 1}\}.$$



Minimal automaton of $L_{ab,\mathfrak{R},2}$

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