Dendric words and morphisms

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Introduction

Words

Concept	**Examples**
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The *length* of a word w is $\|w\|$.	$\|\varepsilon\| = 0$, $\|010\| = 3$
A *bi-infinite word* $(x, y, \dots)$ is an element of $\mathcal{A}^{\mathbb{Z}}$	$\cdots 011100.010 \cdots$, ${}^{\omega}(010).(010)^{\omega}$

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Examples:

- ε, 10 and 010 are factors of 01001
- $\varepsilon, 0, 1, 00, 01, 10, 11, \ldots$ are in $\mathcal{L}(\cdots 011100.010 \cdots)$
- ε and 010 are also prefixes of 01001 and ε is a suffix

Left and right extensions

$$\cdots 011100.010110101101 \cdots$$

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$$\cdots 01110 0.010110101101 \cdots$$

Left and right extensions

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Definition

Let $x \in \mathcal{A}^{\mathbb{Z}}$ and $w \in \mathcal{L}(x)$.

$E_x^L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(x)\}$

Example:

$$E^L(110) = \{1, 0\}$$

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$$E_x(w) = \{(a,b) \in E_x^L(w) \times E_x^R(w) \mid awb \in \mathcal{L}(x)\}$$

Example:

$$E^L(110) = \{1,0\}, \quad E^R(110) = \{0,1\},$$

$$E(110) \supseteq \{(1,0),(0,1)\}$$

Extension graph

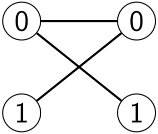
Definition

The *extension graph* of $w \in \mathcal{L}(x)$ is the bipartite graph $\mathcal{E}_x(w)$ with vertices $E^L_x(w) \sqcup E^R_x(w)$ and edges $E_x(w)$.

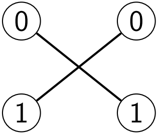
Example:

$$^{\omega}(010).(010)^{\omega}$$

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(0)$



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Definition (Dolce, Perrin '19)

A bi-infinite word x is *eventually* *dendric* if all its long enough factors are dendric.

Examples:
The words $\cdots 011100.010110101101 \cdots$ and ${}^{\omega}(010).(010)^{\omega}$ are not dendric.
But ${}^{\omega}(010).(010)^{\omega}$ is eventually dendric.

First restriction

For the empty word:

- left extensions = all the letters
- right extensions = all the letters
- bi-extensions = the factors of length 2

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So if ε is dendric,

$$\#(\mathcal{L}(x) \cap \mathcal{A}^2) = 2 \times \#\mathcal{A} - 1$$

or, in other words,

$$\#(\mathcal{L}(x) \cap \mathcal{A}^2) - \#(\mathcal{L}(x) \cap \mathcal{A}^1) = \#(\mathcal{L}(x) \cap \mathcal{A}^1) - \#(\mathcal{L}(x) \cap \mathcal{A}^0).$$

Factor complexity

The factor complexity of $x \in \mathcal{A}^{\mathbb{Z}}$ is the function

$$p_x(n) : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \#(\mathcal{L}(x) \cap \mathcal{A}^n).$$

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Proposition (Berthé *et al.*, Dolce, Perrin)

If $x \in \mathcal{A}^{\mathbb{Z}}$ is dendric, then

$$p_x(n) = (\#\mathcal{A} - 1)n + 1.$$

If $x \in \mathcal{A}^{\mathbb{Z}}$ is eventually dendric, then, for all large enough n,

$$p_x(n) = Sn + C.$$

Definition

Definition

A *morphism* is a monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, i.e. for any $u, v \in \mathcal{A}^*$,

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Example:

$$\sigma : \{0, 1, 2\}^* \to \{0, 1\}^*, \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

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$$\sigma(\cdots 1.20 \cdots) = \cdots 10.0\,001 \cdots$$

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Assumptions: the **image alphabet is minimal** and the morphism is **non erasing**.

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- If x is dendric, what can we say about the **factor complexity of** $\sigma(x)$?
  If x is eventually dendric, what can we say about the **factor complexity of** $\sigma(x)$?
- If x is dendric, under what conditions is $\sigma(x)$ **dendric**?

Factor complexity

Intuition

$$\sigma : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

$$\begin{aligned} x : &\quad \ldots 2.001210 \ldots \\ \sigma(x) : &\quad \ldots 0.001\ 001\ 10\ 0\ 10\ 001 \ldots \end{aligned}$$

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A *covering* of $u \in \mathcal{B}^n$ is a pair $(w, k) \in \mathcal{L}(x) \times \mathbb{Z}_{\geq 0}$ where $u = \sigma(w)_{[k+1,k+n]}$ and w is minimal, i.e.

$$k + 1 \leq |\sigma(w_1)| \quad \text{and} \quad k + n \geq \left|\sigma(w_{[1,|w|[})\right| + 1$$

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Proposition

If the set of coverings of words of length n is denoted $C_{x,\sigma}(n)$, we have

$$p_{\sigma(x)}(n) \leq \#C_{x,\sigma}(n).$$

Link between $C_{x,\sigma}(n)$ and $C_{x,\sigma}(n+1)$

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- $(00, 0)$ is a covering of 0010

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- $(00, 0)$ is a covering of 0010 and of 00100

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- $(00, 0)$ is a covering of 0010 and of 00100
- $(121, 1)$ is a covering of 0010 but $(121, 1) \notin C_{x,\sigma}(5)$

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- $(00, 0)$ is a covering of 0010 and of 00100
- $(121, 1)$ is a covering of 0010 but $(121, 1) \notin C_{x,\sigma}(5)$
- $(1210, 1)$ is a covering of 00100

We have

$$\#C_{x,\sigma}(n+1) - \#C_{x,\sigma}(n) = \sum_{w \in W_n} (\#E_x^R(w) - 1)$$

where $W_n = \{w \in \mathcal{L}(x) \mid |\sigma(w_{[2,|w|]})| < n \leq |\sigma(w)|\}$.

Number of coverings

Proposition

- *If $x \in \mathcal{A}^{\mathbb{Z}}$ is eventually dendric, then there exists $C \in \mathbb{Z}$ such that, for all n large enough,*

$$\#C_{x,\sigma}(n) = p_x(n) + C.$$

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- *If $x \in \mathcal{A}^{\mathbb{Z}}$ is dendric, then, for all $n \geq 1$,*

$$\#C_{x,\sigma}(n) = \sum_{a \in \mathcal{A}} |\sigma(a)| + (\#\mathcal{A} - 1)(n - 1).$$

Factor complexity and alphabet sizes

Theorem

If x is eventually dendric and σ is non-erasing, then

$$p_{\sigma(x)}(n) \leq p_x(n) + C$$

for some $C \in \mathbb{N}$.

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Corollary

If $x \in \mathcal{A}^{\mathbb{Z}}$ and $\sigma(x) \in \mathcal{B}^{\mathbb{Z}}$ are dendric, then $\#\mathcal{B} \leq \#\mathcal{A}$.

Preserving dendricity

Unary alphabets

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  $\longrightarrow$ **always dendric**

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- If $\mathcal{A} = \{a\}$ and $\sigma(a) = v$, then $\sigma(x) = {}^{\omega}v.v^{\omega}$
  $\longrightarrow$ **dendric iff** $\#\mathcal{B} = 1$

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  $\longrightarrow$ **dendric iff** $\#\mathcal{B} = 1$

From now on, we assume that the alphabets are of size at least 2.

Definition

In general, it is difficult to know if $\sigma(x)$ is dendric, even if we know that x is dendric.

Definition

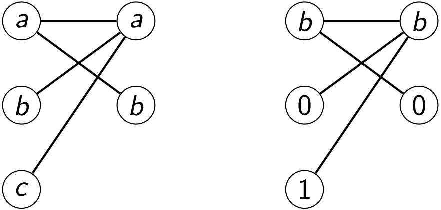
A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is *dendric preserving* if, for all dendric $x \in \mathcal{A}^{\mathbb{Z}}$, $\sigma(x)$ is dendric.

What are the dendric preserving morphisms?

Bijective codings

$$\sigma : \{a, b, c\}^* \to \{b, 0, 1\}^*, \quad \begin{cases} a \mapsto b \\ b \mapsto 0 \\ c \mapsto 1 \end{cases}$$

$\mathcal{E}_x(ba)$ $\qquad$ $\mathcal{E}_{\sigma(x)}(0b)$



Arnoux-Rauzy morphisms

The *Arnoux-Rauzy morphisms* are defined by

$$\alpha_\ell^L : \begin{cases} \ell \mapsto \ell \\ a \mapsto \ell a & \text{if } a \neq \ell \end{cases} \qquad \alpha_\ell^R : \begin{cases} \ell \mapsto \ell \\ a \mapsto a\ell & \text{if } a \neq \ell \end{cases}$$

for any letter ℓ.

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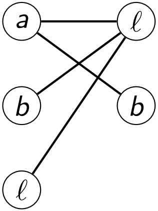
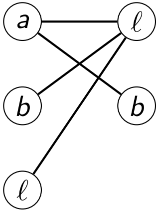


$\mathcal{E}_{\alpha_\ell^L(x)}(\ell b \ell a \ell)$



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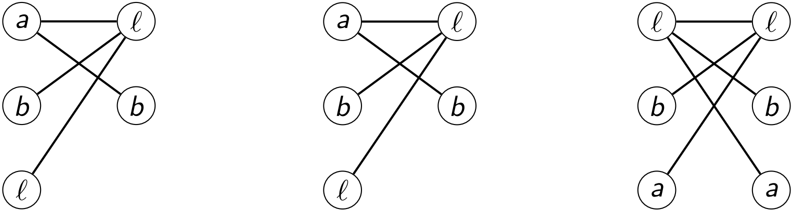
$\mathcal{E}_x(ba)$ $\mathcal{E}_{\alpha_\ell^L(x)}(\ell b \ell a \ell)$ $\mathcal{E}_{\alpha_\ell^L(x)}(\varepsilon)$



First result

Proposition

If σ is a bijective coding or an Arnoux-Rauzy morphism, then x is dendric if and only if $\sigma(x)$ is dendric.

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Corollary

If σ is as above, for any morphism τ, τ is dendric preserving if and only if $\sigma \circ \tau$ is dendric preserving.

Properties of bijective codings and AR morphisms

$$\sigma : \begin{cases} a \mapsto b \\ b \mapsto 0 \\ c \mapsto 1 \end{cases} \qquad \sigma : \begin{cases} a \mapsto a \\ b \mapsto ab \\ c \mapsto ac \end{cases} \qquad \sigma : \begin{cases} a \mapsto ab \\ b \mapsto b \\ c \mapsto cb \end{cases}$$

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$$\sigma : \begin{cases} a \mapsto b \\ b \mapsto 0 \\ c \mapsto 1 \end{cases} \qquad \sigma : \begin{cases} a \mapsto aa \\ b \mapsto aba \\ c \mapsto aca \end{cases} \qquad \sigma : \begin{cases} a \mapsto ab \\ b \mapsto b \\ c \mapsto cb \end{cases}$$

If p_σ is the longuest common prefix of all $\sigma(\ell)p_\sigma$, $\ell \in \mathcal{A}$, then the letters that follow it in $\sigma(\ell)p_\sigma$ are different for all ℓ.

Properties of bijective codings and AR morphisms

$$\sigma : \begin{cases} a \mapsto b \\ b \mapsto 0 \\ c \mapsto 1 \end{cases} \qquad \sigma : \begin{cases} a \mapsto a \\ b \mapsto ab \\ c \mapsto ac \end{cases} \qquad \sigma : \begin{cases} a \mapsto ab \\ b \mapsto b \\ c \mapsto cb \end{cases}$$

If p_σ is the longuest common prefix of all $\sigma(\ell)p_\sigma$, $\ell \in \mathcal{A}$, then the letters that follow it in $\sigma(\ell)p_\sigma$ are different for all ℓ.

We have a similar result with suffixes.

First result on dendric preserving morphisms

Lemma

If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, for each $a \in \mathcal{A}$, the letter b such that $p_\sigma b$ is a prefix of $\sigma(a)p_\sigma$ is different.

Corollary

If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, then $\#\mathcal{A} = \#\mathcal{B}$.

We have a similar result with suffixes.

Induction on $|s_\sigma p_\sigma|$

Lemma

If σ is dendric preserving and $s_\sigma p_\sigma = \varepsilon$, then σ is a bijective coding.

Lemma

If σ is dendric preserving and $|s_\sigma p_\sigma| > 0$, then there exists a morphism τ such that $\sigma \in \{\alpha_\ell^L \circ \tau, \alpha_\ell^R \circ \tau\}$ and $|s_\tau p_\tau| < |s_\sigma p_\sigma|$.

Induction on $|s_\sigma p_\sigma|$

Lemma

If σ is dendric preserving and $s_\sigma p_\sigma = \varepsilon$, then σ is a bijective coding.

Lemma

If σ is dendric preserving and $|s_\sigma p_\sigma| > 0$, then there exists a morphism τ such that $\sigma \in \{\alpha_\ell^L \circ \tau, \alpha_\ell^R \circ \tau\}$ and $|s_\tau p_\tau| < |s_\sigma p_\sigma|$.

Ideas of the proofs:
If it is not the case, we can build dendric words whose images are not dendric.

Characterization of dendric preserving morphisms

Proposition

A morphism is dendric preserving if and only if

- *the image alphabet is of size 1*
- *or it is, up to a bijective coding, in the monoid generated by the Arnoux-Rauzy morphisms.*

Conclusion

Open questions

- Can we characterize when the image of a dendric word x under a morphism σ is dendric?
- Is the image of an eventually dendric word always eventually dendric?

Thank you for your attention!