Morphic images of (eventually) dendric words

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Given a bi-infinite word x, we can associate to each factor w its extension graph in x. This graph is a bipartite graph describing the letters that we can see to the left and to the right of w in x. If this graph is a tree, then we say that w is dendric, and if all the factors of x are dendric, we say that x itself is dendric.

The family of bi-infinite dendric words generalizes well-known families of words such as the Sturmian words, the Arnoux-Rauzy words and the codings of regular interval exchanges. These words were introduced via their languages under the terminology of tree sets in a series of papers [2, 3, 4]. In these papers and in the ones that followed [1, 5], recurrent dendric words have been shown to possess several interesting combinatorial, algebraic and dynamical properties.

Dendric bi-infinite words were later generalized in [7] with the notion of eventually dendric words which are the bi-infinite words such that all their long enough factors are dendric. Some of the properties of dendric words can be extended to this new family.

In this talk, I will focus on the link between dendricity and morphisms. Not much is known about images of dendric and eventually dendric words. For example, only a partial answer to the following question is known (see [8]): given a dendric bi-infinite word x and a morphism σ, under which conditions is the image $\sigma(x)$ dendric? One could also wonder what restrictions on the structure of the images of (eventually) dendric words can be obtained.

As a first approach to these broader problems, we can tackle more specific questions. In this talk, I will consider two of these.

The first one revolves around the classical factor complexity as it is often the first step when studying the structure of a word. Due to the specific structure of its language, we can easily prove that the factor complexity of a bi-infinite dendric word x is given by

$$p_n(x) = (k-1)n + 1 \tag{1}$$

where k is the number of different letters appearing in the word. While there is no exact formula for the factor complexity of an eventually dendric word over a given alphabet, it is still possible to prove that, eventually, it is linear and the main coefficient is closely related to the structure of the language.

In [6], the authors proved that when applying a non-erasing morphism σ, the factor complexity of the image is restricted by that of the initial word. More specifically, factor complexity grows at most by a multiplicative constant given by the morphism, i.e. there exists $C_\sigma \in \mathbb{N}$ such that $p_n(\sigma(x)) \leq C_\sigma p_n(x)$ for all words x. Using basic combinatorics on words tools, we can improve this result in the case where the initial word is eventually

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dendric and show that the factor complexity grows at most by an additive constant, which depends on the initial word and on the morphism.

The second focus of this talk is the dendric morphisms, i.e. the morphisms such that the image of a dendric word is always dendric. It is well known that Sturmian words or, more generally, Arnoux-Rauzy words can be defined using a finite set of morphisms called the Arnoux-Rauzy morphisms. Moreover, a morphism preserves the Arnoux-Rauzy words, i.e. the image of an Arnoux-Rauzy word is an Arnoux-Rauzy word, if and only if it is generated by the Arnoux-Rauzy morphisms. As dendric words generalize Arnoux-Rauzy words, it is quite natural to ask whether there exists such a family of morphisms for dendric words. It turns out that the dendric morphisms are also exactly those generated by the Arnoux-Rauzy morphisms, as shown in this talk.

References

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