Morphic images of (eventually) dendric words

France Gheeraert

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Introduction

Left and right extensions

Let $x \in \mathcal{A}^{\mathbb{Z}}$ and $w \in \mathcal{L}(x)$.

$$E_x^L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(x)\}, \quad E_x^R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}(x)\},$$

$$E_x(w) = \{(a, b) \in E_x^L(w) \times E_x^R(w) \mid awb \in \mathcal{L}(x)\}$$

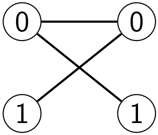
Definition

The *extension graph* of $w \in \mathcal{L}(x)$ is the bipartite graph $\mathcal{E}_x(w)$ with vertices $E_x^L(w) \sqcup E_x^R(w)$ and edges $E_x(w)$.

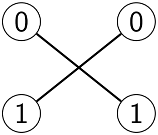
Example:

$$^{\omega}(010).(010)^{\omega}$$

$\mathcal{E}(\varepsilon)$



$\mathcal{E}(0)$



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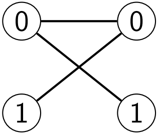

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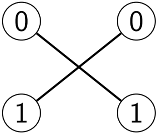

Definition (Berthé *et al.* '15)

A word $w \in \mathcal{L}(x)$ is *dendric* (in x) if $\mathcal{E}_x(w)$ is a tree.

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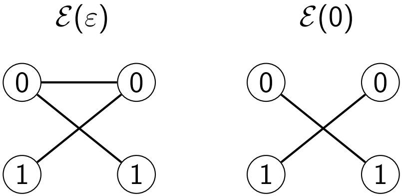

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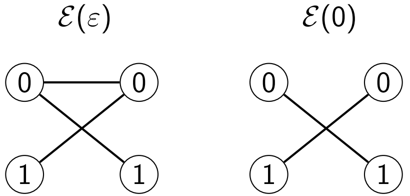

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A bi-infinite word x is *dendric* if all its factors are dendric.
A bi-infinite word x is *eventually dendric* if all its long enough factors are dendric.

Factor complexity

The factor complexity of $x \in \mathcal{A}^{\mathbb{Z}}$ is the function

$$p_x(n) : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \#\mathcal{L}(x) \cap \mathcal{A}^n.$$

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Proposition

If $x \in \mathcal{A}^{\mathbb{Z}}$ is dendric, then

$$p_x(n) = (\#\mathcal{A} - 1)n + 1.$$

If $x \in \mathcal{A}^{\mathbb{Z}}$ is eventually dendric, then, for all large enough n,

$$p_x(n) = Sn + C.$$

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A *morphism* is a monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, i.e. for any $u, v \in \mathcal{A}^*$,

$$\sigma(uv) = \sigma(u)\sigma(v).$$

Assumptions: the **image alphabet is minimal** and the morphism is **non erasing**.

$$\sigma : \{0,1,2\}^* \to \{0,1\}^*, \quad \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

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- What are the morphisms σ that **preserve dendricity**, i.e. if x is dendric, then $\sigma(x)$ is dendric?
  Such a morphism is called *dendric preserving*.

Factor complexity

Intuition

$$\sigma : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

$$\begin{aligned} x : &\quad \ldots 2.001210 \ldots \\ \sigma(x) : &\quad \ldots 0.001\ 001\ 10\ 0\ 10\ 001 \ldots \end{aligned}$$

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A *covering* of $u \in \mathcal{B}^n$ is a pair $(w, k) \in \mathcal{L}(x) \times \mathbb{Z}_{\geq 0}$ where $u = \sigma(w)_{[k+1,k+n]}$ and w is minimal, i.e.

$$k + 1 \leq |\sigma(w_1)| \quad \text{and} \quad k + n \geq \left|\sigma(w_{[1,|w|[})\right| + 1$$

The set of coverings of words of length n is denoted $C_{x,\sigma}(n)$.

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Proposition

We have

$$p_{\sigma(x)}(n) \leq \#C_{x,\sigma}(n).$$

Link between $C_{x,\sigma}(n)$ and $C_{x,\sigma}(n+1)$

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- $(121, 1)$ is a covering of 0010 but $(121, 1) \notin C_{x,\sigma}(5)$
- $(1210, 1)$ is a covering of 00100

We have

$$\# C_{x,\sigma}(n+1) - \# C_{x,\sigma}(n) = \sum_{w \in W_n} (\# E_x^R(w) - 1)$$

where $W_n = \{w \in \mathcal{L}(x) \mid |\sigma(w_{[2,|w|]})| < n \leq |\sigma(w)|\}$.

Number of coverings

Proposition

1. *If $x \in \mathcal{A}^{\mathbb{Z}}$ is eventually dendric, then there exists $C \in \mathbb{Z}$ such that, for all n large enough,*

$$\# C_{x,\sigma}(n) = p_x(n) + C.$$

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2. *If $x \in \mathcal{A}^{\mathbb{Z}}$ is dendric, then, for all $n \geq 1$,*

$$\#C_{x,\sigma}(n) = \sum_{a \in \mathcal{A}} |\sigma(a)| + (\#\mathcal{A} - 1)(n - 1).$$

Factor complexity and alphabet sizes

Theorem

If x is eventually dendric and σ is non-erasing, then

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for some $C \in \mathbb{N}$.

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Corollary

If $x \in \mathcal{A}^{\mathbb{Z}}$ and $\sigma(x) \in \mathcal{B}^{\mathbb{Z}}$ are dendric, then $\#\mathcal{B} \leq \#\mathcal{A}$.

Dendric preserving morphisms

Unary alphabets

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  $\longrightarrow$ **always dendric**

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- If $\mathcal{A} = \{a\}$ and $\sigma(a) = v$, then $\sigma(x) = {}^{\omega}v.v^{\omega}$
  $\longrightarrow$ **dendric iff $\#\mathcal{B} = 1$**

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  $\longrightarrow$ **dendric iff** $\#\mathcal{B} = 1$

From now on, we assume that the alphabets are of size at least 2.

Bijective codings and Arnoux-Rauzy morphisms

The *bijective codings*, i.e. the bijections between alphabets, are dendric preserving.

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If σ is a bijective coding, then x is dendric iff $\sigma(x)$ is dendric.

In particular, for any morphism τ, τ is dendric preserving iff $\sigma \circ \tau$ is dendric preserving.

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The *Arnoux-Rauzy morphisms* are defined by

$$L_\ell : \begin{cases} \ell \mapsto \ell \\ a \mapsto \ell a \quad \text{if } a \neq \ell \end{cases} \qquad R_\ell : \begin{cases} \ell \mapsto \ell \\ a \mapsto a\ell \quad \text{if } a \neq \ell \end{cases}$$

for any letter ℓ.

Bijective codings and Arnoux-Rauzy morphisms

The *bijective codings*, i.e. the bijections between alphabets, are dendric preserving.

Proposition

If σ is a bijective coding or an Arnoux-Rauzy morphism, then x is dendric iff $\sigma(x)$ is dendric.
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Definition of p_σ

If σ is aperiodic and non-erasing, p_σ is the longest word p such that p is a prefix of $\sigma(a)p$ for all $a \in \mathcal{A}$.

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Example:

$$\sigma : \begin{cases} 0 \mapsto 02010 \\ 1 \mapsto 0200 \\ 2 \mapsto 02 \end{cases} \qquad p_\sigma =$$

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Similarly, we can define s_σ using suffixes.

First result on dendric preserving morphisms

Proposition

If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, for each $a \in \mathcal{A}$, the letter b such that $p_\sigma b$ is a prefix of $\sigma(a)p_\sigma$ is different.

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If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, then $\#\mathcal{A} = \#\mathcal{B}$.

We have a similar result for s_σ.

Induction on $|s_\sigma p_\sigma|$

Lemma

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Lemma

If σ is dendric preserving and $|s_\sigma p_\sigma| = n > 0$, then

1. $(s_\sigma p_\sigma)_1 = (s_\sigma p_\sigma)_n =: \ell$ *and it is such that, for any* x,
$$E_{\sigma(x)}(\varepsilon) = (\ell \times \mathcal{B}) \cup (\mathcal{B} \times \ell);$$
2. *there exists a morphism* τ *such that* $\sigma \in \{L_\ell \circ \tau, R_\ell \circ \tau\}$ *and* $|s_\tau p_\tau| < |s_\sigma p_\sigma|$.

Characterization of dendric preserving morphisms

Proposition

A morphism is dendric preserving if and only if

- *the image alphabet is of size 1*
- *or it is, up to a bijective coding, in the monoid generated by the Arnoux-Rauzy morphisms.*

Conclusion

Open questions

1. Is the image of an eventually dendric word always eventually dendric?
2. Can we characterize when the image of a dendric word x under a morphism σ is dendric?

Thank you for your attention!