Alternate bases where all rational numbers have periodic representations.

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Representations in base β

For a number $\alpha \in [0, 1[$, a sequence $(a_i)_{i \in \mathbb{N}}$ such that

$val_\beta((a_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = \alpha$ is called a β-representation of α.

It is called the β-expansion of α if furthermore $\frac{1}{\beta^j} > \sum_{i=j+1}^{\infty} \frac{a_i}{\beta^i} \forall j$.

A way to obtain it : define

$$a_0 = 0, \rho^{(0)} = \alpha, a_{i+1} = \lfloor \beta \rho^{(i)} \rfloor, \rho^{(i+1)} = \{\beta \rho^{(i)}\} = T_\beta(\rho^{(i)}).$$

Some basic properties

We have

$$\beta\alpha = \lfloor\beta\alpha\rfloor + \{\beta\alpha\} = a_1 + \rho^{(1)}$$

$$\beta^2\alpha = \beta a_1 + \lfloor\beta\rho^{(1)}\rfloor + \{\beta\rho^{(1)}\} = \beta a_1 + a_2 + \rho^{(2)}$$

$$\ldots$$

$$\beta^n\alpha = \beta^{n-1} a_1 + \ldots + \beta a_{n-1} + a_n + \rho^{(n)}$$

Hence $(a_i)_{i\in\mathbb{N}}$ is indeed a representation of α . We have

$$\rho^{(n)} = \beta^n(\alpha - \sum_{i=1}^{n} \frac{a_i}{\beta^i}),$$

$$\rho^{(n)} = val_\beta((a_{i+n})_{i\in\mathbb{N}})$$

and

$$\rho^{(n+1)} = \beta\rho^{(n)} - a_{n+1}.$$

An example

Take $\beta = \sqrt{2}, \alpha = 2/3$. We have

i	$\sqrt{2}\rho^{(i-1)}$	a_i	$\rho^{(i)}$
0			$2/3$
1	$2\sqrt{2}/3 \approx 0.94$	0	$2\sqrt{2}/3$
2	$4/3$	1	$1/3$
3	$\sqrt{2}/3$	0	$\sqrt{(2)}/3$
4	$2/3$	0	$2/3$

hence the $\sqrt{2}$-expansion of $2/3$ is $(0100)^\omega$.

Another example

Take now $\alpha = 4/5$. We have

i	$\sqrt{2}\rho^{(i-1)}$	a_i	$\rho^{(i)}$
0			$4/5$
1	$4\sqrt{2}/5 \approx 1.13$	1	$4\sqrt{2}/5 - 1$
2	$8/5 - \sqrt{2} \approx 0.18$	0	$8/5 - \sqrt{2}$
3	$8\sqrt{2}/5 - 2 \approx 0.26$	0	$8\sqrt{2}/5 - 2$
4	$16/5 - 2\sqrt{2} \approx 0.37$	0	$16/5 - 2\sqrt{2}$
5	$16\sqrt{2}/5 - 4 \approx 0.53$	0	$16\sqrt{2}/5 - 4$
6	$32/5 - 4\sqrt{2} \approx 0.74$	0	$32/5 - 4\sqrt{2}$
7	$32\sqrt{2}/5 - 8 \approx 1.05$	1	$32\sqrt{2}/5 - 9$

and here we can prove that the $\sqrt{2}$-expansion of $4/5$ is not periodic

Periodicity?

The following propositions hold :

Proposition

If β is an integer, all rational numbers have a periodic expansion.

Proposition

If α has a periodic expansion, then $\alpha \in \mathbb{Q}(\beta)$.

Proposition (1)

If $1/q$ has a periodic expansion for any $q \in \mathbb{N}$, then β is an algebraic integer.

Can we give necessary and sufficient conditions on β for all rational numbers to have a periodic expansion?

Necessary condition : decomposition of remainders in $\mathbb{Q}(\beta)$

Let β have minimal polynomial $x^d + \sum_{i=0}^{d-1} b_i x^i$. Any $\alpha \in \mathbb{Q}(\beta)$ can be written as $\alpha = \frac{1}{q} \sum_{i=0}^{d-1} p_i \beta^i$ with integers p_i. If q is fixed, there exists at most one such decomposition.

Lemma (1)

With the notations above, for every $n \in \mathbb{N}$ there exists a unique d-uple of integers $(r_1^{(n)}, ..., r_d^{(n)})$ such that

$$\rho^{(n)}(\alpha) = \frac{1}{q} \sum_{i=1}^{d} r_k^{(n)} \beta^{-k}.$$

Necessary condition : link to Galois conjugates

Let $\beta = \beta_1, ..., \beta_d$ be the roots of the minimal polynomial of β. Let $\mathrm{Per}(\beta)$ be the set of real numbers in $[0, 1[$ with ultimately periodic β-expansion.

Lemma (2)

For every $n \in \mathbb{N}$, $m \in \{1, ..., d\}$, we have

$$\beta_m^n \left(\frac{1}{q} \sum_{i=0}^{d-1} p_i \beta_m^i - \sum_{k=0}^{n} a_k \beta_m^{-k} \right) = \frac{1}{q} \sum_{k=1}^{d} r_k^{(n)} \beta_m^{-k}.$$

In particular, if $\alpha \in \mathrm{Per}(\beta)$ and $|\beta_m| > 1$,

$$\frac{1}{q} \sum_{i=0}^{d-1} p_i \beta_m^i = \sum_{k=0}^{\infty} a_k \beta_m^{-k}.$$

Note that if $\alpha \in \mathbb{Q}$, then $p_1 = ... = p_{d-1} = 0$.

Necessary condition

Recall that a Pisot number is an algebraic integer whose Galois conjugates all have modulus < 1, and a Salem number is an algebraic integer whose Galois conjugates all have modulus ≤ 1, with equality in at least one case.

Theorem (1)

If $\beta > 1$ is a real number such that $\mathbb{Q} \cap [0, 1[\subset \mathrm{Per}(\beta)$, then β is either a Pisot number or a Salem number.

Sufficient condition

Let $\rho_m^{(n)} = \frac{1}{q}\sum_{k=1}^{d} r_k^{(n)} \beta_m^{-k}$.

Lemma (3)

The following are equivalent :

- $\alpha \in \mathrm{Per}(\beta)$.
- $\rho^{(n)}$ *is periodic.*
- $r_k^{(n)}$ *is bounded independently of* k *and* n.
- $\rho_m^{(n)}$ *is bounded independently of* m *and* n.

Theorem (2)

If β *is a Pisot number, then* $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1[$.

Alternate bases

Expansions in an alternate base $(\beta_0, ..., \beta_{p-1})$ are obtained by alternating the transformations $T_{\beta_0}, ..., T_{\beta_{p-1}}$ rather than iterating T_β :

$$a_{-1} = 0, \varrho^{(0)} = \alpha, a_i = \lfloor \beta_{i\%p} \varrho^{(i)} \rfloor, \varrho^{(i+1)} = \{\beta_{i\%p} \varrho^{(i)}\}$$

where $i\%p$ denotes i modulo p.
The value of a sequence of digits $(a_i)_{i\in\mathbb{N}}$ is

$$\sum_{i=0}^{\infty} \frac{a_i}{\beta_0\beta_1....\beta_{p-1}\beta_0...\beta_{i\%p}},$$

with $i+1$ factors in the denominator.
We wish to find necessary and sufficient conditions on $\beta_0, ..., \beta_{p-1}$ for all rational numbers to have a periodic expansion, and we wish to adapt Schmidt's work if possible.
We let $\delta = \prod_{i=0}^{p-1} \beta_i$ and $\beta^{(i)}$ denotes the alternate base $(\beta_i, ..., \beta_{p-1}, \beta_0, ..., \beta_{i-1})$.

Example

Let us consider $p = 2$, $\beta_0 = \dfrac{1+\sqrt{13}}{2}$, $\beta_1 = \dfrac{5+\sqrt{13}}{6}$ and $\alpha = 1$.

i	$\beta_{i\%2}\rho^{(i)}$	a_i	$\rho^{(i+1)}$
-1			1
0	$\frac{1+\sqrt{13}}{2} \approx 2.30$	2	$\frac{1+\sqrt{13}}{2} - 2$
1	$(\frac{1+\sqrt{13}}{2} - 2) * \frac{5+\sqrt{13}}{6} \approx 0.43$	0	$(\frac{1+\sqrt{13}}{2} - 2) * \frac{5+\sqrt{13}}{6}$
2	$(\frac{1+\sqrt{13}}{2} - 2) * (\frac{5+\sqrt{13}}{6}) * (\frac{1+\sqrt{13}}{2}) = 1$	1	0
3	0	0	0

Adapting Schmidt : First steps

Proving that δ is an algebraic integer is substantially harder than in the $p = 1$ case.

Proposition (a twist on Charlier, Cisternino, Maśaková, and Pelantová)

If for all $i \in \{0, ..., p-1\}$ there exists $q_i \in \mathbb{Z}$ and an ultimately periodic sequence $a^{(i)} \in \mathbb{Z}^{\mathbb{N}}$ that evaluates to $\frac{1}{q_i}$ in $\beta^{(i)}$, then δ is an algebraic integer. If furthermore, for all $i \in \{0, ..., p-1\}$, $a^{(i)}$ is in $\mathbb{N}^{\mathbb{N}}$ and $a^{(i)}_{np} \geq 1$ for some $n \geq 0$, then $\beta_i \in \mathbb{Q}(\delta) \forall i \in \{0, ..., p-1\}$.

In the following, we assume that δ is an algebraic integer and that $\beta_i \in \mathbb{Q}(\delta) \forall i \in \{0, ..., p-1\}$.

Adapting Schmidt : grouping digits

Let $f_{\underline{\beta}}(a_0, ..., a_{p-1}) = a_0\beta_1...\beta_{p-1} + a_1\beta_2...\beta_{p-1} + ... + a_{p-1}$.
Then, let $\eta_k(\alpha) = f(a_{kp}(\alpha), ..., a_{kp+p-1}(\alpha))$. We have
$$\alpha = \sum_{i=0}^{\infty} \frac{\eta_k(\alpha)}{\delta^{k+1}}.$$
We have $\eta_k(\alpha) \in \mathrm{Digits}(\underline{\beta})$, where this last set is
$f_{\underline{\beta}}(\{0, ..., \lfloor\beta_0\rfloor\} \times ... \times \{0, ..., \lfloor\beta_{p-1}\rfloor\})$
Additionally, let $\rho^{(n)}(\alpha) = (T_{\beta_{p-1}} \circ ... \circ T_{\beta_0})^n(\alpha)$. We get
$$\rho^{(n)}(\alpha) = \delta^n(\alpha - \sum_{i=0}^{n-1} \frac{\eta_k(\alpha)}{\delta^{k+1}}) \text{ and } \rho^{(n+1)} = \delta\rho^{(n)} - \eta_n(\alpha).$$
Finally, let the minimal polynomial of δ be $x^d + \sum_{i=0}^{d-1} b_i x^i$.

Necessary condition

Every number γ in $\mathbb{Q}(\delta)$ can be decomposed as $\gamma = \frac{1}{q}\sum_{i=0}^{d-1} p_i\delta^i$. This decomposition is unique when q is fixed. For a given α, we consider this decomposition, choosing q to be the least common multiple of all the minimal q for α and the members of $\mathrm{Digits}(\underline{\beta})$.

Lemma

For all $n \in \mathbb{N}$, there exists a unique d-uple $(r_1^{(n)}, ..., r_d^{(n)})$ of integers such that

$$\rho^{(n)}(\alpha) = \frac{1}{q}\sum_{i=1}^{d} r_k^{(n)}\delta^{-k}.$$

Necessary condition

Let $\delta, ..., \delta_d$ be the roots of the minimal polynomial of δ, and let ψ_k be the isomorphism from $\mathbb{Q}(\delta)$ to $\mathbb{Q}(\delta_k)$ that fixes $\mathbb{Q}$ and maps δ to δ_k.

Lemma

For all $\alpha \in \mathbb{Q}$ and $m \in \{1, ..., d\}$, we have

$$\delta_m^n \left(\frac{1}{q} \sum_{i=0}^{d-1} p_i \delta_m^i - \sum_{k=0}^{n-1} \psi_m(\eta_k(\alpha)) \delta_m^{-k-1} \right) = \frac{1}{q} \sum_{k=1}^{d} r_k^{(n)} \delta_m^{-k}.$$

In particular, if $\alpha \in \mathrm{Per}(\underline{\beta})$ and $|\delta_m| > 1$,

$$\frac{1}{q} \sum_{i=0}^{d-1} p_i \delta_m^i = \sum_{k=0}^{\infty} \psi_m(\eta_k(\alpha)) \delta_m^{-k-1}.$$

Necessary condition

Theorem

If for all $i \in \{0, ..., p-1\}$ *there exists* $q_i \in \mathbb{Z}$ *and an ultimately periodic sequence* $a^{(i)} \in \mathbb{Z}^{\mathbb{N}}$ *that evaluates to* $\frac{1}{q_i}$ *in* $\beta^{(i)}$*, and for all* $i \in \{0, ..., p-1\}$*,* $a^{(i)}$ *is in* $\mathbb{N}^{\mathbb{N}}$ *and* $a_{np}^{(i)} \geq 1$ *for some* $n \geq 0$*, and if additionally we have* $\mathbb{Q} \cap [0, 1[\subset \mathrm{Per}(\underline{\beta})$*, then* δ *is a Pisot number or a Salem number and* $\beta_i \in \mathbb{Q}(\delta) \forall i \in \{1, ..., d\}$*.*

Sufficient condition

Lemma

The representation $(a_i)_{i\in\mathbb{N}}$ is periodic if and only if the representation $(\eta_k)_{k\in\mathbb{N}}$ is periodic.

Theorem

If δ is a Pisot number and $\beta_i \in \mathbb{Q}(\delta) \forall i \in \{0, ..., p-1\}$, then

$$\mathbb{Q}(\delta) \cap [0, 1[\subset \cap_{i=0}^{p-1} \mathrm{Per}(\beta^{(i)}).$$

Open questions

- The case where β is a Salem number is still not well understood, even when $p = 1$.
- Is there any way to only use $\underline{\beta}$ rather than all the $\underline{\beta^{(i)}}$?

Thank you for your attention!