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ON THE POINTWISE REGULARITY OF THE MULTIFRACTIONAL BROWNIAN MOTION AND SOME EXTENSIONS

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Abstract. We study the pointwise regularity of the Multifractional Brownian Motion and, in particular, we obtain the existence of so-called slow points of the process, that is points which exhibit a slower oscillation than the a.e regularity. This result entails that a non self-similar process can also exhibit such a behavior. We also consider various extensions with the aim of imposing weaker regularity assumptions on the Hurst function without altering the regularity of the process.

Introduction

Given a compact subset $[a, b]$ of $(0, 1)$ and a function $H : \mathbb{R} \to [a, b]$, the Multifractional Brownian Motion (MBM) is the process defined in [8] by the harmonizable representation

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}} \, d\widehat{W}(\xi), \tag{0.1}$$

where $d\widehat{W}$ is the "Fourier transform" of the real-valued white noise measure dW. A slightly different MBM has been defined alternatively and independently in [26] by the moving average representation

$$B'_H(t) = \int_{\mathbb{R}} \left(|t-s|^{H(t)-\frac{1}{2}} - |s|^{H(t)-1/2} \right) dW(s). \tag{0.2}$$

When $H(\cdot) = h$ is a constant function, we recover the equivalent definitions of the Fractional Brownian Motion (FBM) of Hurst parameter h. For this reason, H is usually called the Hurst function. Note that the fundamental equality

$$\int_{\mathbb{R}} f(s) \, dW(s) = \int_{\mathbb{R}} \widehat{f}(\xi) \, d\widehat{W}(\xi),$$

which holds almost surely for all function $f \in L^2(\mathbb{R})$, ensures that the processes (0.1) and (0.2) are identical, up to a multiplicative deterministic smooth, bounded and non-vanishing function, see [13].

Generally, one requires that the function $H : \mathbb{R} \to [a, b]$ is β-Hölderian, for some $\beta > b$. With this assumption, one can recover several fundamental properties of FBM, namely

(a) **Local asymptotic self-similarity** [8, 20, 21]**:** for all $t \in \mathbb{R}$,

$$\lim_{\rho \to 0^+} \mathrm{Law} \left\{ \frac{B_H(t+\rho s) - B_H(t)}{\rho^{H(t)}}, \, s \in \mathbb{R} \right\} = \mathrm{Law}\{B_{H(t)}(s), \, s \in \mathbb{R}\},$$

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where $\{B_{H(t)}(s),\, s\in\mathbb{R}\}$ is FBM with constant Hurst parameter $H(t)$. Here, the convergence holds for the finite-dimensional distributions but also in the space of continuous functions over an arbitrary compact subset of $\mathbb{R}$, see [1, Definitions 1.69 and 1.70]. In particular, if the function H is non constant, the process $\{B_H(t),\, t\in\mathbb{R}\}$ is not self-similar.

(b) **Uniform modulus of continuity** [8]**:** on an event of probability 1, for every open bounded subset D of $\mathbb{R}$, one has

$$\limsup_{s,t\in D,|s-t|\to 0}\frac{|B_H(s)-B_H(t)|}{|s-t|^{\underline{H}_D}\sqrt{\log|s-t|^{-1}}}=\sqrt{2}C_D$$

where $\underline{H}_D=\inf_{t\in D}H(t)$ and $C_D=\sup_{t\in H^{-1}(\underline{H}_D)\cap\overline{D}}C(t)$, where $H^{-1}(\underline{H}_D)$ denotes the reciprocal image of $\underline{H}_D$, with

$$C(t)=\sqrt{\int_{\mathbb{R}}\frac{1-\cos^2(xt)}{|x|^{1+2H(t)}}\,dx},\quad \forall t\in\mathbb{R}.$$

(c) **Law of the iterated logarithm** [8]**:** on an event of probability 1, for all $t\in\mathbb{R}$,

$$\limsup_{s\to t}\frac{|B_H(s)-B_H(t)|}{|s-t|^{H(t)}\sqrt{\log\log|s-t|^{-1}}}=\sqrt{2}C(t)$$

(d) **Existence of local-time**, see e.g. [9, 10].

We also refer to the book [1] for a very complete overview on the topic.

In the present paper, we will be particularly interested in the pointwise regularity of MBM. Using the terminologies introduced by Kahane [22], item (c) above means that almost surely, almost every point $t\in\mathbb{R}$ is ordinary while item (b) ensures for some points, called the rapid points, exhibits faster oscillation. Moreover, concerning the Brownian Motion (BM), Kahane pointed out in [22] the existence of a third family of points, called slow points, presenting a slower oscillation. Recently, in [19], we showed that FBM also exhibits these three types of points and we also showed that this property is somehow exceptional using two different notions of genericity. A natural question is therefore to understand where this particularity comes from. The extension from BM to FBM has underlined that the existence of slow points does not depend on the Markovian property of the process. The results in [16] highlighted that the existence of slow points does not depend on the Gaussianity of the process either: indeed, the (generalized) Rosenblatt process, which is known to be non-Gaussian, also presents slow points in its pointwise regularity. In this paper, we prove that the existence of slow points does not depend on the self-similarity either, because MBM presents slow points.

The paper is organized as follows. Section 1 is devoted to the proof of upper bounds for the regularity of the MBM, while the optimality of these bounds is studied in Section 2. Note that these sections rely deeply on a wavelet-type expansion of MBM first given in [6]. In Section 3, by slightly modifying this expansion, we show that one can relax some hypothesis made on the function H without altering the pointwise regularity properties of the process. This model could be of interest for simulation purposes, when we have to consider multifractional phenomena with few regularity assumptions for the Hurst function.

1. A sharp upper bound for some oscillations

The definition (0.1) of MBM naturally leads to considering the following Gaussian field.

Definition 1.1. The generator of Multifractional Brownian motion (gMBM) is the Gaussian field $\{B(t,\theta) : (t,\theta) \in \mathbb{R} \times (0,1)\}$ defined as

$$B(t,\theta) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{\theta + \frac{1}{2}}} \, d\widehat{W}(\xi).$$

From (0.1), it is clear that, for all $t \in \mathbb{R}$, we have $B_H(t) = B(t, H(t))$.

A wavelet-type expansion of gMBM is the crucial point for our analysis of the pointwise regularity of MBM. Let us introduce it in a few words. Details can be found in the paper [6]. In what follows, $\{2^{j/2}\psi(2^j \cdot -k) : (j,k) \in \mathbb{Z}^2\}$ stands for the Lemarié-Meyer orthonormal wavelet base of the Hilbert space $L^2(\mathbb{R})$, introduced in [24]. Its particular features are the fact that the mother wavelet ψ belongs to the Schwartz class of C^∞ functions whose derivatives of any order have fast decay and that $\widehat{\psi}$ is compactly supported and is vanishing in a neighbourhood of 0. Thanks to these facts, the function

$$\Psi : (t,\theta) \in \mathbb{R}^2 \mapsto \int_{\mathbb{R}} \frac{e^{it\xi}\widehat{\psi}(\xi)}{|\xi|^{\theta + \frac{1}{2}}} \, d\xi$$

is well-defined, see [1, Definition 5.3 and Proposition 5.10]. Moreover, one can check that Ψ belongs to $C^\infty(\mathbb{R}^2)$ and satisfies the following fast decay property: for all $a, b \in \mathbb{R}$ and $L, m, n \in \mathbb{N}$,

$$\sup_{\theta \in [a,b]} \sup_{t \in \mathbb{R}} (3 + |t|)^L |D_t^m D_\theta^n \Psi(t,\theta)| < \infty, \tag{1.1}$$

see [6, Lemma 2.1]. The function Ψ leads to the following expansion for gMBM, for all $\mathbb{R} \times (0,1)$,

$$B(t,\theta) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-j\theta} \varepsilon_{j,k} \left(\Psi(2^j t - k, \theta) - \Psi(-k, \theta) \right), \tag{1.2}$$

where $(\varepsilon_{j,k})_{(j,k) \in \mathbb{Z}^2}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables. The convergence of this series holds in $L^2(\Omega)$, as a consequence of Wiener isometry, but also, most importantly in our case, almost surely uniformly on every compact subset of $\mathbb{R} \times (0,1)$, as can be deduced from the following estimate.

Lemma 1.2. [5] *Let $(\varepsilon_{j,k})_{(j,k) \in \mathbb{Z}^2}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables. There are an event Ω_0^* of probability 1 and a positive random variable C_1 of finite moment of every order such that, for all $\omega \in \Omega_0^*$ and $(j,k) \in \mathbb{Z}^2$, the inequality*

$$|\varepsilon_{j,k}(\omega)| \leq C_1(\omega) \sqrt{\log(3 + j + |k|)} \tag{1.3}$$

holds.

This last Lemma is also useful to show that the sample paths of the field

$$\overline{B}(t,\theta) := \sum_{j=-\infty}^{-1} \sum_{k \in \mathbb{Z}} 2^{-j\theta} \varepsilon_{j,k} \left(\Psi(2^j t - k, \theta) - \Psi(-k, \theta) \right)$$

are almost surely C^∞ functions.

In the sequel, we assume the following condition for the Hurst function. It is slightly less restrictive than the original uniform Hölder regularity assumption required in [8, 26]. Note that it is the condition used in [1, Theorem 1.89] to study the pointwise Hölder exponent of MBM.

Condition 1.3. The Hurst function $H : \mathbb{R} \to [a,b]$, with $0 < a < b < 1$, is such that for all $t \in \mathbb{R}$, there exists $\gamma \geq H(t)$ such that H belongs to the pointwise Hölder space $C^\gamma(t)$, which means that there exist $R_t > 0$ and $c_t > 0$ such that

$$|H(s) - H(t)| \leq c_t |s - t|^\gamma$$

for all $s \in \mathbb{R}$ with $|s-t| \leq R_t$.

Our main result of this section can be stated as follows.

Theorem 1.4. *If the function $H : \mathbb{R} \to [a,b]$ satisfies the Condition 1.3 then almost surely, for every interval I of $\mathbb{R}$ with non-empty interior, there exists $t \in I$ such that*

$$\limsup_{s\to t} \frac{|B_H(s)-B_H(t)|}{|s-t|^{H(t)}} < \infty. \tag{1.4}$$

The proof of Theorem 1.4 uses the wavelet series representation (1.2) that gives for all $t \in \mathbb{R}$

$$B_H(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-jH(t)}\varepsilon_{j,k}\left(\Psi(2^j t-k,H(t)) - \Psi(-k,H(t))\right). \tag{1.5}$$

First, note that Condition 1.3 and the fact that the trajectories of the field $\overline{B}$ are almost surely C^∞ functions entail that almost surely

$$\limsup_{s\to t} \frac{|\overline{B}(s,H(s))-\overline{B}(t,H(t))|}{|s-t|^{H(t)}} < \infty \tag{1.6}$$

for all $t \in \mathbb{R}$. Therefore, we only need to analyse the high frequency part of MBM, which can be done through the field

$$\widetilde{B}(t,\theta) := \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-j\theta}\varepsilon_{j,k}\left(\Psi(2^j t-k,\theta) - \Psi(-k,\theta)\right). \tag{1.7}$$

Remark 1.5. Before going further, let us remark that one can reduce our work to the proof of the existence of a point satisfying (1.4) in the interval $[0,1)$. Indeed, let us recall that any open interval in $\mathbb{R}$ can be written as a countable union of dyadic intervals $(\lambda_{j,k} = [k2^{-j},(k+1)2^{-j}))_{j\in\mathbb{N},k\in\mathbb{Z}}$. Therefore, in order to prove Theorem 1.4, it is sufficient to show that, for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, there exists an event $\Omega_{j,k}$ of probability 1 such that, for all $\omega \in \Omega_{j,k}$, there exists $t \in \lambda_{j,k}$ which satisfies (1.4). Now, up to dilatations and translations, it suffices to consider the dyadic interval $\lambda_{0,0} = [0,1)$.

Let us come back to the field (1.7). This last one has been largely considered in [2], where an alternative wavelet-type expansion of MBM is given. Let us already mention that we will also be interested in this representation in the last section of this paper. From now on, as in [2], for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, the notation $g_{j,k}$ stands for the function

$$g_{j,k} : (t,\theta) \mapsto 2^{-j\theta}\left(\Psi(2^j t-k,\theta) - \Psi(-k,\theta)\right).$$

Lemma 1.6. *For all compact interval K of $[0,1)$ and for all $n \in \mathbb{N}$, there exists a deterministic constant $c_{K,n} > 0$ such that, for all $\omega \in \Omega^*$*

$$\sup_{t\in K,\theta\in[a,b]} \left(\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} |D^n_\theta g_{j,k}(t,\theta)||\varepsilon_{j,k}(\omega)|\right) \leq c_{K,n}C_1(\omega).$$

In particular, for every $t \in K$ and every $\theta_1,\theta_2 \in [a,b]$, we have

$$|\widetilde{B}(t,\theta_1) - \widetilde{B}(t,\theta_2)| \leq c_{K,1}C_1(\omega)|\theta_1-\theta_2|.$$

Proof. The first part is a direct consequence of Lemma 1.2 and [2, Lemma 2]. The second part is obtained by applying the Mean Value Theorem to each function $g_{j,k}(t,\cdot)$. □

The strategy to prove the existence of slow points relies on a procedure which allows to deduce a sharper estimate than the inequality (1.3) obtained in Lemma 1.2. This procedure was initiated by Kahane for BM [22] and we have generalized it for FBM [19]. This generalized version can also be applied in the present setting and can be summarized in the following theorem. Throughout this paper, given $t \in \mathbb{R}$ and $j \in \mathbb{N}$, $k_j(t)$ stands for the unique integer such that $t \in [k_j(t)2^{-j}, (k_j(t)+1)2^{-j})$.

Theorem 1.7. [19] *Let us fix $m > 0$. There exists an event Ω_1^* of probability 1 such that for every $\omega \in \Omega_1^*$, there are $\mu > 0$ and $t \in (0,1)$ such that*

$$|\varepsilon_{j,k}(\omega)| \leq 2^l \mu \tag{1.8}$$

for every $j \in \mathbb{N}_0$ and every $k \in \Lambda^l_{j,m}(t)$, where

$$\Lambda^0_{j,m}(t) = \{k \in \mathbb{Z} : \, |k_j(t) - k| \leq 1\}$$

and for all $l \geq 1$,

$$\Lambda^l_{j,m}(t) = \{k \in \mathbb{Z} : \, 2^{m(l-1)} < |k_j(t) - k| \leq 2^{ml}\}.$$

The set of such points t is denoted $S^\mu_{low,m}$.

From now on and until the end of this section, we fix $m > 0$ such that $\frac{1}{m} < a$ and denote

$$\Omega^* = \Omega_0^* \cap \Omega_1^* \tag{1.9}$$

the event of probability 1 obtained as the intersection of the events of probability 1 given by Lemma 1.2 and Theorem 1.7 respectively. For all $j \in \mathbb{N}$, let us set

$$\widetilde{B}_j(t,\theta) := \sum_{k\in\mathbb{Z}} 2^{-j\theta} \varepsilon_{j,k} \left(\Psi(2^j t - k, \theta) - \Psi(-k,\theta) \right).$$

Note that on Ω^*, since inequality (1.3) holds, it is straightforward to check that the trajectories of the field $\widetilde{B}_j$ are continuously differentiable, using the fast decay property (1.1).

Lemma 1.8. *On the event Ω^* of probability 1, there exists a deterministic constant $c_m > 0$ such that, for all $n \in \mathbb{N}$ and $\mu > 0$, if $t \in S^\mu_{low,m}$ and $\varepsilon > 0$ is such that $I_\varepsilon(t) := [t-\varepsilon, t+\varepsilon] \subset (0,1)$, then*

$$\left| \sum_{j=0}^{n} (\widetilde{B}_j(t,\theta_1) - \widetilde{B}_j(s,\theta_2)) \right| \leq c_m \mu 2^{-\theta_1 n} 2^{n|\theta_1-\theta_2|} + C^* \, c_{I_\varepsilon(t),1} |\theta_1 - \theta_2|$$

for all $s \in I_\varepsilon(t)$ with $|s-t| \leq 2^{-n+1}$ and $\theta_1, \theta_2 \in [a,b]$.

Proof. If $t \in S^\mu_{\mathrm{low,m}}$, $s \in [t-\varepsilon, t+\varepsilon]$ with $|s-t| \leq 2^{-n+1}$ and $\theta_1, \theta_2 \in [a,b]$, then by the Taylor formula at first order, there exist x between s and t, and ξ between θ_1 and θ_2 such that

$$\begin{aligned} \sum_{j=0}^{n} (\widetilde{B}_j(t,\theta_1) - \widetilde{B}_j(s,\theta_2)) &= (t-s) \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} \varepsilon_{j,k} D_t g_{j,k}(x,\xi) \\ &\quad + (\theta_1 - \theta_2) \sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} \varepsilon_{j,k} D_\theta g_{j,k}(x,\xi). \end{aligned} \tag{1.10}$$

The second series on the right-hand side of equality (1.10) is bounded by Lemma 1.6. In order to control the first series on the right-hand side of (1.10), as $D_t g_{j,k}(x,\xi) = 2^{j(1-\xi)} D_t \Psi(2^j x - k, \xi)$, we use the fast decay property (1.1) to get, for all $0 \leq j \leq n$,

$$\sum_{k\in\mathbb{Z}} |\varepsilon_{j,k} D_t g_{j,k}(x,\xi)| \leq c_1 2^{j(1-\xi)} \sum_{l\in\mathbb{N}} \sum_{k\in\Lambda^l_{j,m}(t)} |\varepsilon_{j,k}| \frac{1}{(3+|2^j x - k|)^4}, \tag{1.11}$$

for a deterministic positive constant c_1. Now, note that for all $l \geq 1$ and $k \in \Lambda^l_{j,m}(t)$, we have

$$\begin{aligned}|2^j x - k| &\geq |k_j(t) - k| - |2^j x - k_j(t)| \\ &\geq |k_j(t) - k| - (|2^j x - 2^j t| + |k_j(t) - 2^j t|) \\ &\geq 2^{m(l-1)} - 3,\end{aligned}$$

because $j \leq n$. Together with (1.11) and inequality (1.8), this implies that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\varepsilon_{j,k} D_t g_{j,k}(x,\xi)| &\leq c_1 2^{j(1-\xi)} \sum_{l\in\mathbb{N}} \sum_{k\in\Lambda^l_{j,m}(t)} 2^l \mu \frac{1}{(3+|2^j x - k|)^4} \\ &\leq c_2 2^m \mu 2^{j(1-\xi)} \sum_{k\in\mathbb{Z}} \frac{1}{(3+|2^j x - k|)^3} \\ &\leq c_3 \mu 2^{j(1-\xi)}\end{aligned}$$

where c_2 and c_3 are positive deterministic constants only depending on m. Thus,

$$\left|\sum_{j=0}^{n} \sum_{k\in\mathbb{Z}} \varepsilon_{j,k} D_t g_{j,k}(x,\xi)\right| \leq c_3 \mu \sum_{j=0}^{n} 2^{j(1-\xi)} \leq c_4 2^{n(1-\xi)} \leq c_4 2^{-\theta_1 n} 2^{n|\theta_1-\theta_2|}$$

for a deterministic constant c_4 only depending on m, since $\xi \in (0,1)$ is between θ_1 and θ_2. □

Lemma 1.9. *On the event Ω^* of probability 1, there exists a deterministic constant $c_m > 0$ such that, for all $\mu > 0$, $\theta \in [a,b]$ and $t \in S^\mu_{low,m}$*

(1) one has

$$\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k} \Psi(2^j t - k, \theta)\right| \leq c_m \mu,$$

(2) if $\varepsilon > 0$ is such that $I_\varepsilon(t) := [t-\varepsilon, t+\varepsilon] \subset (0,1)$, then for all $n \in \mathbb{N}$, $s \in I_\varepsilon(t)$ with $|s-t| \leq 2^{-n+1}$ and $j \geq n$, one has

$$\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k} \Psi(2^j s - k, \theta)\right| \leq c_m \mu 2^{\frac{1}{m}(j-n)}.$$

Proof. The first bound is obtained exactly as in (1.11), partitioning the sum over $k \in \mathbb{Z}$ with the subsets $\Lambda^l_{j,m}(t)$ and using the fast decay property (1.1) for Ψ. Concerning the second bound, we note that, if l is the greatest integer for which $|s-t| \geq 2^{ml} 2^{-j}$ then, for all $l' \in \mathbb{N}$ and $k \in \Lambda^{l'}_{j,m}(s)$, the construction gives

$$|\varepsilon_{j,k}| \leq 2^{l'+l} \mu.$$

As $|s-t| \leq 2^{-n+1}$, we deduce $l \leq \frac{1}{m}(j+1-n)$ and we obtain the desired upper bound by partitioning the sum over $k \in \mathbb{Z}$ using the subsets $\Lambda^{l'}_{j,m}(s)$. □

Let us recall that, for all $L > 1$, there exists a deterministic constant $c > 0$ such that, for all $j \in \mathbb{Z}$ and $x \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+|j|+|k|}}{(3+|2^j x - k|)^L} \leq c\sqrt{\log(3+|j|+2^j|x|)}, \tag{1.12}$$

see for instance [3, Lemma 4.2] for a proof.

Lemma 1.10. *On the event Ω_0^* of probability 1, there exists a deterministic constant $c_1 > 0$ such that, for all $\theta_1, \theta_2 \in [a, b]$ and $j \in \mathbb{N}$,*

$$\left| \sum_{k\in\mathbb{N}} \varepsilon_{j,k} \left(2^{-j\theta_1}\Psi(-k,\theta_1) - 2^{-j\theta_2}\Psi(-k,\theta_2)\right) \right| \leq C^*\, c_1|\theta_1 - \theta_2| 2^{-ja}\sqrt{\log(3+j)}.$$

Proof. If inequality (1.3) holds, we know from the fast decay property (1.1), that the function

$$\theta \mapsto \sum_{k\in\mathbb{N}} \varepsilon_{j,k} 2^{-j\theta}\Psi(-k,\theta)$$

is smooth. Therefore, using the Taylor formula at first order, we obtain the existence of ξ between θ_1 and θ_2 such that

$$\sum_{k\in\mathbb{N}} \varepsilon_{j,k} \left(2^{-j\theta_1}\Psi(-k,\theta_1) - 2^{-j\theta_2}\Psi(-k,\theta_2)\right) = (\theta_1-\theta_2)\sum_{k\in\mathbb{N}} \varepsilon_{j,k} 2^{-j\xi}\Psi(-k,\xi).$$

Using (1.3), the fast decay property (1.1) and inequality (1.12), we get

$$\begin{aligned}\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k} 2^{-j\xi}\Psi(-k,\xi)\right| &\leq C^* 2^{-j\xi} \sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3+j+|k|)}}{(3+|k|)^4} \quad \leq C^* c_1 2^{-j\xi}\sqrt{\log(3+j)} \\ &\leq C^* c_1 2^{-ja}\sqrt{\log(3+j)},\end{aligned}$$

for a positive deterministic constant c_1. The conclusion follows immediately. □

We have now enough material to prove Theorem 1.4.

Proof of Theorem 1.4. In view of (1.6) and Remark 1.5, it suffices to show that on the event Ω^* of probability 1 defined in (1.9) there exists $t \in (0,1)$ such that

$$\limsup_{s\to t} \frac{|\widetilde{B}(s,H(s)) - \widetilde{B}(t,H(t))|}{|s-t|^{H(t)}} < \infty.$$

Let us recall that we have fixed $m \in \mathbb{N}$ such that $\frac{1}{m} < a$. Theorem 1.7 allows to consider $t \in S^{\mu}_{\text{low,m}}$ for some $\mu > 0$. Now, if $s \in (0,1)$ is such that $2^{-n} \leq |s-t| \leq 2^{-n+1}$, we write

$$\begin{aligned}\left|\widetilde{B}(t,H(t)) - \widetilde{B}(s,H(s))\right| \leq & \left|\sum_{j=0}^{n}(\widetilde{B}_j(t,H(t)) - \widetilde{B}_j(s,H(s)))\right| \\ &+ \sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k}2^{-jH(t)}\Psi(2^jt-k,H(t))\right| \\ &+ \sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k}2^{-jH(s)}\Psi(2^js-k,H(s))\right| \\ &+ \sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}} \varepsilon_{j,k}\left(2^{-jH(t)}\Psi(-k,H(t)) - 2^{-jH(s)}\Psi(-k,H(s))\right)\right|.\end{aligned}$$

From Condition 1.3 we know that, if n is large enough,

$$|H(t) - H(s)| \leq c_t|t-s|^{H(t)}. \tag{1.13}$$

Thus, Lemma 1.8 and (1.13) combined with the fact that $2^{-n} \leq |s-t| \leq 2^{-n+1}$ give

$$\begin{aligned}\left|\sum_{j=0}^{n}(\widetilde{B}_j(t,H(t)) - \widetilde{B}_j(s,H(s)))\right| &\leq c_m\mu|t-s|^{H(t)}2^{c_t n 2^{-(n-1)H(t)}} + C^* c_{I_\varepsilon(t),1}c_t|t-s|^{H(t)} \\ &\leq (c_m\mu + C^* c_{I_\varepsilon(t),1})|t-s|^{H(t)}.\end{aligned}$$

Using $2^{-n} \leq |s-t| \leq 2^{-n+1}$, Lemma 1.9 and (1.13) imply

$$\sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}}\varepsilon_{j,k}2^{-jH(t)}\Psi(2^j t-k,H(t))\right| \leq 2c_m\mu 2^{-nH(t)} \leq 2c_m\mu|s-t|^{H(t)}$$

while, as $\frac{1}{m} < a$,

$$\begin{aligned}\sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}}\varepsilon_{j,k}2^{-jH(s)}\Psi(2^j s-k,H(s))\right| &\leq c_m\mu\sum_{j\geq n+1}2^{(\frac{1}{m}-H(s))(j-n)}2^{-H(s)n} \\ &\leq c_m\mu\sum_{j\geq n+1}2^{(\frac{1}{m}-a)(j-n)}2^{-H(s)n} \\ &\leq c_m\mu|t-s|^{H(t)}2^{|H(t)-H(s)|n} \\ &\leq c_m\mu|t-s|^{H(t)}.\end{aligned}$$

Finally, by Lemma 1.10 and (1.13)

$$\sum_{j\geq n+1}\left|\sum_{k\in\mathbb{N}}\varepsilon_{j,k}\left(2^{-jH(t)}\Psi(-k,H(t)) - 2^{-jH(s)}\Psi(-k,H(s))\right)\right| \leq C^*\, c_2|t-s|^{H(t)},$$

for a deterministic positive constant c_2. ☐

Remark 1.11. Theorem 1.4 implies that if t is a point which satisfies (1.4), then $r \mapsto |r|^{H(t)}$ is a pointwise modulus of continuity for B_H at t. Let us remark that our strategy can also be applied to recover the upper bounds for the well-known uniform modulus of continuity as well as the law of iterated logarithm. Let us explain how to adapt our proofs on this purpose.

Concerning the uniform modulus of continuity, if $s,t \in [0,1]$ are such that $2^{-n} \leq |s-t| \leq 2^{-n+1}$ and θ_1 and θ_2 are fixed in $[a,b]$, we know that, almost surely, one can write (1.10). Therefore, if inequality (1.3) holds, using $D_t g_{j,k}(x,\xi) = 2^{j(1-\xi)}D_t\Psi(2^j x-k,\xi)$, the fast decay property (1.1) and (1.12), one has

$$\begin{aligned}\left|\sum_{j=0}^{n}\sum_{k\in\mathbb{Z}}\varepsilon_{j,k}D_t g_{j,k}(x,\xi)\right| &\leq c\,C_1\sum_{j=0}^{n}\sum_{k\in\mathbb{Z}}2^{j(1-\xi)}\frac{\sqrt{\log(3+|j|+|k|)}}{(3+|2^j x-k|)^L} \\ &\leq c\,C_1\sum_{j=0}^{n}2^{j(1-\xi)}\sqrt{\log(3+|j|+2^j|x|)} \\ &\leq c\,C_1\sum_{j=0}^{n}2^{j(1-\xi)}\sqrt{j} \\ &\leq c\,C_1 2^{n(1-\xi)}\sqrt{n} \qquad (1.14)\end{aligned}$$

where c is a positive deterministic constant whose value may differ from a line to another but does not depend on any relevant quantities. Similarly, for all $j > n$, we have

$$\left| \sum_{k\in\mathbb{N}} \varepsilon_{j,k} \Psi(2^j t - k, \theta_1) \right| \leq c\, C_1 \sqrt{j} \tag{1.15}$$

and

$$\left| \sum_{k\in\mathbb{N}} \varepsilon_{j,k} \Psi(2^j s - k, \theta_2) \right| \leq c\, C_1 \sqrt{j}. \tag{1.16}$$

Therefore, gathering the expression (1.10), Lemma 1.6, the inequalities (1.14), (1.15) and (1.16), Lemma 1.10 and Condition 1.3, we get

$$\begin{aligned}
|B_H(s) - B_H(t)| \leq c\, C_1 \Bigg(& (|t-s| 2^{n(1-\xi)} \sqrt{n} + |H(t) - H(s)| \\
& + \sum_{j>n} \left(2^{-jH(t)} \sqrt{j} + 2^{-jH(s)} \sqrt{j} + |H(t) - H(s)| 2^{-ja} \sqrt{\log(3+j)} \right) \Bigg) \\
& \leq c\, C_1 \left(|t-s| 2^{n(1-\xi)} \sqrt{n} + 2^{-nH(t)} \sqrt{n} + 2^{-nH(s)} \sqrt{n} + |H(t) - H(s)| \right) \\
& \leq c\, C_1 \left(2^{-nH(t)} \sqrt{n} + |t-s|^{H(t)} \right) \\
& \leq |t-s|^{H(t)} \sqrt{\log |s-t|^{-1}}.
\end{aligned}$$

Thus, we have shown that, almost surely, for all $t \in [0,1]$

$$\limsup_{s\to t} \frac{|B_H(s) - B_H(t)|}{|t-s|^{H(t)} \sqrt{\log |s-t|^{-1}}} < \infty. \tag{1.17}$$

Concerning the law of iterated logarithm, by an indexing argument, one can note that for all $t \in [0,1]$, there exits an event Ω_t^* of probability 1 and a positive random variable C_t of finite moment of any order such that, for all $\omega \in \Omega_t^*$,

$$|\varepsilon_{j,k}(\omega)| \leq C_t(\omega) \sqrt{\log(3 + |j| + |k - k_j(t)|)}. \tag{1.18}$$

Then we use [16, Lemma 3.22] which gives, for all L, for all $n \in \mathbb{N}$,

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3 + j + |k - k_j(t)|}}{(3 + |2^j x - k|)^L} \leq c \sqrt{\log(3+j)} \tag{1.19}$$

if $0 \leq j \leq n$, and

$$\sum_{k\in\mathbb{Z}} \frac{\sqrt{\log(3 + j + |k - k_j(t)|}}{(3 + |2^j x - k|)^L} \leq c \sqrt{j - n + 1} \sqrt{\log(3+j)} \tag{1.20}$$

for all $j > n$. Adapting what has been done for (1.14), (1.15) and (1.16) by using inequality (1.18) instead of (1.3) and inequalities (1.19) and (1.20) instead of (1.12), we get, for all $t, s \in [0,1]$,

$$|B_H(s) - B_H(t)| \leq c C_t' |t-s|^{H(t)} \sqrt{\log\log |s-t|^{-1}}$$

for some positive random variable C_t' of finite moment of any order. In particular, by Fubini theorem, almost surely, for almost every $t \in [0,1]$,

$$\limsup_{s\to t} \frac{|B_H(s) - B_H(t)|}{|t-s|^{H(t)} \sqrt{\log\log |s-t|^{-1}}} < \infty. \tag{1.21}$$

2. Optimality of the upper bound

Let us now focus on the optimality of the previously obtained modulus of continuity. It is worth mentioning that the optimality of the pointwise modulus of continuity $r \mapsto r^{H(t)}$ for some points t of the MBM given in Theorem 1.4 has been already obtained in [1, Theorem 6.17], under the following assumption on the Hurst function H, which is a little bit stronger than Condition 1.3.

Condition 2.1. The Hurst function $H : \mathbb{R} \to [a, b]$, with $0 < a < b < 1$, is such that for all $t \in \mathbb{R}$, there exists $\gamma > H(t)$ such that H belongs to the pointwise Hölder space $C^\gamma(t)$.

We show in this section how to extend the proof of [1, Theorem 6.17] to get the optimality of the two other pointwise modulus of continuity $r \mapsto r^{H(t)}\sqrt{\log r^{-1}}$ and $r \mapsto r^{H(t)}\sqrt{\log\log r^{-1}}$. Using the terminology of [19] inspired by the work of Kahane [22], the points t for which the modulus of continuity given by (1.4) is optimal are called slow points. Similarly, those for which (1.17) is optimal are called fast points. Finally, we will get the existence of the so-called ordinary points: for almost every point t, the modulus of continuity given in (1.21) is optimal.

Theorem 2.2. *If the function $H : \mathbb{R} \to [a, b]$ satisfies Condition 2.1 then almost surely, for every non-empty interval I of $\mathbb{R}$,*

- *there exists $t \in I$ such that*
$$0 < \limsup_{s\to t} \frac{|B_H(s) - B_H(t)|}{|s - t|^{H(t)}\sqrt{\log|s - t|^{-1}}} < \infty.$$
Such a point is called a rapid point.
- *almost every point $t \in I$ is such that*
$$0 < \limsup_{s\to t} \frac{|B_H(s) - B_H(t)|}{|s - t|^{H(t)}\sqrt{\log\log|s - t|^{-1}}} < \infty.$$
Such a point is called an ordinary point.
- *there exists $t \in I$ such that*
$$0 < \limsup_{s\to t} \frac{|B_H(s) - B_H(t)|}{|s - t|^{H(t)}} < \infty.$$
Such a point is called a slow point.

The method for the proof of Theorem 2.2 is based on a wavelet-type argument and relies on a biorthogonality property of sequences of functions defined via Ψ: For any fixed $\theta \in \mathbb{R}$, the two sequences of functions

$$\{2^{j/2}\Psi(2^j \cdot -k, \theta) : (j, k) \in \mathbb{Z}^2\} \quad \text{and} \quad \{2^{j/2}\Psi(2^j \cdot -k, -\theta - 1) : (j, k) \in \mathbb{Z}^2\}$$

are biorthogonal in $L^2(\mathbb{R})$, see [1, Proposition 5.13 (i)]. This result allows to express coefficients appearing in the decomposition (1.2) in terms of B, see [1, Lemma 6.22].

Lemma 2.3. [1] *On the event Ω_0^* of probability 1, for every $\theta \in (0, 1)$ and for all $(j, k) \in \mathbb{Z}^2$, one has*

$$2^{-j\theta}\varepsilon_{j,k} = 2^j \int_{\mathbb{R}} B(u, \theta)\Psi(2^j u - k, -\theta - 1)du$$

where $\varepsilon_{j,k}$ is given by the representation (1.2).

In the case of the MBM B_H, Lemma 2.3 allows to write

$$2^{-jH(t)}\varepsilon_{j,k_j(t)} = 2^j \int_{\mathbb{R}} B(u,H(t))\Psi(2^j u - k_j(t), -H(t)-1)du. \tag{2.1}$$

In order to state the next result, we recall that a modulus of continuity is an increasing function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\sigma(0) = 0$ and for which there is $C > 0$ such that $\sigma(2x) \leq C\sigma(x)$ for all $x \in \mathbb{R}^+$. We say that σ is submultiplicative if $\sigma(xy) \leq \sigma(x)\sigma(y)$ for all $x, y \in \mathbb{R}^+$.

Remark 2.4. It is very classical that

$$\sqrt{1+x+y} \leq \sqrt{1+x}\sqrt{1+y} \quad \text{and} \quad \log(3+x+y) \leq \log(3+x)\log(3+y)$$

for all $x, y \in \mathbb{R}^+$. It follows that the modulus of continuity of interest in Theorem 2.2 for the rapid and ordinary points are asymptotically equivalent as $r \to 0^+$ to the submultiplicative modulus of continuity given respectively by

$$r \mapsto r^{H(t)}\sqrt{1+\log r^{-1}} \quad \text{and} \quad r \mapsto r^{H(t)}\sqrt{1+\log(3+\log r^{-1})}.$$

Remark 2.5. The assumption that σ is submultiplicative can be slightly weakened by imposing the existence of a constant $C > 0$ such that $\sigma(xy) \leq C\sigma(x)\sigma(y)$ for all $x, y \in \mathbb{R}^+$.

Proposition 2.6. *Let us consider $t \in \mathbb{R}$ and a submultiplicative modulus of continuity σ with polynomial growth. Assume that the Hurst function $H : \mathbb{R} \to [a,b]$, with $0 < a < b < 1$, is such that there exists $\gamma > 0$ such that H belongs to the pointwise Hölder space $C^\gamma(t)$. Then on the event Ω_0^* of probability 1, one has for every j large enough*

$$2^{-jH(t)}|\varepsilon_{j,k_j(t)}| \leq C\Big(\sup\Big\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\Big\}\sigma(2^{-j}) + 2^{-\gamma j}\Big)$$

for a deterministic constant $c > 0$ and a positive random variable C, where the variables $\varepsilon_{j,k}$ are given by (1.5) and where the supremum may take the value $+\infty$.

Proof. As the first moment of Ψ vanishes, see [1, Remark 5.12], we get by using equality (2.1) and the change of variables $y = 2^j u - k$

$$\begin{aligned}
& 2^{-jH(t)}|\varepsilon_{j,k}| \\
\leq\ & 2^j \int_{\mathbb{R}} \big|B(u,H(t)) - B(t,H(t))\big|\big|\Psi(2^j u - k, -H(t)-1)\big|du \\
=\ & \int_{|y|\leq 2^{j/2}} \big|B(\tfrac{k+y}{2^j}, H(t)) - B(\tfrac{k+y}{2^j}, H(\tfrac{k+y}{2^j}))\big|\big|\Psi(y,-H(t)-1)\big|dy \\
& + \int_{|y|\leq 2^{j/2}} \big|B(\tfrac{k+y}{2^j}, H(\tfrac{k+y}{2^j})) - B(t,H(t))\big|\big|\Psi(y,-H(t)-1)\big|dy \\
& + \int_{|y|> 2^{j/2}} \big|B(\tfrac{k+y}{2^j}, H(t)) - B(t,H(t))\big|\big|\Psi(y,-H(t)-1)\big|dy
\end{aligned} \tag{2.2}$$

where $k := k_j(t)$. Let us now provide an appropriate upper bound for each term on the right-hand side of (2.2). Note that the assumption of regularity on H implies that there is a neighborhood I of t and a constant $c_0 > 0$ such that

$$|H(s) - H(t)| \leq c_0|s-t|^\gamma \quad \forall t \in I. \tag{2.3}$$

Now, for the first term we notice that

$$|t - \frac{k+y}{2^j}| \leq 2^{-j}|y + (k - 2^j t)| \leq 2^{-j}(|y|+1) \tag{2.4}$$

and in particular if $|y| \leq 2^{j/2}$, then $\frac{k+y}{2^j} \in I$ for large j. It follows that

$$
\begin{aligned}
& \int_{|y|\leq 2^{j/2}} \big|B(\frac{k+y}{2^j}, H(t)) - B(\frac{k+y}{2^j}, H(\frac{k+y}{2^j}))\big|\big|\Psi(y,-H(t)-1)\big|dy \\
\leq\ & c_{I,1}C_1\Big(2^j \int_{|y|\leq 2^{j/2}} \big|H(t)-H(\frac{k+y}{2^j})\big|\big|\Psi(y,-H(t)-1)\big|dy \\
\leq\ & C_2\Big(2^j \int_{|y|\leq 2^{j/2}} \big|t-\frac{k+y}{2^j}\big|^{\gamma}\big|\Psi(y,-H(t)-1)\big|dy \\
\leq\ & C_2 2^{-\gamma j}\int_{\mathbb{R}} \big(1+|y|\big)^{\gamma}\big|\Psi(y,-H(t)-1)\big|dy \\
\leq\ & C_3 2^{-\gamma j} \qquad (2.5)
\end{aligned}
$$

for some positive random constants C_2, C_3, by using successively Lemma 1.6, equations (2.3), (2.4) and (1.1).

For the second term, if $|y| \leq 2^{j/2}$, inequality (2.4) gives the existence of a constant $c > 0$ such that $|t - \frac{k+y}{2^j}| \leq c2^{-j/2}$. Hence

$$
\begin{aligned}
& \int_{|y|\leq 2^{j/2}} \big|B(\frac{k+y}{2^j}, H(\frac{k+y}{2^j})) - B(t,H(t))\big|\big|\Psi(y,-H(t)-1)\big|dy \\
& \leq \sup\left\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\} \int_{|y|\leq 2^{j/2}} \sigma(|t-\frac{k+y}{2^j}|)\big|\Psi(y,-H(t)-1)\big|dy \\
& \leq \sup\left\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\} \sigma(2^{-j}) \int_{\mathbb{R}} \sigma(|y|+1)\big|\Psi(y,-H(t)-1)\big|dy \\
& \leq c_2 \sup\left\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\} \sigma(2^{-j}) \qquad (2.6)
\end{aligned}
$$

for a constant $c_2 > 0$, using (2.4), the submultiplicativity property of σ, (1.1) and the polynomial growth of σ.

The upper bound for the last term is obtained using again the fast decay (1.1) of the wavelet for $L \geq 2\gamma$ together with the boundedness of the process B. Indeed, on can write

$$
\begin{aligned}
& \int_{|y|> 2^{j/2}} \big|B(\frac{k+y}{2^j}, H(t)) - B(t,H(t))\big|\big|\Psi(y,-H(t)-1)\big|dy \\
\leq\ & C_3 \int_{|y|> 2^{j/2}} \frac{1}{(1+|y|)^{2L}} du \\
\leq\ & C_3 2^{-Lj/2} \int_{\mathbb{R}} \frac{1}{(1+|y|)^{L}} dy \\
\leq\ & C_3' 2^{-\gamma j} \qquad (2.7)
\end{aligned}
$$

for some positive random constants C_3, C_3'. Putting together equations (2.2), (2.5), (2.6) and (2.7) leads to the conclusion. □

In order to prove Theorem 2.2, it suffices now to provide convenient asymptotic lower bounds for the coefficients $\varepsilon_{j,k}$. We summarize the relevant known results of [1], [4] and [19] in the following Lemma.

Lemma 2.7. [1, 4, 19] *Let $(\varepsilon_{j,k})_{(j,k)\in\mathbb{Z}^2}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables. There exists an event Ω_2^* of probability 1 on which*

(1) for every $t \in \mathbb{R}$, one has

$$\limsup_{j\to+\infty} |\varepsilon_{j,k_j(t)}| \geq 2^{-3/2}\sqrt{\pi}\,,$$

(2) for every non-empty open interval I of $\mathbb{R}$, there is $t \in I$ such that

$$\limsup_{j\to+\infty} \frac{|\varepsilon_{j,k_j(t)}|}{\sqrt{j}} > 0\,,$$

(3) for almost every $t \in \mathbb{R}$, one has

$$\limsup_{j\to+\infty} \frac{|\varepsilon_{j,k_j(t)}|}{\sqrt{\log j}} > 0\,.$$

The proof of the main result of this section is now straightforward.

Proof of Theorem 2.2. From Theorem 1.4, equation (1.21) and equation (1.17), it suffices to prove the three lower bounds.

Let us work on the event $\Omega^* \cap \Omega_2^*$ of probability 1. In each case, if σ denotes the corresponding modulus of continuity, we know from Proposition 2.6 and Remark 2.4 that

$$2^{-jH(t)}|\varepsilon_{j,k_j(t)}| \leq C\Big(\sup\left\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\}\sigma(2^{-j}) + 2^{-\gamma j}\Big)$$

with $\gamma > H(t)$ by Condition 2.1. Lemma 2.7 then implies that

$$0 < \limsup_{j\to+\infty} \frac{|\varepsilon_{j,k_j(t)}|}{\sigma(2^{-j})} \leq C \lim_{j\to+\infty} \sup\left\{\frac{|B_H(s)-B_H(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\}$$

since $\frac{2^{-\gamma j}}{\sigma(2^{-j})}$ tends to 0 as j tends to infinity. □

3. Extensions

The methodology developed in the previous sections can easily be adapted to study very general random wavelet series of the form

$$f_H = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \varepsilon_{j,k} 2^{-H(k2^{-j})j}\psi(2^j\cdot -k)$$

where $(\varepsilon_{j,k})_{(j,k)\in\mathbb{Z}^2}$ still denotes a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables. Among the families of wavelet basis that exist, we will work with two classes: The Lemarié-Meyer wavelets for which ψ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$, or Daubechies wavelets for which ψ is a compactly supported function (see [15]). In both cases, the first moment of the wavelet ψ vanishes. We will also include the setting given by biorthogonal wavelet basis, [11, 12].

Clearly, as soon as we work with a compactly supported wavelet or a wavelet which decays sufficiently fast, one can make sure that the function f_H is almost surely well-defined, exploiting Lemma 1.2. The process f_H gives a multifractal version of the random series studied in [19], by substituting the exponent h at level (j,k) by $H(k2^{-j})$ as done in [7]. Of course, this model can not be used to represent MBM. Nevertheless, we believe that it can have its own interest as it can be used to numerically simulate multifractional signals more efficiently than by considering the random series (1.5) since one can avoid the computation of the fractional primitives. Moreover, concerning the pointwise regularity, we will show that we do not alter the results obtained in the previous sections.

The biorthogonality of the wavelets allows to state in our present context the following result similar to Proposition 2.6.

Proposition 3.1. *Let us consider $t \in \mathbb{R}$ and a submultiplicative modulus of continuity σ. Assume that*

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k}\psi(2^j\cdot -k)$$

is a bounded function and that the wavelet ψ satisfies

$$\sup_{y\in\mathbb{R}}(1+|y|)^{2L}|\psi(y)| < +\infty$$

for some $L > 0$, and

$$\int_{\mathbb{R}} \sigma(1+|y|)|\psi(y)|dy < +\infty.$$

Then for every j large enough, one has

$$|c_{j,k_j(t)}| \leq c\Big(\sup\left\{\frac{|f(s)-f(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\}\sigma(2^{-j}) + 2^{-Lj/2}\Big)$$

for a constant $c > 0$, where the supremum may take the value $+\infty$.

Proof. The (bi)orthogonality of the wavelets allows to write

$$c_{j,k} = 2^j\int_{\mathbb{R}} f(u)\psi(2^j u - k)\,du.$$

Using similar arguments as in Proposition 2.6, for $k = k_j(t)$, we can write

$$\begin{aligned}
|c_{j,k}| &\leq 2^j\int_{\mathbb{R}}\big|f(u)-f(t)\big|\big|\psi(2^j u-k)\big|du\\
&= \int_{|y|\leq 2^{j/2}}\big|f(\frac{k+y}{2^j})-f(t)\big|\big|\psi(y)\big|dy + \int_{|y|>2^{j/2}}\big|f(\frac{k+y}{2^j})-f(t)\big|\big|\psi(y)\big|dy\\
&\leq \sup\left\{\frac{|f(s)-f(t)|}{\sigma(|s-t|)} : |s-t| < c2^{-j/2}\right\}\sigma(2^{-j})\int_{\mathbb{R}}\sigma(1+|y|)\big|\psi(y)\big|dy\\
&\quad +2\|f\|_\infty 2^{-Lj/2}\int_{\mathbb{R}}\frac{1}{(1+|y|)^L}dy
\end{aligned}$$

hence the conclusion. □

This section aims at showing that f_H still shares the same features as MBM when one considers its pointwise regularity. Moreover, in this context, we can significantly reduce the condition made on the regularity of the function H to obtain the results. In the sequel, Condition 1.3 is replaced by the following.

Condition 3.2. The Hurst function $H : \mathbb{R} \to [a,b]$, with $0 < a < b < 1$, is such that for all $t \in \mathbb{R}$ there exist $R_t > 0$ and $c_t > 0$ such that

$$|H(s)-H(t)| \leq \frac{c_t}{\log|s-t|^{-1}}$$

for all $s \in \mathbb{R}$ with $|s-t| \leq R_t$.

Remark 3.3. Of course, any function H satisfying Condition 3.2 is necessarily continuous and any Hölder-continuous function satisfies Condition 3.2. In particular, Condition 3.2 is weaker than Condition 1.3.

Remark 3.4. In [14], it is proved that if H is the function "Hölder exponent" of a continuous function, then there exists a sequence $(P_j)_{j\in\mathbb{N}_0}$ of polynomials such that

$$\begin{cases} H(t) = \liminf_{j\to+\infty} P_j(t) \\ \|DP_j\|_\infty \leq j, \quad \forall j \in \mathbb{N}_0. \end{cases} \tag{3.1}$$

Because of Condition 3.2, our function H is not enough general, but if a function H satisfies (3.1) and if we assume the existence of a constant $C > 0$ such that, for all $t \in \mathbb{R}$ and $j \in \mathbb{N}_0$,

$$|H(t) - P_j(t)| \leq c_t j^{-1} \tag{3.2}$$

then Condition 3.2 is satisfied. Note that the authors in [23] assume a condition similar to (3.2) to prove a law of the iterated logarithm for a multifractional extension of BM defined using the Faber-Schauder base.

Our result concerning the pointwise regularity of the process f_H can then be stated as follows.

Theorem 3.5. *If the function $H : \mathbb{R} \to [a, b]$ satisfies the Condition 3.2 then almost surely, for every interval I of $\mathbb{R}$ with non-empty interior,*

- *there exists $t \in I$ such that*

$$0 < \limsup_{s \to t} \frac{|f_H(s) - f_H(t)|}{|s-t|^{H(t)}\sqrt{\log|s-t|^{-1}}} < \infty, \tag{3.3}$$

- *almost every point $t \in I$ is such that*

$$0 < \limsup_{s \to t} \frac{|f_H(s) - f_H(t)|}{|s-t|^{H(t)}\sqrt{\log\log|s-t|^{-1}}} < \infty, \tag{3.4}$$

- *there exists $t \in I$ such that*

$$0 < \limsup_{s \to t} \frac{|f_H(s) - f_H(t)|}{|s-t|^{H(t)}} < \infty. \tag{3.5}$$

Proof. We will slightly modify the proofs done in the previous sections for the MBM. As previously mentioned, it suffices to work on $[0, 1)$. Let us first focus on the three upper bounds. On this purpose, for all $j \in \mathbb{N}$, we define the random series

$$f_{H,j} := \sum_{k \in \mathbb{Z}} 2^{-jH(k2^{-j})} \varepsilon_{j,k} \psi(2^j \cdot -k).$$

Let us start by showing the existence of slow points (3.5). As previously, we take $m \in \mathbb{N}$ such that $\frac{1}{m} < a$ and on an event of probability 1, Theorem 1.7 allows to consider $t \in S^{\mu}_{\text{low,m}}$, for some $\mu > 0$. Now, if $s \in (0, 1)$ is such that $2^{-n} \leq |s-t| \leq 2^{-n+1}$, we write

$$|f_H(t) - f_H(s)| \leq \left| \sum_{j=0}^{n} (f_{H,j}(t) - f_{H,j}(s)) \right| + \sum_{j \geq n+1} |f_{H,j}j(t)| + \sum_{j \geq n+1} |f_{H,j}(s)|.$$

As in Lemma 1.8, we have

$$\left| \sum_{j=0}^{n} (f_{H,j}(t) - f_{H,j}(s)) \right| \leq |s-t| \left(\sum_{j=0}^{n} \sum_{k \in \mathbb{Z}} |\varepsilon_{j,k}| 2^{j(1-H(k2^{-j}))} |D_t \psi(2^j x - k)| \right)$$

for some x between s and t. Then, similarly to (1.11), we deduce, using the fast decay property (1.1),

$$\sum_{j=0}^{n} \sum_{k \in \mathbb{Z}} |\varepsilon_{j,k}| 2^{j(1-H(k2^{-j}))} |D_t \psi(2^j x - k)| \leq c_1 \mu \sum_{j=0}^{n} \sum_{k \in \mathbb{Z}} \frac{2^{j(1-H(k2^{-j}))}}{(3 + |2^j x - k|)^L},$$

for L sufficiently large and whose value will be specified later and a deterministic positive constant c_1 which only depends on ψ, L and m. Now, if $k \in \mathbb{Z}$ is such that $|t - k2^{-j}| \leq 2^{-j/2}$ then, by Condition 3.2, $|H(t) - H(k2^{-j})| \leq 2c_t j^{-1}$ and

$$\frac{2^{j(1-H(k2^{-j}))}}{(3+|2^j x - k|)^L} \leq 2^{j(1-H(t))} \frac{2^{2c_t}}{(3+|2^j x - k|)^L}.$$

On the other hand, if $|t - k2^{-j}| > 2^{-j/2}$, we write

$$\frac{2^{j(1-H(k2^{-j}))}}{(3+|2^j x - k|)^L} \leq 2^{j(1-H(t))} \frac{2^{j(b-a)}}{(3+|2^j x - k|)^L}$$

and then, if $L \geq 2$ is such $b - a < L/2 - 1$, as $|2^j x - k| > 2^{j/2} - 2$, we get

$$\frac{2^{j(1-H(k2^{-j}))}}{(3+|2^j x - k|)^L} \leq 2^{j(1-H(t))} \frac{1}{(3+|2^j x - k|)^2}.$$

In total, we obtain

$$\begin{aligned}
\left| \sum_{j=0}^{n} (f_{H,j}(t) - f_{H,j}(s)) \right| &\leq c_2 \mu |t-s| \sum_{j=0}^{n} 2^{j(1-H(t))} \sum_{k \in \mathbb{Z}} \frac{1}{(3+|2^j x - k|)^2} \\
&\leq c_3 \mu |t-s| \sum_{j=0}^{n} 2^{j(1-H(t))} \\
&\leq c_4 \mu |t-s| 2^{n(1-H(t))} \\
&\leq c_5 |t-s|^{H(t)},
\end{aligned}$$

where c_2, c_3, c_4 and c_5 are deterministic positive constants not depending on any relevant quantity. Modifying the proofs of Lemma 1.9 and Theorem 1.4 in exactly the same way, we obtain

$$\sum_{j \geq n+1} |f_{H,j}(t)| \leq c_6 |t-s|^{H(t)}$$

and, as $|s - t| \leq 2^{-n+1}$, by Condition 3.2,

$$\sum_{j \geq n+1} |f_{H,j}(s)| \leq c_6 |t-s|^{H(t)} 2^{|H(t)-H(s)|n} \leq 2^{2c_t} c_6 |t-s|^{H(t)},$$

with c_6 a deterministic positive constant which does not depend on any relevant quantity.

Inequalities (3.3) and (3.4) are proved in a similar way, exploiting the alternative arguments given in Remark 1.11.

The lower bounds are obtained by combining Lemma 2.7 together with Proposition 3.1. □

Remark 3.6. In the particular case where the function H is constant, we recover the random wavelet series studied in [19]. Note however that, even in this simple case, we improve here [19, Theorem 2.4] since we obtain that the three above limsup are strictly positive, even if the wavelet is *not* compactly supported, see [19, Remark 5.2].

Remark 3.7. A careful look at the proofs shows that the random series f_H could also be defined through a biorthogonal system of vaguelets, see [18, 25, 17]. Recall that a family of functions $\Psi_{j,k}$ is called vaguelets if it satisfies a localization property

$$|\Psi_{j,k}(t)| \leq C 2^{j/2} (1 + |2^j t - k|)^{-1-\alpha_1} \quad \forall\, t \in \mathbb{R},$$

an oscillation property

$$\int_{\mathbb{R}} \Psi_{j,k}(t) dt = 0$$

and a regularity property

$$|\Psi_{j,k}(t) - \Psi_{j,k}(s)| \leq C2^{j(\alpha_2+1/2)}|t-s|^{\alpha_2} \quad \forall\, s,t \in \mathbb{R}$$

for some constant $C > 0$ and $0 < \alpha_2 < \alpha_1 < 1$.

Let us end this section by considering a third process. In [2], the authors have proved that the process $\{Z(t) : t \in \mathbb{R}\}$ defined for each $t \in \mathbb{R}$ as

$$Z(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-jH(k2^{-j})}\varepsilon_{j,k}\left(\Psi(2^j t - k, H(k2^{-j})) - \Psi(-k, H(k2^{-j})\right) \tag{3.6}$$

shares common properties with MBM. Namely,

(a) when the function H is constant, it reduces to FBM.
(b) the trajectories of the process

$$\overline{Z}(t) := \sum_{j=-\infty}^{-1}\sum_{k\in\mathbb{Z}} 2^{-jH(k2^{-j})}\varepsilon_{j,k}\left(\Psi(2^j t - k, H(k2^{-j})) - \Psi(-k, H(k2^{-j})\right)$$

are almost surely C^∞ functions. Thus, the regularity of Z is only determined by the process

$$\widetilde{Z}(t) := \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-jH(k2^{-j})}\varepsilon_{j,k}\left(\Psi(2^j t - k, H(k2^{-j})) - \Psi(-k, H(k2^{-j})\right). \tag{3.7}$$

(c) the process Z is also locally asymptotically self-similar.
(d) almost surely, for all t, the pointwise Hölder exponent of Z at t is $H(t)$.
(e) if a and b satisfy the condition

$$1 - b > (1-a)(1-ab^{-1}) \tag{3.8}$$

then there exists an exponent $d \in (b, 1]$ such that, almost surely, the process $Z - B_H$ is uniformly Hölder of exponent d. In other words, there exists a process X more regular than B_H and Z such that

$$B_H = Z + X.$$

In some sense, all these facts mean that, up to an additive regular process, if condition (3.8) holds, Z is an appropriate representation of MBM. Of course, even if condition (3.8) does not hold, the process Z has its own interest. Here, we want to show that Z still shares the same features as MBM when one considers its pointwise regularity under the less restrictive condition 3.2.

Theorem 3.8. *If the function $H : \mathbb{R} \to [a,b]$ satisfies the Condition 3.2 then almost surely, for every interval I of $\mathbb{R}$ with non-empty interior,*

- *there exists $t \in I$ such that*

$$\limsup_{s\to t} \frac{|Z(s) - Z(t)|}{|s-t|^{H(t)}\sqrt{\log|s-t|^{-1}}} < \infty. \tag{3.9}$$

- *almost every point $t \in I$ is such that*

$$\limsup_{s\to t} \frac{|Z(s) - Z(t)|}{|s-t|^{H(t)}\sqrt{\log\log|s-t|^{-1}}} < \infty. \tag{3.10}$$

- *there exists $t \in I$ such that*

$$\limsup_{s\to t} \frac{|Z(s) - Z(t)|}{|s-t|^{H(t)}} < \infty. \tag{3.11}$$

Proof. We already know that it suffices to prove (3.9), (3.10) and (3.11) for the process $\widetilde{Z}$. Also, when one considers the increments $\widetilde{Z}(t) - \widetilde{Z}(s)$, $t, s \in \mathbb{R}$, the terms $2^{-jH(k2^{-j})}\Psi(-k, H(k2^{-j}))$ in (3.7) cancel and thus we just need to study the pointwise regularity of the random series

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} 2^{-jH(k2^{-j})}\varepsilon_{j,k}\Psi(2^j\cdot -k, H(k2^{-j})).$$

Then, it suffices to replace ψ by $\Psi(\cdot, H(k2^{-j}))$ in the proof of Theorem 3.5.

□

When one wants to prove the positiveness of the limits in Theorems 2.2 and 3.5, the strategy is to consider the biorthogonality property of the basis to express the coefficients in terms of the increments of the process. In the present situation, it seems that there is no obvious connection between the random coefficients in the series (3.6) and the oscillations of the process. In particular, one can not apply this strategy anymore. Therefore, the positiveness of the three limits remains an interesting open question which needs different tools than those developed in this paper.

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