Chaos in Bose-Hubbard systems: Lyapunov exponents, Ehrenfest time, and scrambling

T. Michel[1], J. D. Urbina[2], and P. Schlagheck[1]

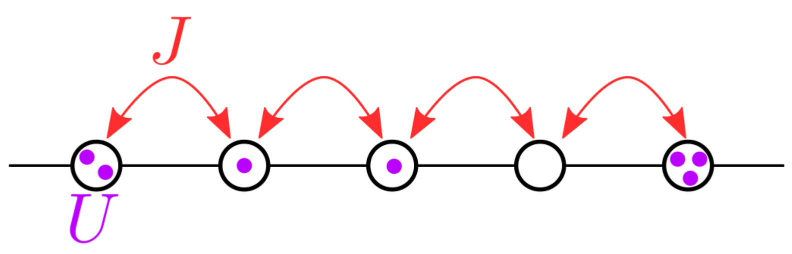
[1]CESAM research unit, Université de Liège, Belgium
[2] Institut für Theoretische Physik, Universität Regensburg, Germany

Motivation

We study one-dimensional Bose-Hubbard systems, or chains, which are describing many-body ultracold bosonic atoms. Our goal is to study quantum chaos in such systems, and see if it can be linked to classical chaos by considering the (semi)classical equivalent of the system. In addition to purely quantum time-evolution simulations, the numerical tools used for this purpose are a mean-field Gross-Pitaevskii description of the system and a truncated Wigner (tW) approach. We will see if signatures of chaos can be found in out-of-time-ordered correlators (OTOCs), and how it can be linked to the deviation of classical trajectories.

Bose-Hubbard chains

- Quantum system:



$$\hat{H} = \sum_{l=1}^{L} \left(E_l \hat{n}_l + \frac{U}{2}\hat{n}_l(\hat{n}_l - 1) \right) - J \sum_{l=1}^{L-1} \left(\hat{b}_l^\dagger \hat{b}_{l+1} + \hat{b}_{l+1}^\dagger \hat{b}_l \right)$$

with:
  - E_l the on-site interaction
  - J the hopping parameter between adjacent sites
  - U the 2-body interaction parameter
  - $\hat{b}_l$, $\hat{b}_l^\dagger$ the bosonic annihilation and creation operators on site l
  - $\hat{n}_l = \hat{b}_l^\dagger \hat{b}_l$ the population operator on site l
- Classical analogue:

$\frac{1}{2}\left(\hat{a}_l^\dagger \hat{a}_l + \hat{a}_l \hat{a}_l^\dagger\right) = \psi_l^* \psi_l$

$\rightarrow$ Discretised Gross-Pitaevskii equation:

$$i\hbar \frac{d}{dt}\psi_l = E_l \psi_l - J(\psi_{l+1} + \psi_{l-1}) + U|\psi_l|^2 \psi_l$$

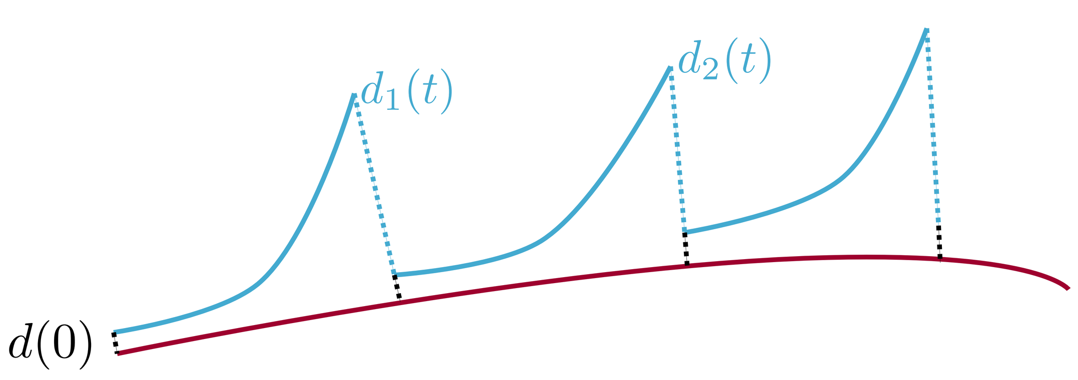
Lyapunov exponent

- Characterises the rate of separation of initially close trajectories in phase space

$$|\boldsymbol{\delta}(t)| \sim e^{\lambda t} |\boldsymbol{\delta}_0|$$

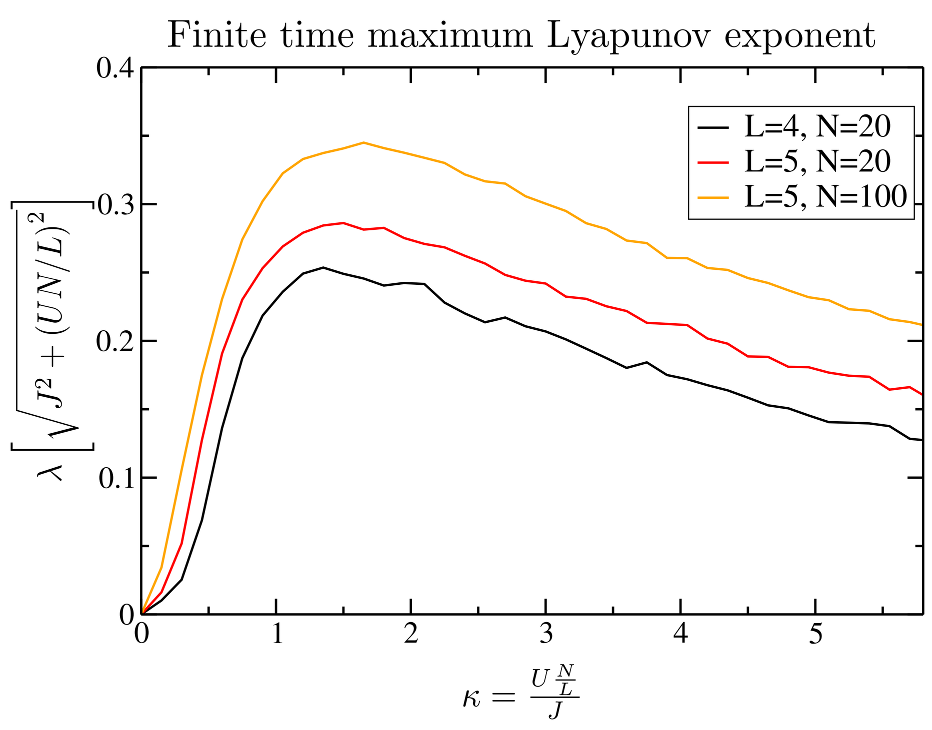
with $\lambda = \lim_{t\to\infty} \lim_{|\boldsymbol{\delta}_0|\to 0} \frac{1}{t} \log \frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}_0|}$

- Only finite differences are achievable
  $\rightarrow$ necessity of a numerical approach: Wolf algorithm



$$\lambda_{\text{FTMLE}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_n = \lim_{N,K\to\infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{t} \log\left(\frac{d_k(t)}{d(0)}\right)$$

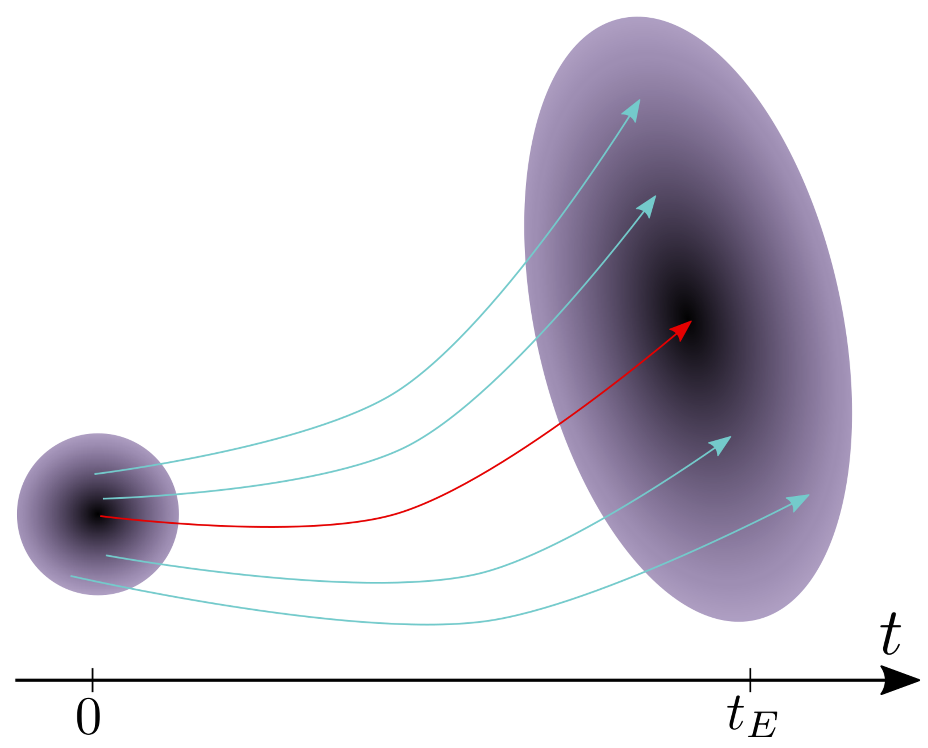
Lyapunov exponent (2)



- In good agreement with [A. C. Cassidy et al., Phys. Rev. Lett. **102**, 025302 (2009)]

Ehrenfest time

- Breakdown of one-to-one correspondence between quantum wave packet and classical trajectories

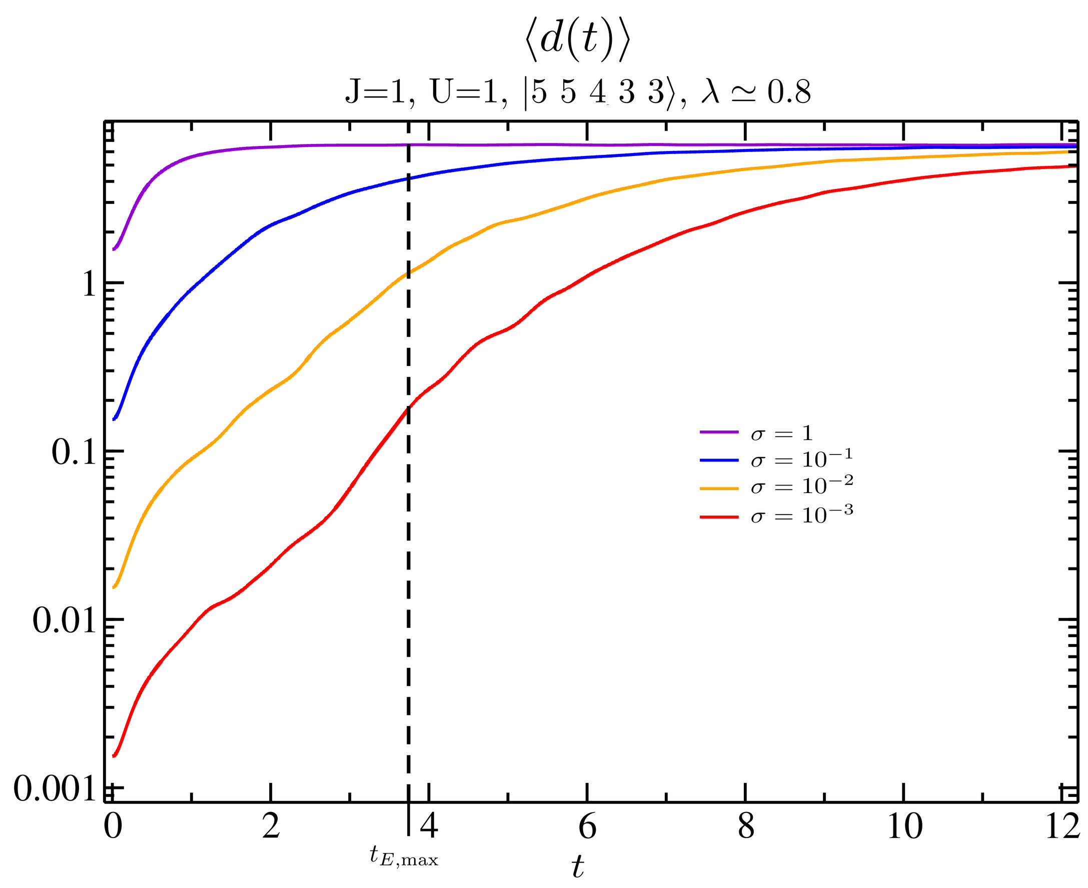


The Ehrenfest time in the hypothesis of a perfectly ergodic system is given by

$$t_{E,\max} \sim \frac{1}{\lambda} \log\left(\frac{1}{\hbar_{eff}}\right)$$

As we don't know whether a system is or not ergodic in general, this can be seen as an upper bound for the actual t_E.

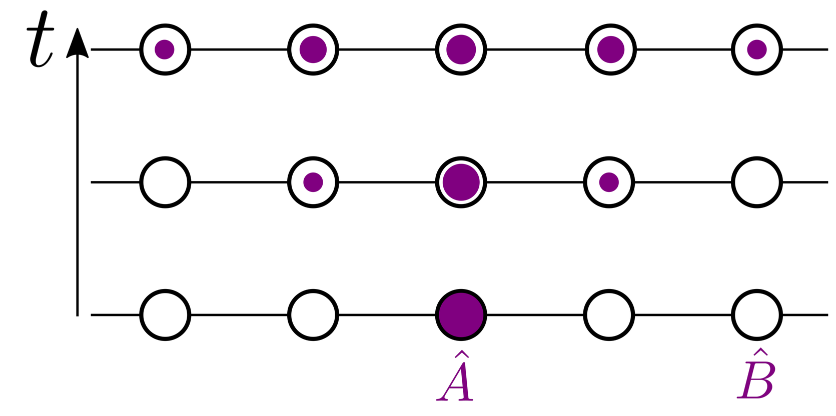
- To quantify the spreading of the wave packet classically, we sampled a coherent state centred around $|5\ 5\ 4\ 3\ 3\rangle$ with a tW approach, and computed the spreading as the average distance between the centre of the distribution and the sampling points, for different initial extensions σ of the packet.



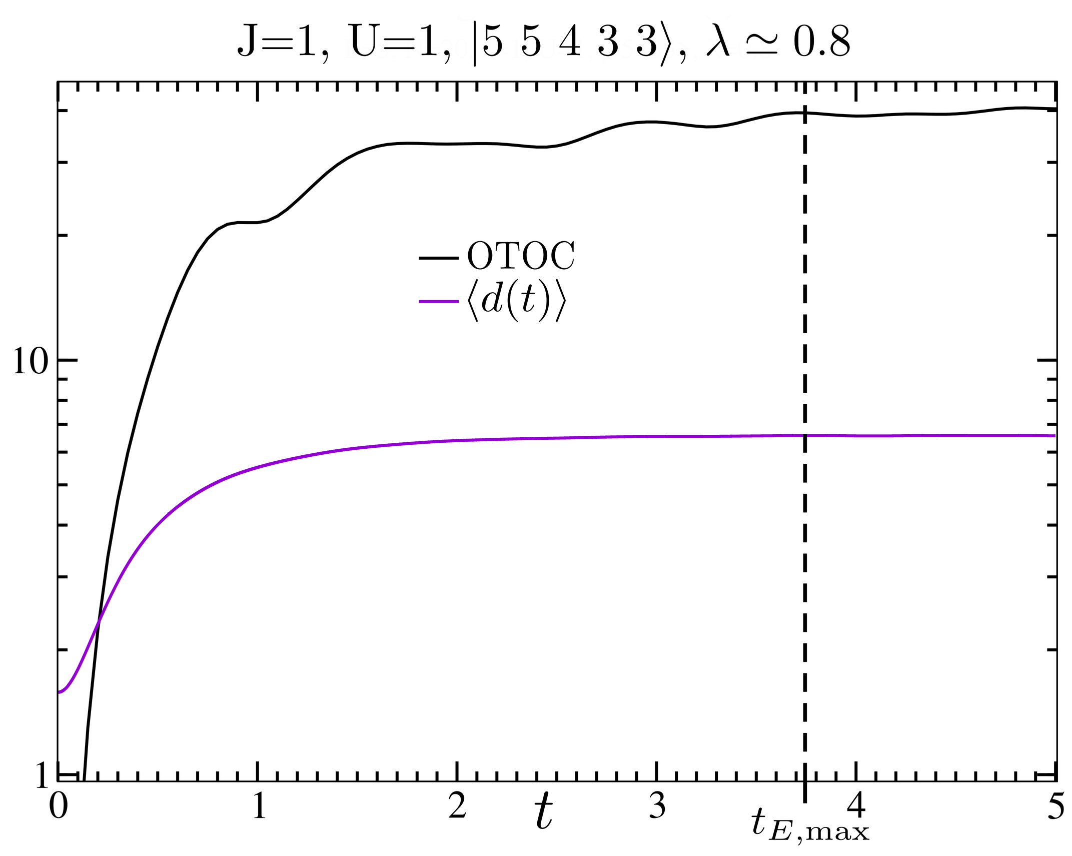
- The sampling-averaged distance saturates after a certain time, translating the spreading of the wave packet over the accessible region of phase space. Saturation at $t = t_E$ is expected when $\sigma \sim 1$.

Out-of-time-ordered correlators (OTOCs)

- $C(t) = \left\langle \left| \left[\hat{A}(t), \hat{B}(0) \right] \right|^2 \right\rangle$
- Characterises the propagation of quantum information in complex quantum systems, *i.e.* scrambling



- Initially zero, then 3 regimes:
  - power-law growth $\sim t^{2s}$ with s the operators separation
  - exponential regime $\sim e^{2\lambda t}$
  - saturation at $t \sim t_E$
- Choice of operators: $\hat{A} = \hat{n}_1$, $\hat{B} = \hat{n}_2$, applied on Fock state $|5\ 5\ 4\ 3\ 3\rangle$



- Both curves saturate at $t \sim \mathcal{O}(t_E)$

Conclusion and challenges

We managed to successfully estimate the Ehrenfest time of a quantum Bose-Hubbard system using its classical equivalent by comparing the time it takes to classically explore the whole accessible region of phase space, using a tW approach, with the expectation value of a quantum operator: an OTOC. We showed that the spreading of the wave packet saturates at the same time as the OTOC, which is known to saturate at the Ehrenfest time.

Several challenges remain:

- Can we compute t_E classically using Fock states?
- What are the effects of scars/dynamical localisation?
- Can t_E be computed directly with classical OTOC calculation?
- Generalisation for Fermi-Hubbard systems?
  [T. Engl et al, Phys. Rev. A 98, 013630 (2018)]

References

- [A. Wolf, et al., Physica 16D (1985)285-317]
- [A. C. Cassidy et al., Phys. Rev. Lett. **102**, 025302 (2009)]
- [T. Engl, J. D. Urbina, K. Richter, Phys. Rev. E**92**, 062907 (2015)]

t.michel@uliege.be