Third-order spectral analysis of an oscillator subjected to wind loads with complex-valued bispectrum

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Abstract
The third-order spectral analysis of an oscillator subjected to time-irreversible wind loads is conducted in this paper. It is proposed to evaluate the statistics of the oscillator's response otherwise than through the numerical integration of the corresponding spectra, in order to avoid the computational burden associated with this operation. To this aim, the Multiple Timescale Spectral Analysis is used. This framework generalizes the Background/Resonant decomposition and offers a rapid yet accurate way to estimate the statistics of the structural responses at any order. In particular, it recently allowed to derive simple approximate expressions for the third central moments of the structural responses obtained under such a specific loading whose bispectrum is actually complex-valued. These formulas are presented in this paper and are eventually shown to provide precise results for the computation of statistics, 100 times faster than before.

Keywords
Bispectrum, Skewness, Background, Bi-Resonant, Time-Irreversible, Non-Gaussian

1. Introduction

In a spectral context, the second and third cumulants of real-valued processes are typically obtained by integrating the real parts of their power spectral density and their bispectrum over a one- and a two-dimensional frequency space, respectively. When dealing with the response of a slightly damped oscillator whose natural frequency is much higher than the characteristic frequency of the loading as it is typical in wind engineering, these spectra are however expected to feature sharp peaks and their numerical integration requires using a lot of points to provide sufficiently accurate results.

From a perturbation perspective, the distinctness of the peaks in the functions to integrate can fortunately be turned into an advantage [10]. The contributions of such separated peaks to the integral can be considered sequentially and be expressed by easily interpretable semi-analytical formulas. Regarding the variance, it yielded the famous background/resonant decomposition, which is widely used by the wind engineering community [2].

Then, it allowed to formalize the general framework of the Multiple Timescale Spectral Analysis, which helps to find similar expressions for higher order or crossed statistics [4]. They are eventually computed much faster than before thanks to the resulting reduction in the dimension of the integrals.

In previous works [5, 3], the third cumulant of a linear oscillator's response to non-Gaussian wind forces has already been decomposed into two parts. To do so, the imaginary part was completely discarded in the bispectrum of loading. This assumption is licit provided that it is a time-reversible process, e.g. a memoryless transformation of a Gaussian input such as the wind velocity. In this event, the imaginary part in the bispectrum of loading is in fact exactly equal to zero everywhere [12].

But in a more general context, it can be significant, although no wind load model is currently able to produce a complex-valued bispectrum. Experimental evidences show that the imaginary part might even be of the same order of magnitude as the real part [6], see the example in Figure 2 which is based on the data collected by [1].

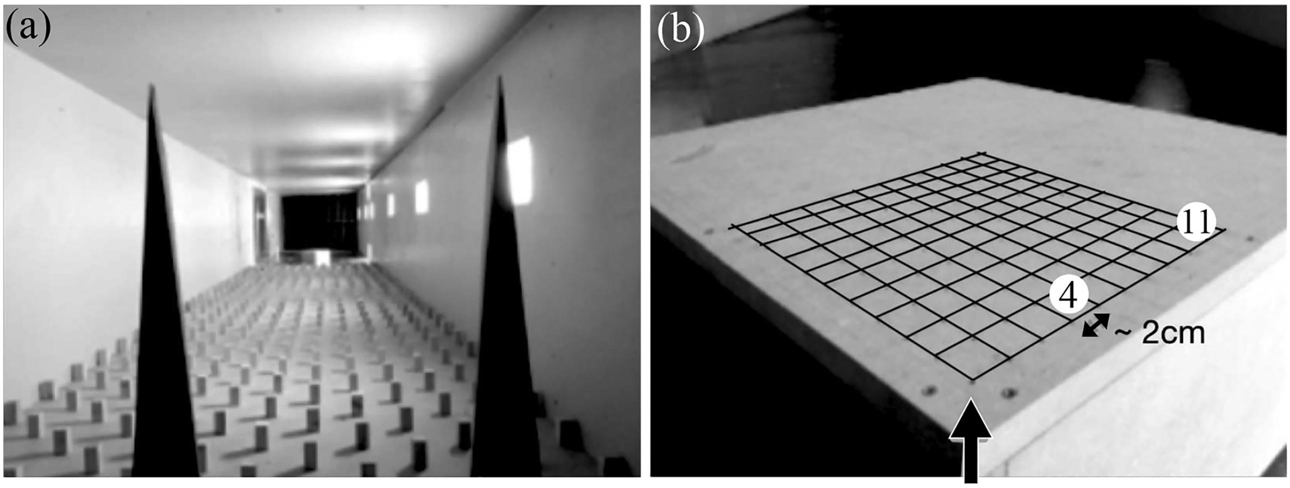

Figure 1. **(a)** Experimental setup in the wind tunnel. **(b)** Location of pressure taps and direction of the wind. [1]

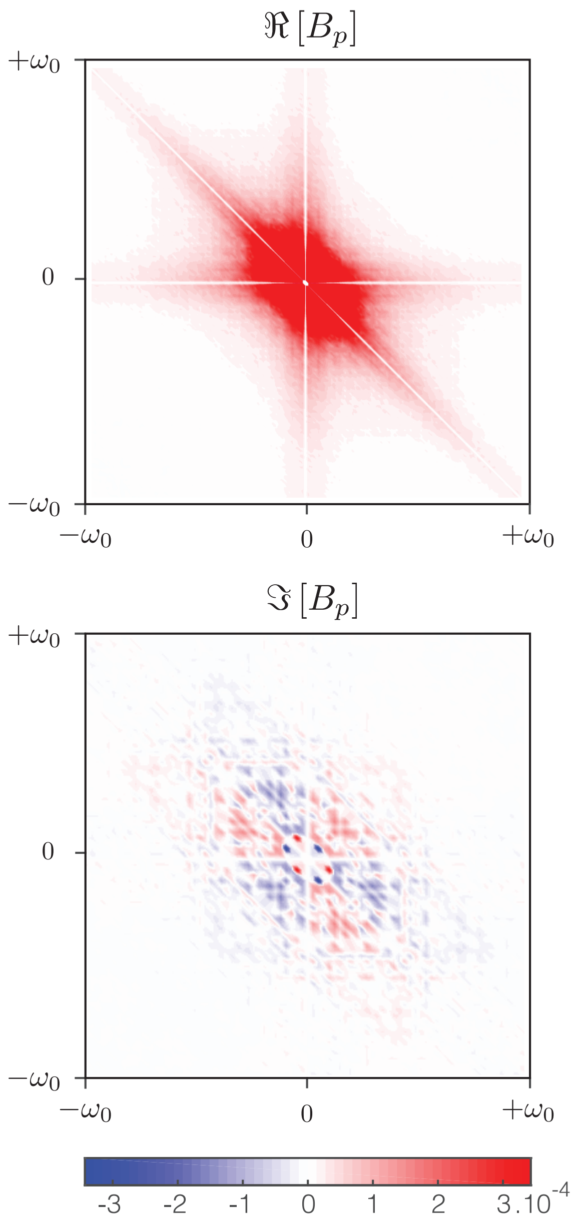

Figure 2. Bispectrum of the wind pressures recorded on the experimental setup on tap n°4.

2. Motivation

In the context described above, the bispectrum of the oscillator's response is theoretically given by

$$B_q(\omega_1,\omega_2) = B_p(\omega_1,\omega_2)K_3(\omega_1,\omega_2) \tag{1}$$

where $B_p(\omega_1,\omega_2)$ is the bispectrum of the loading and $K_3(\omega_1,\omega_2)$ is the structural kernel of the oscillator at third order [3]. It is more specifically given by

$$K_3(\omega_1,\omega_2) = H_x(\omega_1)H_x(\omega_2)H_x^*(\omega_1+\omega_2) \tag{2}$$

where the star superscript denotes the conjugate operation, and depends on the frequency response function

$$H_x(\omega) = (k_s - m_s\omega^2 + 2\imath\omega\xi\sqrt{k_s m_s})^{-1} \tag{3}$$

of the oscillator with k_s being its stiffness, m_s being its mass, ξ being its damping ratio, and hence $\omega_0 = \sqrt{k_s/m_s}$ being its natural frequency.

The third cumulant of the oscillator's response can then be obtained by integrating the corresponding bispectrum as follows

$$\kappa_{3,q} = \iint \Re[B_q(\omega_1,\omega_2)]\mathrm{d}\omega_1\mathrm{d}\omega_2 \tag{4}$$

where

$$\Re[B_q(\omega_1,\omega_2)] = \Re[B_p(\omega_1,\omega_2)]\Re[K_3(\omega_1,\omega_2)] - \Im[B_p(\omega_1,\omega_2)]\Im[K_3(\omega_1,\omega_2)] \tag{5}$$

while the imaginary part of $B_q(\omega_1,\omega_2)$ does not contribute for symmetry reasons. This is in line with the fact that the cumulants of any real-valued process are real-valued as well.

On the one hand, if the imaginary part of the loading bispectrum is negligible, the product of the imaginary parts drops in Eq.(5) and the same conditions as in [3] are met. Using the Multiple Timescale Spectral Analysis in this particular case then, as in [4], shows that the third cumulant can be decomposed into a sum of two major contributions. The first one

$$\kappa_{3,b} = \frac{\kappa_{3,p}}{k_s^3} \tag{6}$$

is referred to as (tri-)background and the second component

$$\kappa_{3,r} = 6\pi \frac{\omega_0^3}{k_s^3} \xi \int \frac{\Re\left[B_p\left(\omega_0, \omega_2\right)\right]}{\left(2\xi\omega_0\right)^2 + \omega_2^2} \, \mathrm{d}\omega_2 \tag{7}$$

is termed background-biresonant. These names indicate the origin of the peaks from which these contributions stem. In particular, the background component is due to the peak of the loading bispectrum which is located in the low frequencies. Meanwhile, the biresonance is associated with the crossing of two lines of poles in the frequency response functions which constitute the structural kernel.

On the other hand, if the imaginary part of the loading bispectrum is significant, an additional background-biresonant contribution has to be considered. It has more recently been expressed as

$$\kappa_{3,i} = 3\pi \frac{\omega_0^2}{k_s^3} \int \frac{\Im\left[B_p\left(\omega_0, \omega_2\right)\right]}{\left(2\xi\omega_0\right)^2 + \omega_2^2} \omega_2 \mathrm{d}\omega_2 \tag{8}$$

by using the Multiple Timescale Spectral Analysis. Details about the establishment of this formula can be found in [7].

However, until now, it was difficult to verify that the latter formula was correct and necessary, since wind models currently generate real-valued bispectra only, while experimental studies show that it can indeed be complex-valued. In this paper, these two questions are therefore addressed by means of an example for which it has been decided to estimate the bispectrum of loading through a parametric approach based on an auto-regressive loading model [11]. It is thus expressed as follows

$$B_p(\omega_1, \omega_2) = H_p(\omega_1) H_p(\omega_2) H_p^*(\omega_1 + \omega_2) \tag{9}$$

with

$$H_p(\omega) = \frac{1}{1 + \alpha i \omega} \tag{10}$$

where $\alpha = \omega_p / \omega_0$ stands for the frequency ratio between the characteristic frequencies of the wind load and the oscillator.

3. Parametric Analysis

Globally, the results presented in Figure 3-(h) and Figure 3-(i) can be explained quite easily thanks to the simple expressions that have been obtained for the main components of the third central moment of the response.

First, the approximations provided by their sums are verified as their ratio with the reference values are close to one and are getting even closer when the frequency ratio and the damping ratio decrease. This is indeed to be expected given that the multiple timescale spectral analysis is based on perturbation methods and its accuracy is consequently conditioned upon the smallness of these two parameters.

Second, the background to bi-resonant ratio decreases with the damping ratio. By contrast, the bi-resonant component increases with the frequency ratio. Small damping ratios and large frequency ratios result in a relatively more resonant structural response, which is less skewed and thus more Gaussian, as shown in Figure 3-(i). This corroborates the central limit theorem.

At last, the bi-resonant component related to the imaginary part of the loading bispectrum is clearly not negligible in the example at stake.

4. Conclusions

Thanks to the meaningful analytical expressions provided by the Multiple Timescale Spectral Analysis for the third central moments of the oscillator's response, it appears that a significant contribution can sometimes be attributed to the existence of an imaginary part in the bispectrum of loading.

This raises a question about the validity of traditional wind load models in some circumstances, given that they are only able to produce real-valued bispectra of forces. New wind load models, which create time-irreversible processes with complex-valued bispectra, should therefore be proposed in the future to solve this problem.

But overall, the Multiple Timescale Spectral Analysis is already able to deal with the response of an oscillator subjected to such a loading process. The expressions detailed in Section 2 help to understand how the response is influenced by the loading and by the structural parameters, respectively. They thus allow to determine how the structure can be modified to exhibit a safer dynamical behavior, based on simple and sound mathematics.

Hinging on the perturbation theory, the discrepancy is also known to remain limited provided that the damping ratio and the frequency ratio are sufficiently

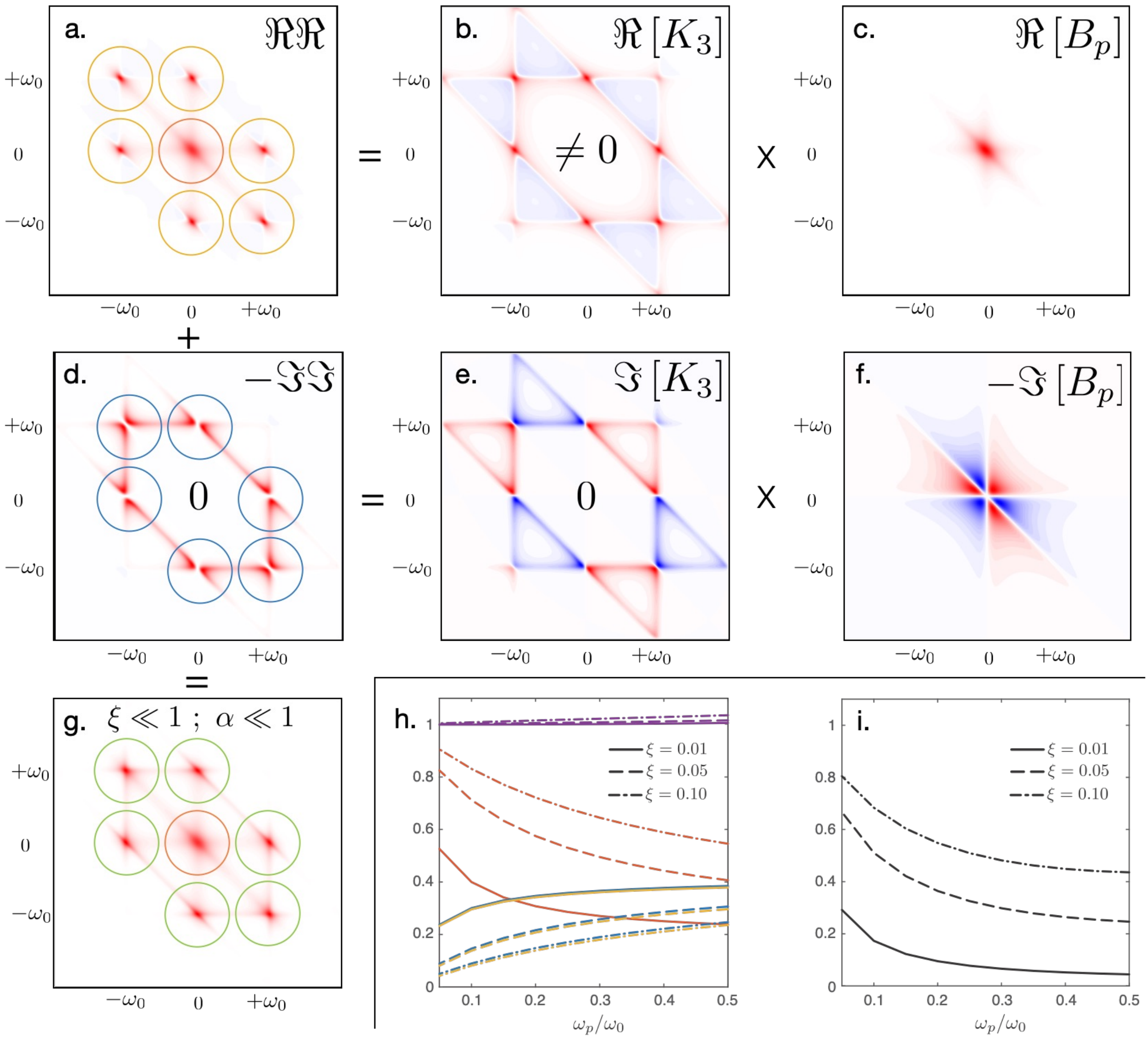

Figure 3. (a) Product of (b) and (c) with (b) the real part of the structural kernel and (c) the real part of the loading bispectrum. (d) Minus the product of (e) and (f) with (e) the imaginary part of the structural kernel and (f) the imaginary part of the loading bispectrum. (g) Sum of (a) and (d), which gives the real part of the response bispectrum. (h) In purple, the ratio between the third central moments of the response obtained through the proposed decomposition and through the numerical integration of the bispectrum, then in orange, yellow and blue, the distribution of the estimated results between the background and the bi-resonant components associated to either the real part, either the imaginary part of the loading bispectrum, respectively. (i) Dynamic amplification of the skewness of the response with respect to the skewness of the loading.

smaller than unity. Last but not least, the computational time is divided by about one hundred at least when using the expressions derived in this paper for the third central moments of the responses. This is explained by the fact that integrating their bispectrum over a two-dimensional frequency space is avoided.

The Multiple Timescale Spectral Analysis has thus proved its worth and is currently being developed to provide such expressions for the statistics of structures with many degrees-of-freedom, subjected to self-excited forces [9], or wave forces which trigger the inertial regime as well [8], on top of the background and the resonant ones.

5. Acknowledgements

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