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TECHNISCHE UNIVERSITÄT DARMSTADT

Mixed Finite Element Formulations for Systems with Superconductors and Ferromagnetic Materials

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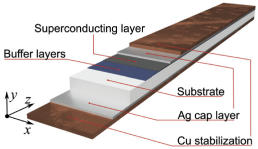
CERN

ACOMEN 2022

Introduction

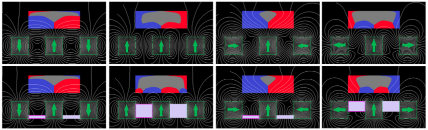
We model eddy current problems for high-temperature superconductors (HTS) and ferromagnetic materials (FM).

Coated HTS tape



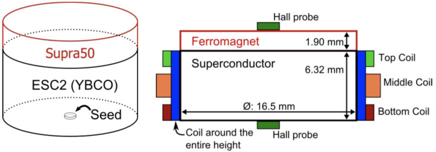
[Solovyov, Supercond. Sci. Technol., 2013]

Magnetic levitation



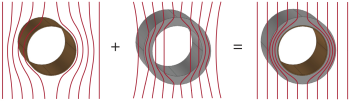
[Huang, Supercond. Sci. Technol., 2015]

Trapped-field magnet



[Philipe, Physica C Superconductivity, 2014]

Magnetic cloak



[Capobianco-Hogan, Nucl. Instrum. Methods Phys. Res., 2018]

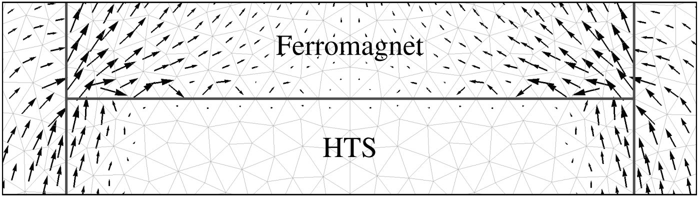
Context

Coupled formulations offer many advantages for HTS-FM modeling:

- improved efficiency for nonlinear system resolution,
- reduced number of DOFs,
- increased flexibility...

However, they enter the framework of mixed formulations, thus requiring to be extremely careful regarding function spaces.

Otherwise, non-physical results must be expected:



Strong form

- Magnetodynamic (quasistatic) equations

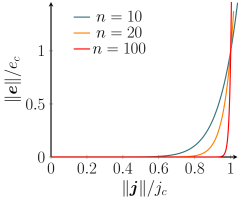
$$\mathrm{div}\, \boldsymbol{b} = 0, \qquad \mathbf{curl}\, \boldsymbol{h} = \boldsymbol{j}, \qquad \mathbf{curl}\, \boldsymbol{e} = -\partial_t \boldsymbol{b}.$$

- Constitutive relationships

High-temperature superconductors (HTS):

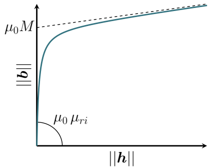
$$\boldsymbol{e} = \rho(\|\boldsymbol{j}\|)\boldsymbol{j} \quad \text{and} \quad \boldsymbol{b} = \mu_0\, \boldsymbol{h},$$

with the power law $\rho(\|\boldsymbol{j}\|) = \frac{e_c}{j_c}\left(\frac{\|\boldsymbol{j}\|}{j_c}\right)^{n-1}$.



Ferromagnetic material (FM):

$$\boldsymbol{b} = \mu(\boldsymbol{b})\, \boldsymbol{h} \quad \text{and} \quad \boldsymbol{j} = \boldsymbol{0}.$$



Dual formulations

Two classes of formulations with the finite element method:

- h-conform, e.g. h-formulation,
  - enforces the continuity of the tangential component of $\boldsymbol{h}$,
  - involves $\boldsymbol{e} = \rho \boldsymbol{j}$ and $\boldsymbol{b} = \mu \boldsymbol{h}$,
  - with $\mathbf{curl}\, \boldsymbol{h} = \mathbf{0}$ in non-conducting domain ("$\boldsymbol{h}$-ϕ"+cuts),

$$\left(\partial_t(\mu \boldsymbol{h}) \,, \boldsymbol{h}'\right)_{\Omega} + \left(\rho \, \mathbf{curl}\, \boldsymbol{h} \,, \mathbf{curl}\, \boldsymbol{h}'\right)_{\Omega_c} - \left\langle \boldsymbol{e} \times \boldsymbol{n} \,, \boldsymbol{h}' \right\rangle_{\Gamma_e} = 0.$$

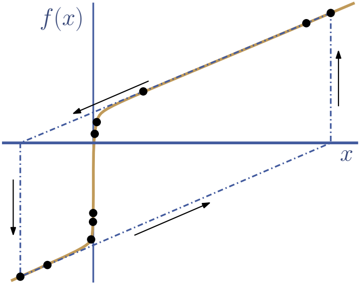
- b-conform, e.g. a-formulation,
  - enforces the continuity of the normal component of $\boldsymbol{b}$,
  - involves $\boldsymbol{j} = \sigma \boldsymbol{e}$ and $\boldsymbol{h} = \nu \boldsymbol{b}$, ($\sigma = \rho^{-1}$, $\nu = \mu^{-1}$)

$$\left(\nu \, \mathbf{curl}\, \boldsymbol{a} \,, \mathbf{curl}\, \boldsymbol{a}'\right)_{\Omega} + \left(\sigma \, \partial_t \boldsymbol{a} \,, \boldsymbol{a}'\right)_{\Omega_c} - \left\langle \boldsymbol{h} \times \boldsymbol{n} \,, \boldsymbol{a}' \right\rangle_{\Gamma_h} = 0.$$

Nonlinear constitutive laws involved in opposite ways $\Rightarrow$ very different numerical behaviors are expected... and observed.

Best choice for HTS only

Cycles in iterations:



In the a-formulation, the diverging slope associated with $\boldsymbol{j} = \sigma \boldsymbol{e}$ for $\boldsymbol{e} \to 0$ is really difficult to handle.

$\Rightarrow$ Among the two simple formulations, the h-formulation is much more efficient for systems with HTS:

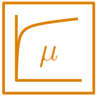
- with an adaptive time-stepping algorithm,
- solved with a Newton-Raphson method.

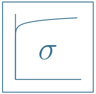
Dular, J., et al. (2020) TAS 30 8200113.

Ferromagnetic materials

The nonlinearity is in the magnetic constitutive law.

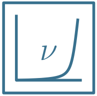
- h-formulation the involved law is $\boldsymbol{b} = \mu \boldsymbol{h}$.

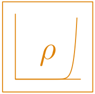
  

  $\approx$
  

  $\Rightarrow$ Often enters cycles with Newton-Raphson.
  OK with fixed point, or N-R with relaxation factors but slow.

- a-formulation the involved law is $\boldsymbol{h} = \nu \boldsymbol{b}$.

  $\approx$
  

  $\Rightarrow$ Efficiently solved with Newton-Raphson.

The a-formulation is more appropriate for dealing with the nonlinearity, whereas for HTS, the h-formulation is best.

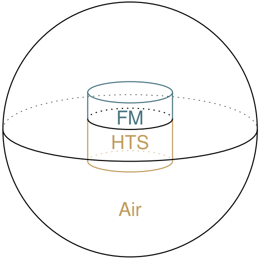
Coupled materials - h-a-formulation

Use the best formulation in each material

Decompose the domain Ω, for example into:

- $\Omega^h = \{\text{HTS, Air}\}$
- $\Omega^a = \{\text{Ferromagnet}\}$

and couple via $\Gamma_\text{m} = \partial(\text{FM})$:



$$\left(\partial_t(\mu \boldsymbol{h})\ , \boldsymbol{h}'\right)_{\Omega^h} + \left(\rho\,\textbf{curl}\,\boldsymbol{h}\ , \textbf{curl}\,\boldsymbol{h}'\right)_{\Omega_\text{c}^h} + \left\langle \partial_t \boldsymbol{a} \times \boldsymbol{n}_{\Omega^h}\ , \boldsymbol{h}' \right\rangle_{\Gamma_\text{m}} = 0,$$

$$\left\langle \boldsymbol{h} \times \boldsymbol{n}_{\Omega^a}\ , \boldsymbol{a}' \right\rangle_{\Gamma_\text{m}} - \left(\nu\,\textbf{curl}\,\boldsymbol{a}\ , \textbf{curl}\,\boldsymbol{a}'\right)_{\Omega^a} = 0.$$

Dular, J., et al. (2020) TAS 30 8200113.
See also: Brambila R. et al, (2018) TAS 28, 5207511.

Perturbed saddle point problem

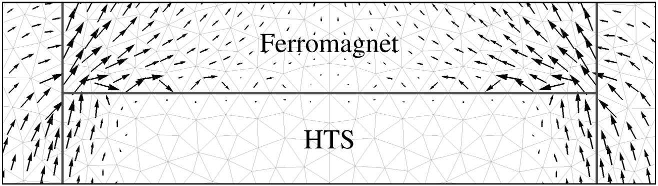
$$\left(\partial_t(\mu \boldsymbol{h}),\boldsymbol{h}'\right)_{\Omega^h} + \left(\rho\,\mathbf{curl}\,\boldsymbol{h},\mathbf{curl}\,\boldsymbol{h}'\right)_{\Omega^h_{\mathrm{c}}} + \left\langle \partial_t \boldsymbol{a}\times\boldsymbol{n}_{\Omega^h},\boldsymbol{h}'\right\rangle_{\Gamma_{\mathrm{m}}} = 0, \quad \forall \boldsymbol{h}'\in\mathcal{H},$$
$$\left\langle \boldsymbol{h}\times\boldsymbol{n}_{\Omega^a},\boldsymbol{a}'\right\rangle_{\Gamma_{\mathrm{m}}} - \left(\nu\,\mathbf{curl}\,\boldsymbol{a},\mathbf{curl}\,\boldsymbol{a}'\right)_{\Omega^a} = 0, \quad \forall \boldsymbol{a}'\in\mathcal{A}.$$

It is a perturbed saddle point problem:

$$\begin{cases} a(u,v)+b(v,p)=\langle f,v\rangle, & \forall v\in V,\\ b(u,q)-c(p,q)=\langle g,q\rangle, & \forall q\in Q, \end{cases} \quad \text{or} \quad \begin{pmatrix}\mathbf{A} & \mathbf{B}^{\mathsf{T}}\\ \mathbf{B} & -\mathbf{C}\end{pmatrix}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{p}\end{pmatrix}=\begin{pmatrix}\boldsymbol{f}\\ \boldsymbol{g}\end{pmatrix}.$$

$\Rightarrow$ Compatibility conditions for numerical stability, otherwise…

First-order functions for $\boldsymbol{h}$ and $\boldsymbol{a}$:



D. Boffi, F. Brezzi, et al., Mixed FE methods and applications, Springer, 2013.

Compatibility conditions

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^\mathsf{T} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}.$$

The solution is stable, i.e., $\|\boldsymbol{u}\|_V + \|\boldsymbol{p}\|_Q \leq C(\|\boldsymbol{f}\|_{V'} + \|\boldsymbol{g}\|_{Q'})$, if $\exists \alpha, \beta, \gamma > 0$ (strictly) such that

$$\boldsymbol{v}^\mathsf{T}\mathbf{A}\boldsymbol{v} \geq \alpha\|\boldsymbol{v}\|_V^2, \ \forall \boldsymbol{v} \in \ker(\mathbf{B}) \qquad \text{(coercivity of } \mathbf{A}\text{)},$$

$$\boldsymbol{q}^\mathsf{T}\mathbf{C}\boldsymbol{q} \geq \gamma\|\boldsymbol{q}\|_Q^2, \ \forall \boldsymbol{q} \in \ker(\mathbf{B}^\mathsf{T}) \qquad \text{(coercivity of } \mathbf{C}\text{)},$$

$$\inf_{\boldsymbol{q} \in (\ker(\mathbf{B}^\mathsf{T}))^\perp} \sup_{\boldsymbol{v} \in V} \frac{\boldsymbol{q}^\mathsf{T}\mathbf{B}\boldsymbol{v}}{\|\boldsymbol{q}\|_Q \|\boldsymbol{v}\|_V} \geq \beta > 0 \qquad \text{(inf-sup condition)}.$$

In our case, the inf-sup condition is the most restrictive.

D. Boffi, F. Brezzi, et al., *Mixed FE methods and applications*, Springer, 2013.

Inf-sup test

The inf-sup condition is not easy to check analytically.

⇒ We perform a numerical inf-sup test.

On progressively refined meshes, for given function spaces:

1. Define suitable norms.
2. Extract matrices $\mathbf{B}$, $\mathbf{N}_V$, and $\mathbf{N}_Q$, from the FE assembly, with

$$\|\boldsymbol{v}\|_V^2 = \boldsymbol{v}^\mathsf{T}\mathbf{N}_V\boldsymbol{v},$$
$$\|\boldsymbol{q}\|_Q^2 = \boldsymbol{q}^\mathsf{T}\mathbf{N}_Q\boldsymbol{q}.$$

3. Solve the eigenvalue problem

$$\left(\mathbf{B}\mathbf{N}_V^{-1}\mathbf{B}^\mathsf{T}\right)\boldsymbol{q} = \lambda\mathbf{N}_Q\boldsymbol{q}.$$

Lowest non-zero eigenvalue $=$ square of the inf-sup value β^δ.

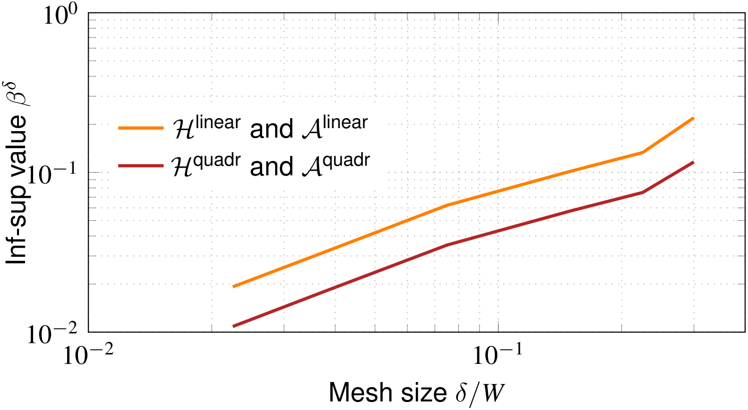
⇒ How does β^δ behave when the mesh is refined?

- It tends to zero ⇒ unstable,
- It is bounded from below ⇒ stable.

D. Chapelle, K.-J. Bathe, *The inf-sup test*, C&S 47, 1993.

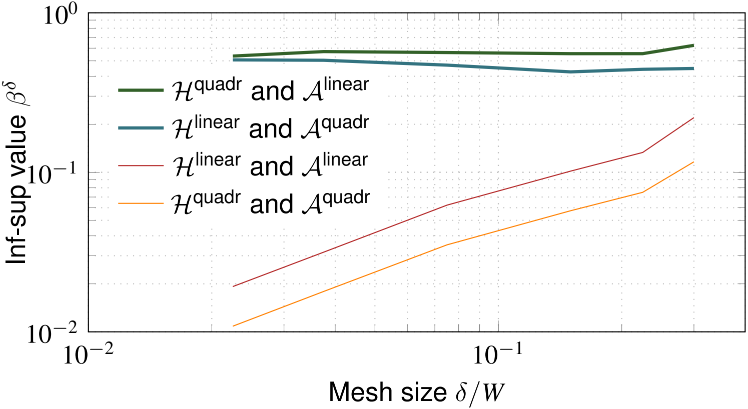
Unstable choices

Linear or quadratic elements for both $\boldsymbol{h}$ and $\boldsymbol{a}$ $\Rightarrow$ Unstable.



h-a-formulation Stable choices

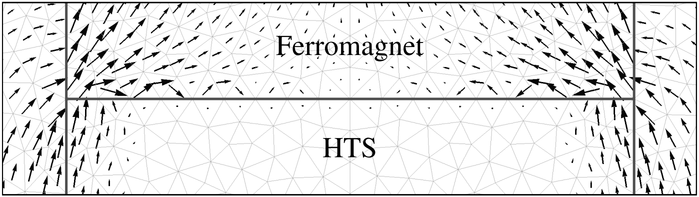
One way to stabilize the problem:
$\Rightarrow$ Increase the discretization order of one field ($\boldsymbol{h}$ or $\boldsymbol{a}$).



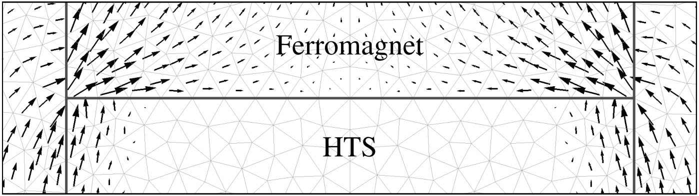
Increasing the order on the coupling interface only is sufficient.

h-a-formulation Stabilization

- First-order functions for $\boldsymbol{h}$ and $\boldsymbol{a}$ (inf-sup KO):

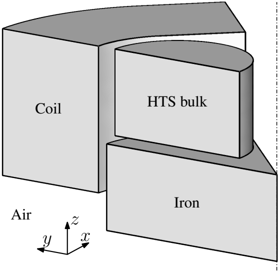


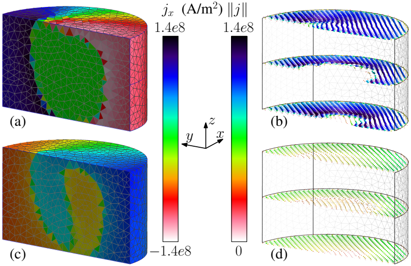
- Second-order for $\boldsymbol{a}$, first-order for $\boldsymbol{h}$ (inf-sup OK):



Application 1: HTS bulk magnetization model (3D)

HTS bulk magnetization with a coil, on top of a FM pellet.

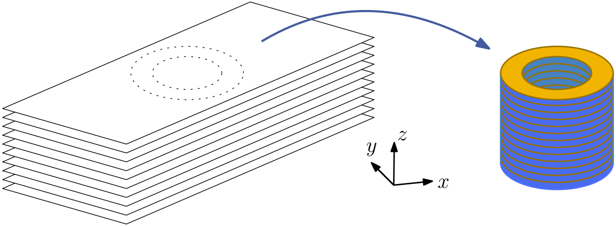




	# DOFs	# iterations	Time/it.	Total time
h-formulation	12,172	3,937	1.4s	1h33
a-formulation	26,964	3,147	2.1s	1h48
h-a-formulation	15,776	1,108	2.1s	**0h39**
h-b-formulation	20,821	1,104	3.2s	0h58

Application 2: magnetic shield model (2D and 3D)

Magnetic shield made up of a stack of tape annuli.



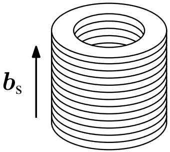
Inner radius: 13 mm. Outer radius: 22.5 mm. Height: 14.9 mm.

S. Hahn, 2011. A. Patel, 2016.

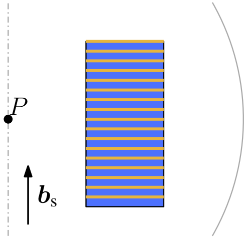
- Number of tapes: $N = 183$. One tape: HTS layer $+$ FM substrate.
- Filling factor of the FM: $f = 0.92$.
- Temperature: 77K.
- Modeled with limited number of tapes.

Shielding configurations

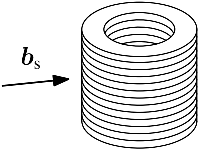
Axial



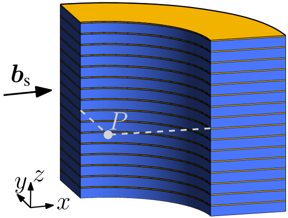
2D-axisymmetric



Transverse



3D



Magnetic shielding application

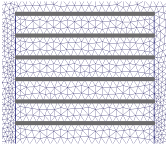
h-*b*-formulation

- h-ϕ in Ω and auxiliary $\boldsymbol{b}$ field in the FM domain Ω_m.
  Volume coupling in Ω_m:

$$\left(\mu_0 \partial_t \boldsymbol{h} \,, \boldsymbol{h}'\right)_{\Omega_\mathrm{m}^\mathrm{C}} + \left(\rho \,\mathbf{curl}\, \boldsymbol{h} \,, \mathbf{curl}\, \boldsymbol{h}'\right)_{\Omega_\mathrm{c}} + \left(\partial_t \boldsymbol{b} \,, \boldsymbol{h}'\right)_{\Omega_\mathrm{m}} = 0$$

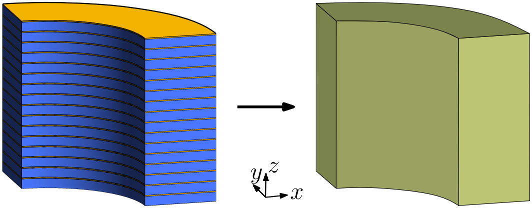
$$\left(\boldsymbol{h} \,, \boldsymbol{b}'\right)_{\Omega_\mathrm{m}} - \left(\nu \boldsymbol{b} \,, \boldsymbol{b}'\right)_{\Omega_\mathrm{m}} = 0$$

- If Ω_m is non-conducting, inf-sup condition satisfied with piecewise constant elements for $\boldsymbol{b}$.
- Much more robust than *h*-formulation.
- More efficient than *h*-*a*-formulation because of large coupling surface:



Homogeneous model: anisotropy

Replace the detailed stack by one homogeneous material.



- Introduce the average $\boldsymbol{h}$ and $\boldsymbol{j}$ fields.
- Introduce anisotropic $\tilde{\rho}(\boldsymbol{j})$ and $\tilde{\mu}(\boldsymbol{h})$ tensors.
- Modified h-formulation:

$$\left(\partial_t(\tilde{\mu}\,\boldsymbol{h})\;,\boldsymbol{h}'\right)_{\Omega} + \left(\tilde{\rho}\,\textbf{curl}\,\boldsymbol{h}\;,\textbf{curl}\,\boldsymbol{h}'\right)_{\Omega_c} = 0$$

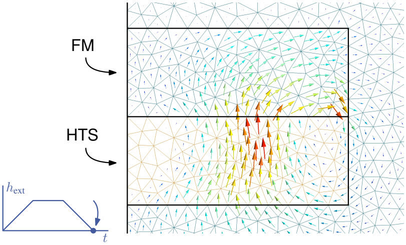
- Not optimal: how to apply the h-b-formulation with anisotropy and conducting Ω_m domain?

Conclusion

Coupled formulations help to model HTS and FM efficiently

- Surface coupling ⇒ h-a-formulation
- Volume coupling ⇒ h-b-formulation
- *Thin HTS tapes ⇒ t-a-formulation (not presented here).*

These formulations are mixed ⇒ Inf-sup condition for stability.



References

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- *Mixed and hybrid finite element methods*,
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- *On the Stability of Mixed Finite-Element Formulations for High-Temperature Superconductors*,
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- *What Formulation Should One Choose for Modeling a 3D HTS Motor Pole with Ferromagnetic Materials?*,
  J. Dular, K. Berger, C. Geuzaine, and B. Vanderheyden, TM 58 (9), 1-11 (2022).