Phase resonance of an oscillator with polynomial stiffness

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Summary. This paper extends the linear concept of phase resonance, which occurs when the damping forces counterbalance exactly the external forces, to oscillators with polynomial stiffness. To this end, a first-order averaging technique is applied to a one degree-of-freedom oscillator with arbitrary polynomial stiffness. We show that phase resonance exists in the vicinity of amplitude resonance and is associated with a phase resonance of $\pi/2$.

Introduction

Modal analysis has been, and continues to be, the dominant dynamical method used in structural design. The goal of modal analysis is to find the vibration modes, resonance frequencies and damping ratios of the considered system [1]. One key assumption of modal analysis is linearity.
In linear theory, the resonant behavior of dynamical systems can be characterized either the amplitude or phase resonance. Amplitude resonance corresponds to a relative maximum in the frequency response function whereas phase resonance is associated with quadrature between the displacement and the external forcing. At phase resonance, the external forcing cancels exactly the damping force with the result that the resonance frequency coincides with the natural frequency of the linear system. The difference between the two resonances remains small for weakly damped systems.
However, real-world structures are intrinsically nonlinear because they may feature advanced materials, friction and contact [2]. In this context, the present study proposes to extend the concept of phase resonance to oscillators with arbitrary polynomial stiffness. To do so, a first-order averaging technique is applied to a one degree-of-freedom oscillator and we show that phase resonance exists in the vicinity of amplitude resonance for a phase lag of $\pi/2$.

Oscillator with polynomial stiffness

The governing equation of motion of a harmonically-forced oscillator with arbitrary polynomial stiffness is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \sum_{d=2}^{\infty} k_d x^d(t) = f \sin \omega t \tag{1}$$

where m, c, k and k_d represent the mass, damping, linear and nonlinear stiffness coefficients, respectively. f is the forcing amplitude whereas ω is the excitation frequency of period T. The natural frequency of the undamped, linearized system is $\omega_0 = \sqrt{\frac{k}{m}}$. Through mass normalization, Equation (1) can be recast into:

$$\ddot{x}(t) + 2\bar{\zeta}\,\omega_0\,\dot{x}(t) + \omega_0^2 x(t) + \sum_{d=2}^{\infty} \bar{\alpha}_d x^d(t) = \bar{\gamma} \sin \omega t \tag{2}$$

where $\bar{\zeta} = \frac{c}{2\sqrt{km}}$, $\alpha_d = k_d/m$ and $\bar{\gamma} = f/m$.

An averaging technique

We consider a weakly nonlinear oscillator of the form:

$$\ddot{x}(t) + \omega_0^2 x(t) = \varepsilon f(x(t), \dot{x}(t)) \tag{3}$$

When $\varepsilon = 0$, the periodic solution of (3) is written as:

$$x(t) = u \cos \omega_0\, t - v \sin \omega_0\, t \tag{4}$$

where u and v are constants. When $\varepsilon \neq 0$, we seek a solution of frequency ω such that $\omega^2 - {\omega_0}^2 = \varepsilon\,\Omega$. The solution is expressed as in Equation (4) but with time-dependent u and v:

$$x(t) = u(t) \cos \omega\, t - v(t) \sin \omega\, t \tag{5}$$

We impose that the velocity should have the same form as in the case $\varepsilon = 0$, *i.e.*,

$$\dot{x}(t) = -u(t)\,\omega \sin \omega\, t - v(t)\,\omega \cos \omega\, t \tag{6}$$

Equation (6) holds if:

$$\dot{u}(t) \cos \omega\, t - \dot{v}(t) \sin \omega\, t = 0 \tag{7}$$

Differentiating Equation (6) and replacing $\ddot{x}(t)$ and $x(t)$ in Equation (3) yields:

$$\dot{u}(t)\,\omega\sin\omega\,t + \dot{v}(t)\,\omega\cos\omega\,t = -\varepsilon\left[f(x(t),\dot{x}(t)) + \Omega\,x(t)\right] \tag{8}$$

Finally, taking into account Equations (7) and (8) and solving for $\dot{u}$ and $\dot{v}$, a system of first-order equations is obtained:

$$\begin{cases} \dot{u} = -\frac{\varepsilon}{\omega}\left[f(x(t),\dot{x}(t)) + \Omega\,x(t)\right]\sin\omega\,t \\ \dot{v} = -\frac{\varepsilon}{\omega}\left[f(x(t),\dot{x}(t)) + \Omega\,x(t)\right]\cos\omega\,t \end{cases} \tag{9}$$

This system has a suitable form to apply first-order averaging, which is performed herein using the Krylov-Bogolyubov technique [3, 4], which consists in integrating these equations over one period of time T, during which u and v are considered to be constants:

$$\begin{cases} \dot{u} = -\frac{\varepsilon}{\omega}\frac{1}{T}\int_0^T\left[f(x(t),\dot{x}(t)) + \Omega\,x(t)\right]\sin\omega\,t\,\mathrm{d}t \\ \dot{v} = -\frac{\varepsilon}{\omega}\frac{1}{T}\int_0^T\left[f(x(t),\dot{x}(t)) + \Omega\,x(t)\right]\cos\omega\,t\,\mathrm{d}t \end{cases} \tag{10}$$

Or alternatively, if we consider $\omega\,t = \theta$:

$$\begin{cases} \dot{u} = -\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi}\left[f(x(\theta),\dot{x}(\theta)) + \Omega\,x(\theta)\right]\sin\theta\,\mathrm{d}\theta \\ \dot{v} = -\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi}\left[f(x(\theta),\dot{x}(\theta)) + \Omega\,x(\theta)\right]\cos\theta\,\mathrm{d}\theta \end{cases} \tag{11}$$

Finally, $x(t)$ is often represented using the polar coordinates r and ϕ such that $x(t) = r(t)\sin\left(\omega\,t - \phi(t)\right)$ with $r = \sqrt{u^2+v^2}$ and $\phi = \mathrm{atan2}(-u,-v)$, where $u = -r\sin\phi$ and $v = -r\cos\phi$. Furthermore, we can express the time derivatives of r and ϕ as:

$$\begin{cases} \dot{r} = \frac{\partial r}{\partial u}\dot{u} + \frac{\partial r}{\partial v}\dot{v} = \frac{u}{r}\dot{u} + \frac{v}{r}\dot{v} \\ \dot{\phi} = \frac{\partial\phi}{\partial u}\dot{u} + \frac{\partial\phi}{\partial v}\dot{v} = \frac{v}{r^2}\dot{u} - \frac{u}{r^2}\dot{v} \end{cases} \tag{12}$$

For conciseness, the time dependence for u, v, r and ϕ is dropped in the remainder of this chapter.

First-order averaging of an oscillator with polynomial stiffness

Scaling of the equation of motion

Considering Equation (2), we scale the system such that $\bar{\zeta} = \varepsilon\zeta$, $\bar{\alpha_d} = \varepsilon\alpha_d$ and $\bar{\gamma} = \varepsilon\gamma$, with ζ, α, $\gamma = \mathcal{O}(1)$, we obtain a weakly nonlinear oscillator:

$$\ddot{x}(t) + \omega_0^2 x(t) = \varepsilon\left(\gamma\sin\omega t - 2\zeta\,\omega_0\,\dot{x}(t) - \sum_{d=2}^{\infty}\alpha_d x^d(t)\right) \tag{13}$$

Assuming a forcing frequency in the vicinity of the natural frequency of the linear system, i.e., $\omega^2 - {\omega_0}^2 = \varepsilon\Omega$, we can apply an averaging technique and the displacement as explained in Section . This consists in solving:

$$\begin{cases} \dot{u} = -\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi}\left[\left(\gamma\sin\theta - 2\zeta\,\omega_0\,\dot{x}(\theta) - \sum_{d=2}^{\infty}\alpha_d x^d(\theta)\right) + \Omega\,x(\theta)\right]\sin\theta\,\mathrm{d}\theta \\ \dot{v} = -\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi}\left[\left(\gamma\sin\theta - 2\zeta\,\omega_0\,\dot{x}(\theta) - \sum_{d=2}^{\infty}\alpha_d x^d(\theta)\right) + \Omega\,x(\theta)\right]\cos\theta\,\mathrm{d}\theta \end{cases} \tag{14}$$

For clarity, the different terms are analysed separately, *i.e.*, the forcing, damping, frequency and stiffness terms. Furthermore, to solve these integrals, we make use of the fact that:

$$\int_0^{2\pi}\cos^a\theta\sin^b\theta\,\mathrm{d}\theta = \frac{1}{2}\left[(-1)^a+1\right]\left[(-1)^b+1\right]\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)} \tag{15}$$

which is always equal to 0 if either a or b is odd. Therefore, we can write:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^{2n}\theta\sin^{2m}\theta\,\mathrm{d}\theta = \frac{1}{\pi}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(n+m+\frac{1}{2}\right)} \tag{16}$$

where Γ is the *Gamma* function.

Forcing term

For $\dot{u}$ and $\dot{v}$, we have respectively:

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma\sin^2\theta\,\mathrm{d}\theta = \frac{\gamma}{2} \tag{17}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\gamma\cos\theta\sin\theta\,\mathrm{d}\theta = 0 \tag{18}$$

Damping term
For $\dot{u}$ and $\dot{v}$, we have respectively:

$$-\frac{1}{2\pi}\int_0^{2\pi} 2\zeta\,\omega_0\left(-u\,\omega\sin^2\theta - v\,\omega\cos\theta\sin\theta\right)\,\mathrm{d}\theta = \zeta\,\omega_0\,\omega\,u \tag{19}$$

and

$$-\frac{1}{2\pi}\int_0^{2\pi} 2\zeta\,\omega_0\left(-u\,\omega\cos\theta\sin\theta - v\,\omega\cos^2\theta\right)\,\mathrm{d}\theta = \zeta\,\omega_0\,\omega\,v \tag{20}$$

Frequency term
For $\dot{u}$ and $\dot{v}$, we have respectively:

$$\frac{1}{2\pi}\int_0^{2\pi} \Omega\left(u\cos\theta\sin\theta - v\sin^2\theta\right)\,\mathrm{d}\theta = -\frac{\Omega\,v}{2} \tag{21}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \Omega\left(u\cos^2\theta - v\cos\theta\sin\theta\right)\,\mathrm{d}\theta = \frac{\Omega\,u}{2} \tag{22}$$

Polynomial stiffness terms
For $\dot{u}$ and $\dot{v}$, we need to solve respectively:

$$-\sum_{d=2}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\alpha_d\left(u\cos\theta - v\sin\theta\right)^d\sin\theta\,\mathrm{d}\theta \tag{23}$$

and

$$-\sum_{d=2}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\alpha_d\left(u\cos\theta - v\sin\theta\right)^d\cos\theta\,\mathrm{d}\theta \tag{24}$$

To do so, we need to expand the polynomial term using the binomial expansion:

$$(u\cos\theta - v\sin\theta)^d = \sum_{p=0}^{d}\binom{d}{p}(u\cos\phi)^{d-p}(-v\sin\theta)^p \tag{25}$$

which thus gives for $\dot{u}$ and $\dot{v}$, respectively:

$$-\sum_{d=2}^{\infty}\alpha_d\sum_{p=0}^{d}\binom{d}{p}u^{d-p}(-v)^p\frac{1}{2\pi}\int_0^{2\pi}\cos^{d-p}\theta\sin^{p+1}\theta\,\mathrm{d}\theta \tag{26}$$

and

$$-\sum_{d=2}^{\infty}\alpha_d\sum_{p=0}^{d}\binom{d}{p}u^{d-p}(-v)^p\frac{1}{2\pi}\int_0^{2\pi}\cos^{d-p+1}\theta\sin^{p}\theta\,\mathrm{d}\theta \tag{27}$$

The result of the integrals depends on the parity of the exponents of the sine and cosine terms and the different possibilities are studied hereafter.

Case 1: d and p are odd.
In this case, we set $d = 2i+1$ and $p = 2j+1$. For $\dot{u}$ we have:

$$\sum_{i=1}^{\infty}\alpha_{2i+1}\sum_{j=0}^{i}\binom{2i+1}{2j+1}u^{2(i-j)}v^{2j+1}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i-j)}\theta\sin^{2(j+1)}\theta\,\mathrm{d}\theta \neq 0 \tag{28}$$

for which the result depends on the values of i and j. For $\dot{v}$, we have:

$$\sum_{i=1}^{\infty}\alpha_{2i+1}\sum_{j=0}^{i}\binom{2i+1}{2j+1}u^{2(i-j)}v^{2j+1}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i-j)+1}\theta\sin^{2j+1}\theta\,\mathrm{d}\theta = 0 \tag{29}$$

since both exponents are odd.

Case 2: d is odd and p is even.
In this case, we set $d = 2i + 1$ and $p = 2j$. For $\dot{u}$ we have:

$$-\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j)} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)+1}\theta \sin^{2j+1}\theta \, \mathrm{d}\theta = 0 \tag{30}$$

since both exponents are odd. For $\dot{v}$, we have:

$$-\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j)+1} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \, \mathrm{d}\theta \neq 0 \tag{31}$$

for which the result depends on the values of i and j.

Case 3: d and p are even.
In this case, we set $d = 2i$ and $p = 2j$. For $\dot{u}$ we have:

$$-\sum_{i=0}^{\infty} \alpha_{2i} \sum_{j=0}^{i} \binom{2i}{2j} u^{2(i-j)} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2j+1}\theta \, \mathrm{d}\theta = 0 \tag{32}$$

since one of the exponents is odd. For $\dot{v}$, we have:

$$-\sum_{i=0}^{\infty} \alpha_{2i} \sum_{j=0}^{i} \binom{2i}{2j+1} u^{2(i-j)} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)+1}\theta \sin^{2j}\theta \, \mathrm{d}\theta = 0 \tag{33}$$

since one of the exponents is odd.

Case 4: d is even and p is odd.
In this case, we set $d = 2i$ and $p = 2j + 1$. For $\dot{u}$ we have:

$$\sum_{i=0}^{\infty} \alpha_{2i} \sum_{j=0}^{i} \binom{2i}{2j+1} u^{2(i-j)-1} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)-1}\theta \sin^{2(j+1)}\theta \, \mathrm{d}\theta = 0 \tag{34}$$

since one of the exponents is odd. For $\dot{v}$, we have:

$$\sum_{i=0}^{\infty} \alpha_{2i} \sum_{j=0}^{i} \binom{2i}{2j+1} u^{2(i-j)-1} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2j+1}\theta \, \mathrm{d}\theta = 0 \tag{35}$$

since one of the exponents is odd.

Summary:
Therefore, we end up with:

$$\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j+1} u^{2(i-j)} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \, \mathrm{d}\theta \tag{36}$$

and

$$-\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j)+1} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \, \mathrm{d}\theta \tag{37}$$

for $\dot{u}$ and $\dot{v}$, respectively. We thus observe that the stiffness of even orders do not participate in the motion around the primary resonance at first order.

Averaged solution around the primary resonance
The average solution for $\dot{u}$ is therefore:

$$\dot{u} = -\frac{\varepsilon}{\omega} \left(\frac{\gamma}{2} + \zeta\,\omega_0\,\omega\,u - \frac{\Omega\,v}{2} + \sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j+1} u^{2(i-j)} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \, \mathrm{d}\theta \right) \tag{38}$$

and for $\dot{v}$:

$$\dot{v} = -\frac{\varepsilon}{\omega} \left(\zeta\,\omega_0\,\omega\,v + \frac{\Omega\,u}{2} - \sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j)+1} v^{2j} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \, \mathrm{d}\theta \right) \tag{39}$$

Those equations can be gathered in order to get $\dot{r}$ and $\dot{\phi}$ using the relations from Equation 12. However, this leads to complex expressions and it is interesting to see if the effect of the polynomial stiffness can be simplified.

Solution for $\dot{r}$

First, for $\dot{r}$, it is possible to show that when we use the relation: $u\dot{u} + v\dot{v}$, then we have for the polynomial stiffness terms:

$$\begin{aligned}\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j+1} u^{2(i-j)+1} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \,\mathrm{d}\theta \\ - \sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j)+1} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \,\mathrm{d}\theta = 0\end{aligned} \tag{40}$$

Taking out the constant terms, we end up with:

$$\sum_{j=0}^{i} \frac{u^{2(i-j)+1} v^{2j+1}}{2\pi} \left(\binom{2i+1}{2j+1} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \,\mathrm{d}\theta - \binom{2i+1}{2j} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \,\mathrm{d}\theta \right) = 0 \tag{41}$$

and therefore, we need to prove that

$$\binom{2i+1}{2j+1} \int_0^{2\pi} \cos\theta^{2(i-j)} \sin^{2(j+1)}\theta \,\mathrm{d}\theta - \binom{2i+1}{2j} \int_0^{2\pi} \cos\theta^{2(i-j+1)} \sin^{2j}\theta \,\mathrm{d}\theta = 0 \tag{42}$$

in order to show that Equation (40) is valid.

The first step is to use the results of the integrals from Equation (16) and rewrite Equation (42) as:

$$\frac{2}{\Gamma(i+2)} \left(\binom{2i+1}{2j+1} \Gamma(i-j+\frac{1}{2}) \Gamma(j+1+\frac{1}{2}) - \binom{2i+1}{2j} \Gamma(i+1-j+\frac{1}{2}) \Gamma(j+\frac{1}{2}) \right) \tag{43}$$

After that, we can make use of the following property of the *Gamma* function:

$$\Gamma(n+\frac{1}{2}) = \binom{n-\frac{1}{2}}{n} n! \sqrt{\pi} \tag{44}$$

for non-negative integer values of n, as well as the following binomial coefficient property:

$$\binom{n}{k} = \frac{n-k+1}{k} \binom{n}{k-1} \tag{45}$$

to rewrite Equation (43) as

$$\frac{2\pi}{\Gamma(i+2)} (i-j)! j! \binom{i-j-\frac{1}{2}}{i-j} \binom{j-\frac{1}{2}}{j} \binom{2i+1}{2j} \left(\frac{2i-2j+1}{2j+1} (j+\frac{1}{2}) - (i-j+\frac{1}{2}) \right) = 0 \tag{46}$$

which proves the relation from Equation (40).

Finally, we can write for $\dot{r}$:

$$\dot{r} = -\frac{\varepsilon}{\omega r} \left(\zeta\, \omega_0\, \omega\, r^2 - \frac{\gamma}{2} r \sin\phi \right) \tag{47}$$

Solution for $\dot{\phi}$

In the case of $\dot{\phi}$, we need to use the relation $v\dot{u} - u\dot{v}$ and therefore, the terms related to the polynomial stiffness can be written as:

$$\begin{aligned}\sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j+1} u^{2(i-j)+1} v^{2(j+1)} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \,\mathrm{d}\theta \\ + \sum_{i=1}^{\infty} \alpha_{2i+1} \sum_{j=0}^{i} \binom{2i+1}{2j} u^{2(i-j+1)} v^{2j+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \,\mathrm{d}\theta\end{aligned} \tag{48}$$

which can be simplified and written under the form:

$$\sum_{i=1}^{\infty} \alpha_{2i+1} C \sum_{j=0}^{i+1} \binom{i+1}{j} u^{2(i+1-j)} v^{2j} = \sum_{i=1}^{\infty} \alpha_{2i+1} C (u^2+v^2)^{i+1} = \sum_{i=1}^{\infty} \alpha_{2i+1} C r^{2(i+1)} \tag{49}$$

where C is a constant to be determined. To demonstrate this, we need to show that:

$$\begin{aligned}\sum_{j=0}^{i} \frac{1}{2\pi} \left(\binom{2i+1}{2j+1} u^{2(i-j)} v^{2(j+1)} \int_0^{2\pi} \cos^{2(i-j)}\theta \sin^{2(j+1)}\theta \,\mathrm{d}\theta + \binom{2i+1}{2j} u^{2(i-j+1)} v^{2j} \int_0^{2\pi} \cos^{2(i-j+1)}\theta \sin^{2j}\theta \,\mathrm{d}\theta \right) \\ = C \sum_{j=0}^{i+1} \binom{i+1}{j} u^{2(i+1-j)} v^{2j}\end{aligned} \tag{50}$$

First, we rearrange the left hand side of Equation (51) such that:

$$\binom{2i+1}{0}u^{2(i+1)}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1)}\theta\,\mathrm{d}\theta \tag{51}$$
$$+\binom{2i+1}{1}u^{2i}v^2\frac{1}{2\pi}\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta\,\mathrm{d}\theta+\binom{2i+1}{2}u^{2i}v^2\frac{1}{2\pi}\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta\,\mathrm{d}\theta$$
$$+\dots$$
$$+\binom{2i+1}{2k-1}u^{2(i+1-k)}v^{2(k)}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1-k)}\theta\sin^{2k}\theta\,\mathrm{d}\theta+\binom{2i+1}{2j}u^{2(i+1-k)}v^{2k}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1-k)}\theta\sin^{2k}\theta\,\mathrm{d}\theta$$
$$+\dots$$
$$\binom{2i+1}{2i+1}v^{2(i+1)}\frac{1}{2\pi}\int_0^{2\pi}\sin^{2(i+1)}\theta\,\mathrm{d}\theta$$

Or simply:

$$\binom{2i+1}{0}u^{2(i+1)}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1)}\theta\,\mathrm{d}\theta \tag{52}$$
$$+\sum_{j=1}^{i}\left(\binom{2i+1}{2j-1}+\binom{2i+1}{2j}\right)u^{2(i+1-j)}v^{2j}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1-j)}\theta\sin^{2j}\theta\,\mathrm{d}\theta$$
$$\binom{2i+1}{2i+1}v^{2(i+1)}\frac{1}{2\pi}\int_0^{2\pi}\sin^{2(i+1)}\theta\,\mathrm{d}\theta$$

Which can be further simplified by making use of the fact that first:

$$\binom{2i+1}{2j-1}+\binom{2i+1}{2j}=\binom{2(i+1)}{2j} \tag{53}$$

and second:

$$\binom{2i+1}{0}=\binom{2i+1}{2i+1}=1=\binom{2(i+1)}{0}=\binom{2(i+1)}{2(i+1)} \tag{54}$$

which leads to

$$\sum_{j=0}^{i+1}\binom{2(i+1)}{2j}u^{2(i+1-j)}v^{2j}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1-j)}\theta\sin^{2j}\theta\,\mathrm{d}\theta \tag{55}$$

The final step consists in showing that:

$$\binom{2(i+1)}{2j}\frac{1}{2\pi}\int_0^{2\pi}\cos^{2(i+1-j)}\theta\sin^{2j}\theta\,\mathrm{d}\theta=C\binom{i+1}{j} \tag{56}$$

To do so, we make use of the fact that a binomial coefficient can be written using the *Gamma* function:

$$\binom{n}{k}=\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \tag{57}$$

Furthermore, using the results from Equation (16), we can write the left hand side of Equation (56) as:

$$\frac{1}{\pi}\frac{\Gamma(2z_1)}{\Gamma(2z_3)\Gamma(2z_2)}\frac{\Gamma(z_2)\Gamma(z_3)}{\Gamma(z_1+\frac{1}{2})} \tag{58}$$

where $z_1=i+1+\frac{1}{2}$, $z_2=i+1-j+\frac{1}{2}$ and $z_3=k+\frac{1}{2}$. Using the Legendre duplication formula:

$$\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\Gamma(2z) \tag{59}$$

it is possible to rewrite Equation (58) as:

$$\frac{\Gamma(2i+3)}{2^{2(i+1)}\Gamma^2(i+2)}\binom{i+1}{j}=C_i\binom{i+1}{j} \tag{60}$$

where $C_i=\frac{\Gamma(2i+3)}{2^{2(i+1)}\Gamma^2(i+2)}$ is a constant that only depends on i. Therefore, we can indeed rewrite Equation (48) as Equation (49). Finally, we can write for $\dot{\phi}$:

$$\dot{\phi}=-\frac{\varepsilon}{\omega r^2}\left(\sum_{i=1}^{\infty}\alpha_{2i+1}C_i r^{2(i+1)}-\frac{\Omega}{2}r^2-\frac{\gamma}{2}r\cos\phi\right) \tag{61}$$

Solution at steady-state

Since we are interested in solutions at steady-state, we have $\dot{r} = \dot{\phi} = 0$ and therefore

$$\begin{cases} r = \frac{\gamma}{2\zeta\,\omega_0\,\omega}\sin\phi \\ \sum_{i=1}^{\infty}\alpha_{2i+1}C_i r^{2i+1} - \frac{\Omega}{2}r = \frac{\gamma}{2}\cos\phi \end{cases} \tag{62}$$

where the first relation shows that the amplitude r does not directly depend on the nonlinear stiffness coefficients α_d. This relation also confirms that even order stiffness do not participate in the motion around the primary resonance.

Amplitude and phase resonances

Amplitude resonance occurs when both $\frac{\partial r}{\partial\,\omega}$ and $\frac{\partial r}{\partial \phi}$ are equal to 0. From Equation (62), we obtain:

$$\begin{cases} \frac{\partial r}{\partial \phi} = \frac{\gamma}{2\zeta\,\omega_0\,\omega}\left(\cos\phi - \frac{\sin\phi}{\omega}\frac{\partial\,\omega}{\partial\phi}\right) = 0 \\ \frac{\partial r}{\partial\,\omega} = \frac{\gamma}{2\zeta\,\omega_0\,\omega}\left(\cos\phi\frac{\partial\phi}{\partial\,\omega} - \frac{\sin\phi}{\omega}\right) = 0 \end{cases} \tag{63}$$

where

$$\frac{\partial\phi}{\partial\,\omega} = -\frac{2}{\gamma\sin\phi}\left(\sum_{i=1}^{\infty}\alpha_{2i+1}C_i(2i+1)r^{2i}\frac{\partial r}{\partial\,\omega} - \frac{1}{2}\left(\frac{2\,\omega}{\varepsilon}r + \frac{\omega^2-{\omega_0}^2}{\varepsilon}\frac{\partial r}{\partial\,\omega}\right)\right) \tag{64}$$

Eventually, we have

$$\frac{\partial r}{\partial\,\omega} = \frac{\gamma^2\sin^2\phi\,(\omega - \varepsilon\zeta\,\omega_0\tan\phi)}{\left(2({\omega_0}^2-\omega^2)\zeta\,\omega_0\,\omega + \varepsilon\left(2\gamma^2\zeta^2\,{\omega_0}^2\,\omega^2\cos\phi + \sum_{i=1}^{\infty}\alpha_{2i+1}C_i(2i+1)\frac{(\gamma\sin\phi)^{2i+1}}{(2\zeta\,\omega_0\,\omega)^{2i-1}}\right)\right)} = 0 \tag{65}$$

This relation is verified when:

$$\tan\phi_a = \frac{\omega_a}{\varepsilon\zeta\,\omega_0} \tag{66}$$

Since we consider a small damping ratio $\bar{\zeta} = \varepsilon\zeta$, the phase lag ϕ_a at amplitude resonance is very close to $\frac{\pi}{2}$.
On the other hand, phase resonance for linear and nonlinear systems occurs when the external forcing counterbalances exactly the damping forces [5]. From the first equation in Equation (62), we see that this happens when the phase lag is $\pi/2$. Phase resonance thus occurs in the immediate vicinity of amplitude resonance.

Numerical validation on a Helmholtz-Duffing oscillator

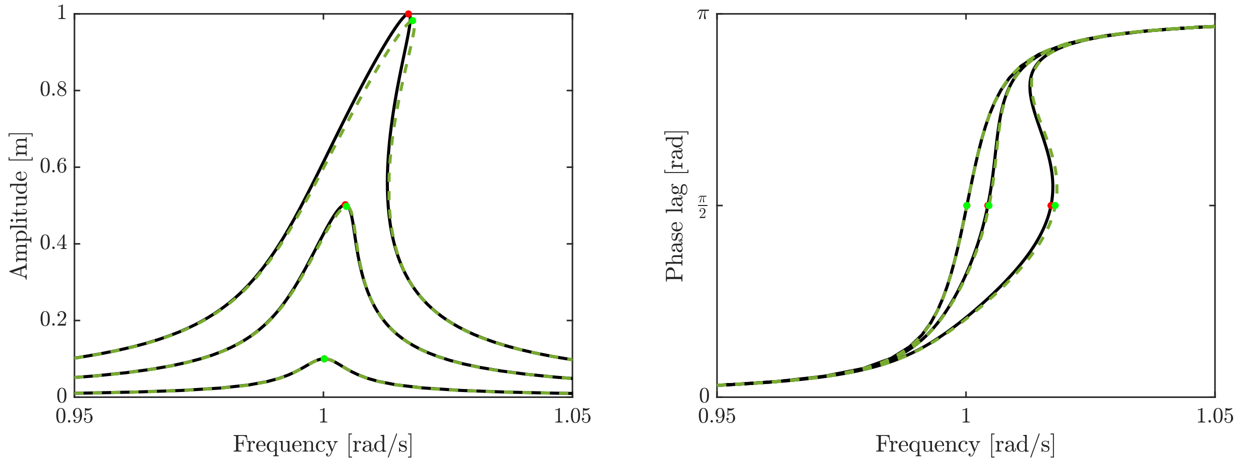
The previous results are applied to a Helmholtz-Duffing oscillator governed by the following equation:

$$\ddot{x}(t) + 2\bar{\zeta}\,\omega_0\,\dot{x}(t) + \omega_0^2 x(t) + \bar{\beta}x^2(t) + \bar{\alpha}x^3(t) = \bar{\gamma}\sin\omega t \tag{67}$$

According to Equation 62, first-order averaging around the primary resonance gives

$$\begin{cases} r = \frac{\gamma}{2\zeta\,\omega_0\,\omega}\sin\phi \\ \frac{3\alpha}{8}r^3 - \frac{\Omega}{2}r = \frac{\gamma}{2}\cos\phi \end{cases} \tag{68}$$

Setting $\bar{\beta} = 0.05$ N/(kg m^2), $\bar{\alpha} = 0.05$ N/(kg m^3) and $\bar{\zeta} = 0.005$, the numerical solution using a harmonic balance continuation procedures with 8 harmonics is compared to the analytical solution from Equation (68) in Figure 1. The two methods give very similar results around the primary resonance. In addition to that, the phase resonance points, which correspond to a phase lag of $\frac{\pi}{2}$ for the first harmonic component of the solution, is also plotted and both techniques show that it is indeed in the vicinity of the amplitude resonance.



(a) (b)

Figure 1: Nonlinear frequency responses (Numerical: black, analytical: green) around the primary resonance of the Helmholtz-Duffing oscillator for forcing amplitudes $\bar{\gamma}$ of 0.001N, 0.005N and 0.01N: (a) amplitude and (b) phase lag. The red (numerical) and green (analytical) dots correspond to a phase lag of $\frac{\pi}{2}$.

Conclusion

A first-order averaging technique was applied around the primary resonance of an oscillator with arbitrary polynomial stiffness. The results show that phase resonance associated with a phase lag of $\frac{\pi}{2}$ exists in the immediate neighborhood of amplitude resonance in the case of weak damping. These results are in agreement with those of Peeters et al. [5] and Haller et al. [6].

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