Conserved quantities in cosmology

Phillip Helbig[1,2,3]

1. Thomas-Mann-Str. 9, 63477 Maintal, Germany
2. helbig@astro.multivax.de
3. Phillip.Helbig@alumni.uliege.be

Abstract

Lake pointed out that a certain combination (which he dubbed α) of Ω and λ is a constant of motion for evolutionary trajectories in the λ–Ω plane and used mapping between α and the λ–Ω plane to demonstrate the lack of a flatness problem for cosmological models with a positive cosmological constant which will expand forever. In such models, the conserved quantity corresponds to the product of λ and the square of the mass of the universe. I investigate other quantities which correspond to α and other constants of motion in the λ–Ω plane.

Introduction

Here, I consider only ideal Friedmann–Robertson–Walker (FRW) models, because historically similar studies have been done within the context of those models and the basic concepts carry over into more-complicated models. I use notation such that $\Omega = \frac{8\pi G\rho}{3H^2}$ refers to the density of matter ('dust') and $\lambda = \frac{\Lambda}{3H^2}$ is the normalized cosmological constant (with dimension time^{-2} so that Λ has the same dimension as $G\rho$); the subscript 0 refers to the current value of a time-dependent parameter. $K = \Omega + \lambda - 1$, $k = \mathrm{sign}(K)$, and $\alpha = \frac{27k\Omega^2\lambda}{4K^3}$. The Hubble constant $H = \frac{\dot{R}}{R}$, where $R = \frac{c}{H\sqrt{|K|}}$. The deceleration parameter $q = -\frac{\ddot{R}}{RH^2} = \frac{\Omega}{2} - \lambda$. The redshift $z = \frac{R_0}{R} - 1$.

Evolutionary trajectories

In general, λ and Ω change with time. Thus, the evolution of cosmological models can be illustrated by trajectories in the λ–Ω plane. Trajectories never cross. The

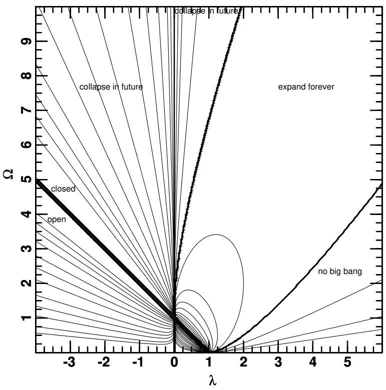

Figure 1: Evolutionary trajectories in the λ–Ω plane

points where two of λ, Ω, and k are 0 are stationary trajectories, *i.e.* cosmological models which start there stay there. The lines $\lambda = 0$, $\Omega = 0$, and $k = 0$ are also trajectories (and thus also not crossed by others). The position of a *point* in the λ–Ω plane (which thus determines a trajectory but also a relative time) can be used to classify cosmological models *via* the corresponding trajectories. A third, dimensionful, parameter is needed in order to set the overall scale. As pointed out by Stabell and Refsdal (1966), such trajectories are useful for classifying cosmological models, which can be summarized as follows, with the numbering illustrated in Fig. 2:

models which will collapse in the future (big-bang models which expand to $R = R_{\max}$ at $t = t_{\max}/2$ and have $\alpha < 1$ and $q > 0$)

- $\lambda < 0$
  - $k = -1$ ($-\infty < \alpha \leq 0$)
    - $\Omega = 0$ ($\alpha = 0$): **1**
    - $\Omega > 0$ ($-\infty < \alpha < 0$): **2**
  - $k = 0$ ($\Omega > 1$, $\alpha = -\infty$, and $q > 0.5$,): **3**
  - $k = +1$ ($\Omega > 1$, $-\infty < \alpha < 0$, and $q > 0.5$,): **4**
- $\lambda = 0$
  - $k = +1$ ($\Omega > 1$, $\alpha = 0$, and $q > 0.5$; preferred by Einstein after he had abandoned Λ): **5**
- $\lambda > 0$
  - $0 < \alpha < 1$
    - $q > 0.5$ ($k = +1$, $\Omega > 1$): **6**

models which will not collapse in the future ($\lambda \geq 0$ and $\alpha > 0$)

- $\lambda = 0$ (big-bang models which expand to $R = \infty$ at $t = \infty$; $\alpha = 0$, $0 \leq q \leq 0.5$)
  - $k = -1$ ($0 \leq q < 0.5$)
    - $\Omega = 0$ (the general-relativistic equivalent of the Milne model; $q = 0$): **7**
    - $\Omega > 0$ ($\Omega < 1$) $0 < q < 0.5$, : **8**
  - $k = 0$ ($\Omega = 1$ and $q = 0.5$; the Einstein–de Sitter model; $\dot{R} \to 0$ for $t \to \infty$): **9**
- $\lambda > 0$ ($0 < \alpha \leq \infty$)
  - $k = -1$ (big-bang models which expand to $R = \infty$ at $t = \infty$; $0 \leq \Omega < 1$, $0 < \lambda < 1$, and $-1 < q < 0.5$)
    - $\Omega = 0$ ($\alpha = 0$ and $q < 0$): **10**
    - $\Omega > 0$ ($\Omega < 1$): **11**
  - $k = 0$ ($\alpha = \infty$)
    - $\Omega = 0$ (the de Sitter model; has a big-bang at $t = -\infty$; $\lambda = 1$ and $q = -1$): **12**
    - $\Omega > 0$ (big-bang models which expand to $R = \infty$) at $t = \infty$; $0 < \lambda < 1$, $0 < \Omega < 1$, and $-1 < q < 0.5$): **13**
  - $k = +1$
    - $\alpha > 1$ (big-bang models which expand to $R = \infty$) at $t = \infty$; $\Omega > 0$): **14**
    - $\alpha = 1$
      - $\infty > q > 0$ (big-bang model which expands forever, but approaches a finite R and hence approaches $\dot{R} = 0$; $\Omega > 1$ and $q > 0.5$): **15**
      - $q = \pm\infty$ (the static Einstein model): **16** $\lambda = \infty$, $\Omega = \infty$;
      - $-\infty < q < 0$ (has a 'big bang' but at $R > 0$ (and hence a maximum redshift) and expands to $R = \infty$ at $t = \infty$); Eddington model; $\lambda > 1$ and $q < -1$): **17**
    - $0 < \alpha < 1$
      - $q < -1$ (expands to $R = \infty$ at $t = \infty$ after having contracted from $R = \infty$ to $R = R_{\min}$ and thus has a maximum redshift; $\lambda > 1$, $\Omega > 0$, and $q < -1$): **18**
    - $\alpha = 0$ (expands to $R = \infty$ at $t = \infty$ after having contracted from $R = \infty$ to $R = 0$; Lanczos model; $\lambda > 1$, $\Omega = 0$, and $q < -1$): **19**

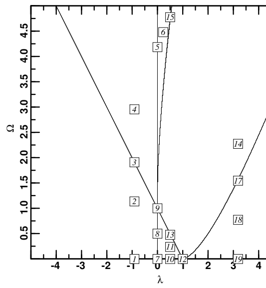

Figure 2: The 19 types of FRW models with (at most) dust and a cosmological constant

Lake's α: M^2 and its importance

Lake (2005) pointed out that α is constant along a trajectory. What is the physical interpretation? The mass of (the dust in) a **closed universe (models 4–6 and 14–19)** is $M = \rho V$; the volume $V = 2\pi^2 R^3$, thus

$$M = \rho\left(2\pi^2\right)\left(\frac{c}{H}\right)^3 |K|^{-\frac{3}{2}} \quad . \tag{1}$$

From the definition of Ω and, since $k = 1$ and thus $K = |K|$, that leads to

$$M = \left(\frac{3H^2\Omega}{8\pi G}\right)\left(2\pi^2\right)\left(\frac{c}{H}\right)^3 K^{-\frac{3}{2}} \quad . \tag{2}$$

Since, from the definition of λ, $H = \sqrt{(\Lambda)/(3\lambda)}$,

$$M = \left(\frac{\Lambda\Omega}{8\pi G\lambda}\right)\left(2\pi^2\right)\left(\frac{3c^2\lambda}{\Lambda}\right)^{\frac{3}{2}} K^{-\frac{3}{2}} \tag{3}$$

and thus

$$M^2 = \left(\frac{\Lambda\Omega}{8\pi G\lambda}\right)^2\left(2\pi^2\right)^2\left(\frac{3c\lambda}{\Lambda}\right)^3 K^{-3} \quad . \tag{4}$$

Placing the cosmologically relevant dimensionful variables on the left-hand side, we have

$$\Lambda M^2 = \left(\frac{\Omega}{8\pi G\lambda}\right)^2\left(2\pi^2\right)^2\left(3c^2\lambda\right)^3 K^{-3} \quad , \tag{5}$$

or

$$\Lambda M^2 = \left(\frac{2\pi^2\Omega}{8\pi G}\right)^2\left(27c^6\lambda\right) K^{-3} \quad . \tag{6}$$

Sorted differently, we have

$$\Lambda M^2 = \left(\frac{27}{16}\right)\left(\frac{\pi^2 c^6}{G^2}\right)\left(\frac{\Omega^2\lambda}{K^3}\right) \tag{7}$$

or

$$\Lambda M^2 = \left(\frac{\pi^2 c^6}{4G^2}\right)\alpha \quad . \tag{8}$$

Thus, up to a constant, $\Lambda M^2 = \alpha$.

Other conserved quantities

Inflection

An obvious conserved quantity for **types 11, 13, and 14**, since they cross the $q = 0$ line, is the point of inflection which occurs when that line is crossed, when $\ddot{R} = 0$. At such a point, $q = \lambda - \Omega/2 = 0$ and hence $\Omega = 2\lambda$. Since $\rho \sim R^{-3}$ and $\lambda \sim R^0$, it follows that $(R_0/R_{\mathrm{infl}})^3 = \Omega_0/2\lambda_0$. Thus,

$$R^3_{\mathrm{infl}} = R^3_0\left(\frac{\Omega_0}{2\lambda_0}\right) \quad . \tag{9}$$

From the definition of R follows

$$R^3_{\mathrm{infl}} = \left(\frac{c}{H_0}\right)^3 |K_0|^{-\frac{3}{2}}\left(\frac{\Omega_0}{2\lambda_0}\right) \quad . \tag{10}$$

Trivially,

$$R^6_{\mathrm{infl}} = \left(\frac{c}{H_0}\right)^6 |K_0|^{-3}\left(\frac{\Omega_0}{2\lambda_0}\right)^2 \quad . \tag{11}$$

From the definition of λ we have $H^2 = (\Lambda)/(3\lambda)$, thus

$$R^6_{\mathrm{infl}} = \left(\frac{c^6(3\lambda)^3}{\Lambda^3}\right)|K_0|^{-3}\left(\frac{\Omega_0}{2\lambda_0}\right)^2 \quad . \tag{12}$$

Thus,

$$R^6_{\mathrm{infl}}\Lambda^3 = \left(\frac{(27{\Omega_0}^2\lambda_0 c^6}{4|K_0|^3}\right) \quad . \tag{13}$$

Since $\alpha = \mathrm{sign}(K)(27\Omega^2\lambda)/(4K^3)$ is constant along a trajectory and thus holds for all values of λ, Ω, and K along a trajectory, not only for the current values ($\alpha = \mathrm{sign}(K_0)(27{\Omega_0}^2\lambda_0)/(4K_0^3)$),

$$R^2_{\mathrm{infl}}\Lambda = c^2\sqrt[3]{\alpha} \quad . \tag{14}$$

$R_{\max}$

In a universe which collapses in the future, $R_{\max}$ is obviously constant along a trajectory, and so could be interpreted as a constant of motion, though that is more interesting if expressed as a fundamental cosmological parameter. At $R = R_{\max}$, $\dot{R} = 0$, thus one can calculate the scale factor (corresponding to the smallest value of R for which $\dot{R} = 0$) of the universe at maximum expansion for **types 1–6 and 15**. That involves solving a cubic equation, and the actual expression depends on the model type. Although that solution depends on quantities such as $\Omega/(2\lambda)$ and $K/(3\lambda)$ (the square of the first divided by the cube of the second is, with the additional factor of $\mathrm{sign}(K)$, the definition of α), I am not aware of any simple expression relating α and $R_{\max}$, even though $R_{\max}$ is obviously constant along a trajectory. However, there are analytic solutions in special cases.

For the **model 1**, $\alpha = 0$ since $\Omega = 0$ and thus we cannot expect any constant of motion expressed in terms of physical quantities to involve α. It thus makes sense to express $R_{\max}$ in terms of Λ by using the definition of λ to eliminate H_0 and λ_0, resulting in $R^2_{\max} = (3c^2)/\Lambda$.

For the **model 3**, $\alpha = \infty$ since the radius of curvature is infinite. If we set $R_0 = c/H_0$, as is commonly done in such cases, then, *via* reasoning similar to that above, one obtains $R^6\Lambda^3 = 27c^6\Omega^2\lambda$ or, up to the constant c^6, $R^6\Lambda^3 = 4\alpha {K_0}^3$. Of course, that is rather meaningless since $\alpha K_0 = \infty * 0$ and results from using $R_0 = c/H_0$ instead of $R_0 = c/(H_0\sqrt{K_0})$. However, $R^6\Lambda^3 = 4\alpha$ is in some sense the limit for $K_0 \to 0$.

For the **model 5**, $\alpha = 0$ since $\lambda = 0$ and thus we cannot expect any constant of motion expressed in terms of physical quantities to involve α. It thus makes sense to express $R_{\max}$ in terms of M, since such models have $k = +1$ and hence a finite volume and finite mass. M is calculated above and $R_{\max} = (c/H_0)(\Omega/(\Omega-1)^{\frac{2}{2}})$, thus $R = (4GM)/(3\pi c^2)$.

Model 15 asymptotically approaches the static Einstein model (**16**). It thus has a maximum scale factor at which not only $\dot{R}$ but also $\ddot{R}$ is zero. Of course, Eq. (8) applies here as well. One can also think of it as an inflection point, though strictly speaking it is an asymptotic limit of **models 15 and 17** and thus the static phase never actually occurs in those models. However, calculating it in the same way as described above, one obtains the result $R^2\Lambda = c^2\sqrt[3]{\alpha} = \sqrt[3]{1} = 1$ which is compatible with the well known relations $R^2 = c^2/(4\pi G\rho)$ and $\Lambda = 4\pi G\rho$.

$R_{\min}$

At $R = R_{\min}$, $\dot{R} = 0$, so one can calculate the scale factor (corresponding to the largest value of R for which $\dot{R} = 0$) of the universe at minimum contraction in the bounce **models (18–19)** and the Eddington model (**17**). As with $R_{\max}$, there seems to be no simple relation between $R_{\min}$ and α.

For $\alpha = 1$, similar arguments apply as for model 15. (One can in some sense think of models 15, 16, and 17 as one trajectory, though it spends an infinite amount of time on each side of, and at, 16.) One obtains the same result, namely $R^2\Lambda = c^2\sqrt[3]{\alpha} = \sqrt[3]{1} = 1$, and thus $z_{\max} = \sqrt[3]{2\lambda/\Omega} - 1$.

For the **model 19**, arguments similar to those for model 1 lead to the same result: $R^2_{\max} = (3c^2)/\Lambda$. Thus, the maximum redshift in this model is $z_{\max} = \sqrt{\lambda/(\lambda-1)} - 1$.

The simplest cases

For the **model 7**, (the general-relativistic equivalent of) the Milne model, $\alpha = 0$ since $\lambda = 0$ and $\Omega = 0$. Note that, in contrast to all other cases except model 9 (see below), this trajectory does not correspond to a set of models specified by an additional parameter (*e.g.* Λ for type 1), but is only one model which evolves in time. In this model, $R = ct$. One can specify only the time since the big bang and quantities trivially related to it, *e.g.* $H \sim 1/t$. It is the only model in which $\Omega + \lambda$ is constant in time except for $k = 0$ models (in which the sum is 1) and the static Einstein model (16) (in which the sum is ∞).

Model 9, the Einstein–de Sitter model, which has $\Omega = 1$ always, is a repulsor; all trajectories in non-empty big-bang models start here. Thus, it does not make sense to define a value of α. Note that, in contrast to all other cases except model 7 (see above), this trajectory does not correspond to a set of models specified by an additional parameter (*e.g.* Λ for type 1), but is only one model which evolves in time. Thus, in such a model either the time or a time-dependent quantity determines the other, *e.g.* the scale factor R is proportional to time as $t^{2/3}$; the age of the universe is 2/3 the Hubble time ($R/\dot{R}$, the inverse of the Hubble constant); the density in kg per m^3 is given by $1/(6\pi G t^2)$ where G is the gravitational constant and t is the age of the universe in seconds. For a generic trajectory, specifying the trajectory and a time-dependent quantity or the time still leaves some leeway, *e.g.* for type 15 only the product ΛM^2, not either individually. Thus, specifying a generic trajectory and, say, ρ does not, without further information, determine the time, and *vice versa*. The Einstein–de Sitter model is spatially infinite and flat and will expand forever (though the rate of expansion, and thus H, approaches 0 for $t \to \infty$), there is no maximum nor (non-zero) minimum scale factor nor point of inflection. There is no non-trivial constant of motion and no defining physical property.

For the **model 12**, the de Sitter model, which has $\lambda = 1$ always, in contrast to the other point-trajectory models 7 and 9 discussed above, there is a set of models determined by Λ. $H = \sqrt{\Lambda/3}$ and is thus constant in time. H is thus the constant of motion in this model; the Hubble sphere—the sphere with radius c/H—is thus constant in time, which is not true in general. Also, the Hubble sphere is an event horizon (Rindler, 1956); in general the Hubble sphere is not associated with any type of horizon (*e.g.* van Oirschot et al., 2010).

The static Einstein model (**17**) being static, has $H = 0$ and hence λ and Ω are infinite. Like all other models except 7–9, there is a set if models determined by Λ. Equivalently, one can characterize those by the density ρ or scale factor $R^2 = c^2/(4\pi G\rho)$; note that $\Lambda = 4\pi G\rho$. It can also be seen as the future (15) or past (17) limit of other models. Like all models with $k = +1$, Eq. (8) applies here as well; that can be verified by using the relations above.

Acknowledgements

This research has made use of NASA's Astrophysics Data System Bibliographic Services.

References

Lake, K. 2005. The Flatness Problem and Λ. *Phys. Rev. Lett.* **94:** 201102

Rindler, W. 1956. Visual horizons in world models. *MNRAS* **116:** 662–677

Stabell, R. and Refsdal, S. 1966. Classification of general relativistic world models. *MNRAS* **132:** 379–388

van Oirschot, P., Kwan, J. and Lewis, G. F. 2010. Through the looking glass: why the 'cosmic horizon' is not a horizon. *MNRAS* **404:** 1633–1638