CONSERVED QUANTITIES IN COSMOLOGY

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Lake pointed out that a certain combination, dubbed α, of Ω and λ is a constant of motion for evolutionary trajectories in the λ–Ω plane and used that to demonstrate the lack of a flatness problem for some cosmological models. I investigate other quantities which correspond to α and other constants of motion in the λ–Ω plane.

Here, I consider only ideal Friedmann–Robertson–Walker (FRW) models, because historically similar studies have been done within the context of those models and the basic concepts carry over into more-complicated models. I use notation such that $\Omega = \frac{8\pi G\rho}{3H^2}$ refers to the density of matter ('dust') and $\lambda = \frac{\Lambda}{3H^2}$ is the normalized cosmological constant (with dimension time^{-2} so that Λ has the same dimension as $G\rho$); the subscript 0 refers to the current value of a time-dependent parameter. $K = \Omega + \lambda - 1$, $k = \mathrm{sign}(K)$, and $\alpha = \frac{27k\Omega^2\lambda}{4K^3}$. The Hubble constant $H = \frac{\dot{R}}{R}$, where $R = \frac{c}{H\sqrt{|K|}}$. The deceleration parameter $q = -\frac{\ddot{R}}{RH^2} = \frac{\Omega}{2} - \lambda$.

In general, λ and Ω change with time. Thus, the evolution of cosmological models can be illustrated by trajectories in the λ–Ω plane [1]. Lake [2] pointed out that α is constant along a trajectory. What is the physical interpretation? The mass of (the dust in) a **closed universe** is $M = \rho V$; the volume $V = 2\pi^2 R^3$. From that, one can derive

$$\Lambda M^2 = \left(\frac{\pi^2 c^6}{4G^2}\right)\alpha \quad .$$

Thus, up to a constant, $\Lambda M^2 = \alpha$.

An obvious conserved quantity for cosmological models which have $q = 0$ at some time is the point of inflection in $R(t)$ which occurs when $\ddot{R} = 0$. At such a point, $q = \Omega/2 - \lambda = 0$ and hence $\Omega = 2\lambda$. Since $\rho \sim R^{-3}$ and $\lambda \sim R^0$, it follows that $(R_0/R_{\mathrm{infl}})^3 = \Omega_0/2\lambda_0$, which leads to

$$R_{\mathrm{infl}}^2\Lambda = c^2\sqrt[3]{\alpha} \quad .$$

In a universe which collapses in the future, R_{max} is obviously constant along a trajectory, and so could be interpreted as a constant of motion, though that is more interesting if expressed as a fundamental cosmological parameter. At $R = R_{\mathrm{max}}$, $\dot{R} = 0$, thus one can calculate the scale factor (corresponding to the smallest value of R for which $\dot{R} = 0$) of the universe at maximum expansion for such models. That involves solving a cubic equation, and the actual expression depends on the model type. Although that solution depends on quantities such as $\Omega/(2\lambda)$ and $K/(3\lambda)$ (the square of the first divided by the cube of the second is, with the additional factor of $\mathrm{sign}(K)$, the definition of α), I am not aware of any simple expression relating α and R_{max}, even though R_{max} is obviously constant along a trajectory. However, there are analytic solutions in special cases ($\lambda_0 = 0$, $\Omega_0 = 0$, $k = 0$). If $\Omega_0 = 0$ then $\alpha = 0$ and thus we cannot expect any constant of motion expressed in terms of physical quantities to involve α. It thus makes sense to express R_{max} (which exists in this case for $\lambda_0 < 0$) in terms of Λ by using the definition of λ to eliminate H_0 and λ_0, resulting in $R^2_{\mathrm{max}} = (3c^2)/\Lambda$. If $k = 0$ and $\lambda_0 < 0$ then $\alpha = \infty$ since the radius of curvature is infinite. If we set $R_0 = c/H_0$, as is commonly done in such cases, then, *via* reasoning similar to that above, one obtains $R^6\Lambda^3 = 27c^6\Omega^2\lambda$ or, up to the constant c^6, $R^6\Lambda^3 = 4\alpha {K_0}^3$. Of course, that is rather meaningless since $\alpha K_0 = \infty * 0$ and results from using $R_0 = c/H_0$ instead of $R_0 = c/(H_0\sqrt{K_0})$. However, $R^6\Lambda^3 = 4\alpha$ is in some sense the limit for $K_0 \rightarrow 0$.

If $\Omega_0 > 1$ and $\lambda_0 = 0$ then $\alpha = 0$ and thus we cannot expect any constant of motion expressed in terms of physical quantities to involve α. It thus makes sense to express R_{max} in terms of M, since such models have $k = +1$ and hence a finite volume and finite mass. M is calculated above and $R_{\mathrm{max}} = (c/H_0)(\Omega/(\Omega - 1)^{\frac{1}{2}})$, thus $R = (4GM)/(3\pi c^2)$.

At $R = R_{\mathrm{min}}$, $\dot{R} = 0$, so one can calculate the scale factor (corresponding to the largest value of R for which $\dot{R} = 0$) of the universe at minimum contraction in the bounce models (which contract from ∞ before expanding) and the Eddington model (which expands from $R_0 > 0$ at $t = -\infty$). As with R_{max}, there seems to be no simple relation between R_{min} and α, though for $\Omega_0 = 0$ one has $R^2_{\mathrm{min}} = (3c^2)/\Lambda$.

For the the de Sitter model, which has $\lambda = 1$ always, in contrast to the other point-trajectory models (the Einstein–de Sitter model with $\lambda_0 = 0$ and $\Omega_0 = 1$ and (the relativistic equivalent of) the Milne model with $\lambda_0 = 0$ and $\Omega_0 = 0$), there is a set of models determined by Λ. $H = \sqrt{\Lambda/3}$ and is thus constant in time. H is thus the constant of motion in this model; the Hubble sphere—the sphere with radius c/H—is thus constant in time, which is not true in general. Also, the Hubble sphere is an event horizon [3]; in general the Hubble sphere is not associated with any type of horizon [4].

The static Einstein model[5] has $H = 0$ and hence λ and Ω are infinite. However, the relations discussed above hold when this model is seen as a limiting case.

The complete poster and some supplementary material can be found at http://www.astro.multivax.de:8000/helbig/research/publications/info/moriond2022_2.html .

References

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