Crank-Nicolson Scheme for Solving Low Mach Number Unsteady Viscous Flows Using an Implicit Preconditioned Dual Time Stepping Technique

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Introduction

Design of a density-based solver for unsteady compressible viscous flows.

- Time and space discretization schemes
- Numerical advective flux function
- Solver for large algebraic systems of non linear equations (Pseudo-transient Newton-GMRes)

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But what about these at the limit of incompressibility ?

$$\lim_{M\to 0} CN = \lim_{M\to 0} \frac{1+M}{M} = \infty$$

Introduction

Use of an artificial speed of sound c' by altering the transient behaviour of Navier-Stokes equations

$$c' \sim \|\mathbf{u}\| \longrightarrow CN \sim 1$$

Local preconditioning methods for steady viscous flows (Chorin [*JCP* 1967], Turkel [*JCP* 1987], Choi & Merkle [*JCP* 1993], Weiss & Smith [*AIAAJ* 1995])

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What about unsteady flows ? (Turkel [*AIAAP* 2003])

- Use of a dual time stepping technique
- Does the condition number depend on the time step ?
- Should the numerical flux function be modified ?

Choice of variables

Use of primitive variables $\mathbf{w}$ rather than conservative ones $\mathbf{s}$

$$p = \frac{p_d - p_0}{\rho_0\, U_0^2} \qquad \mathbf{u} = \frac{\mathbf{u}_d}{U_0} \qquad T = \frac{T_d}{T_0} \quad \rightarrow \rho(p,T) = \frac{\rho_d(p_d,T_d)}{\rho_0}$$

$$Str\, \frac{\partial}{\partial t}\, \mathbf{s} + \nabla \cdot \mathbf{F}_a = \frac{1}{Re}\, \nabla \cdot \mathbf{F}_d$$

$$\mathbf{s} = \begin{pmatrix} \rho \\ \rho\, \mathbf{u} \\ \rho\, H - \chi\, p \end{pmatrix} \qquad \mathbf{F}_a = \begin{pmatrix} \rho\, \mathbf{u}^T \\ \rho\, \mathbf{u}\, \mathbf{u}^T + p\mathbf{I} \\ \rho\, \mathbf{u}^T\, H \end{pmatrix} \qquad \mathbf{F}_d = \begin{pmatrix} \mathbf{0}^T \\ \mathbf{T} \\ -\frac{1}{Pr}\, \mathbf{q}^T + \chi\, \mathbf{u}^T\, \mathbf{T} \end{pmatrix}$$

$$Str = \frac{L_0}{U_0\, t_0} \qquad Re = \frac{\rho_0\, U_0\, L_0}{\mu_0} \qquad \chi = \frac{U_0^2}{Cp_0\, T_0}$$

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For a perfect gas :

$$H = T + \frac{1}{2}\, \chi\, \|\mathbf{u}\|^2 \qquad \rho = \frac{1 + \frac{\gamma}{\gamma - 1}\, \chi\, p}{T} \qquad \chi = (\gamma - 1)\, M_0^2$$

Space and time discretization

- Cell-centered finite volumes
- First order of accuracy on diffusive terms
- Second order of accuracy on advective terms

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$$CFL_c = M^{-1}\,CFL_u$$

$\Longrightarrow$ Second order implicit Crank-Nicolson scheme

Implicit pseudo-transient method

The Crank-Nicolson scheme is written as

$$Str\,\frac{\mathbf{s}\left(\mathbf{w}^{l+1}\right)-\mathbf{s}\left(\mathbf{w}^{l}\right)}{\Delta t^{l}} = \theta\,\mathbf{Rhs}\left(\mathbf{w}^{l+1}\right) + (1-\theta)\,\mathbf{Rhs}\left(\mathbf{w}^{l}\right)$$

Fully implicit pseudo-transient iterations are applied

$$\frac{\mathbf{s}_p\left(\mathbf{w}^{n+1}\right)-\mathbf{s}_p\left(\mathbf{w}^{n}\right)}{\Delta\tau^{n}} + \mathbf{f}\left(\mathbf{w}^{n+1}\right) = -\mathbf{g}\left(\mathbf{w}^{l}\right)$$

At each pseudo-time step, the following system is solved by GMRes iterations

$$\left[\frac{1}{\Delta\tau^{n}}\,\mathbf{P}^{n} + \mathbf{J}^{n}\right]\,\Delta\mathbf{w}^{n} = -\left(\mathbf{f}\left(\mathbf{w}^{n}\right) + \mathbf{g}\left(\mathbf{w}^{l}\right)\right)$$

$$\longrightarrow \mathbf{w}^{n+1} = \mathbf{w}^{n} + \Delta\mathbf{w}^{n}$$

Parameters of preconditioning

- The preconditioning matrix is written as

$$\mathbf{P} = \begin{pmatrix} \rho'_p & \mathbf{0}^T & \rho_T \\ \rho'_p\,\mathbf{u} & \rho\mathbf{I} & \rho_T\,\mathbf{u} \\ \rho'_p\,H + \rho\,h_p - \chi & \chi\,\rho\,\mathbf{u}^T & \rho_T\,H + \rho\,h_T \end{pmatrix}$$

- Density thermodynamical properties are modified : ρ'_p
- The artificial speed of sound c' is defined by

$$c'^2 = \frac{\rho\,h_T}{\rho\,h_T\,\rho'_p + \rho_T\,(\chi - \rho\,h_p)} \longrightarrow \rho'_p = \frac{1}{c'^2} - \frac{(\chi - \rho\,h_p)\,\rho_T}{\rho\,h_T}$$

- Finally the only parameter to be fixed is $M_* = \frac{c'}{c}$

Eigenvalues analysis

Does the condition number depend on the CFL_c number ?

- Pseudo-transient system in its quasi-linear form

$$\mathbf{P}\frac{\partial \mathbf{w}}{\partial \tau} + \frac{Str}{\Delta t}\,\mathbf{S}\,\mathbf{w} + \theta\left(\mathbf{A}_x\,\frac{\partial \mathbf{w}}{\partial x} + \mathbf{A}_y\,\frac{\partial \mathbf{w}}{\partial y} + \mathbf{A}_z\,\frac{\partial \mathbf{w}}{\partial z}\right) = \mathbf{0}$$

- Introducing a Fourier mode

$$\mathbf{w}\,(\tau, x, y, z) = \mathbf{w}_0\,\exp\left(\mathbf{k}^T\mathbf{x} - i\,\omega\,\tau\right)$$

- With $\mathbf{k}$ limited by the grid spacing :

$$||\mathbf{k}|| = \phi/\Delta \text{ with } \phi \in [0, \pi]$$

Eigenvalues analysis

If $u_k = \frac{\mathbf{u}^T\mathbf{k}}{\|\mathbf{k}\|}$ and $\lambda = \frac{\omega}{\|\mathbf{k}\|}$, the eigenvalues are found to be

$$\lambda_{1,2,3} = \theta\ u_k\ \left(1 - i\,CFL_u^{-1}\right)$$

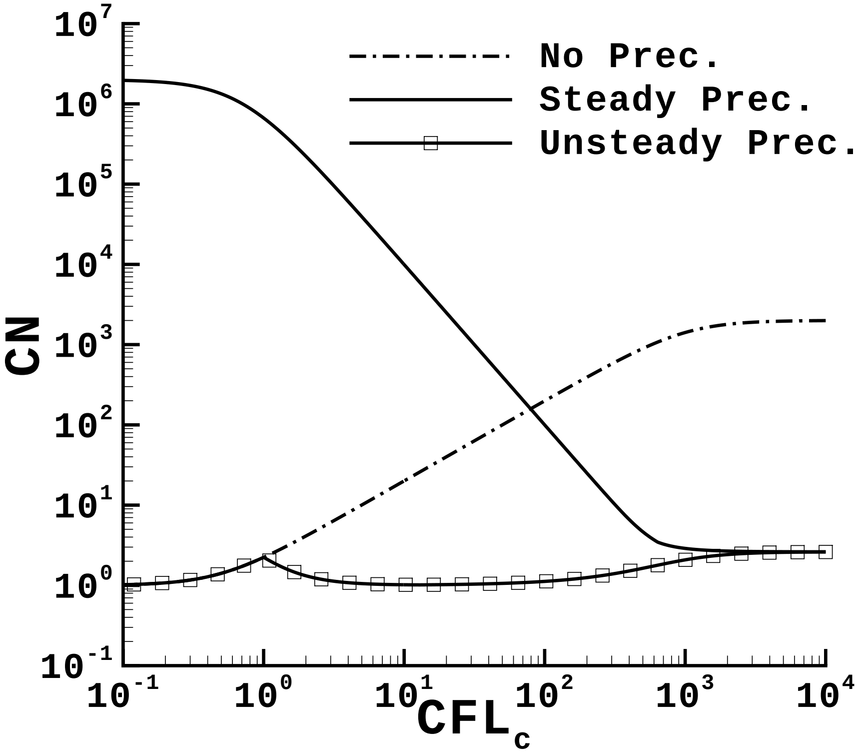
$$\lambda_{4,5} = \theta\ u_k\ \left(1 - i\,CFL_u^{-1}\right)\ T_\pm$$

$$T_\pm = \tfrac{1}{2}\ \left(1 + M_*^2\right) \pm \tfrac{1}{2}\ \sqrt{\left(1 - M_*^2\right)^2 - \frac{4\,M_*^2}{\left(i\,M + CFL_c^{-1}\right)^2}}$$

- Wave Speed $\longrightarrow \mathrm{Re}\,(\lambda)$
- Wave Damping $\longrightarrow \mathrm{Im}\,(\lambda)$

The condition number is $CN = \frac{\max(1,|T_+|,|T_-|)}{\min(1,|T_+|,|T_-|)}$

Preconditioning techniques



No Prec.	$M_* = 1$
Steady Prec.	$M_* = \min\left[1, \max\left(M, M_\epsilon\right)\right]$
Unsteady Prec.	$M_* = \min\left[1, \max\left(\sqrt{M^2 + CFL_c^{-2}}, M_\epsilon\right)\right]$

AUSM+up scheme

Should the numerical flux function be modified ?

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$$\widetilde{\mathbf{F}_a \cdot \mathbf{n}} = c_{\frac{1}{2}}\, M_{\frac{1}{2}} \begin{pmatrix} \rho \\ \rho\,\mathbf{u} \\ \rho\, H \end{pmatrix}_{L/R} + p_{\frac{1}{2}} \begin{pmatrix} 0 \\ \mathbf{n} \\ 0 \end{pmatrix}$$

$$M_{\frac{1}{2}} = \mathcal{M}^{+}_{(4)}\,(M_L) + \mathcal{M}^{-}_{(4)}\,(M_R) - M_p$$

$$p_{\frac{1}{2}} = \mathcal{P}^{+}_{(5)}\,(M_L)\; p_L + \mathcal{P}^{-}_{(5)}\,(M_R)\; p_R - p_u$$

$$M_p = K_p\;\max\left(1 - \overline{M}^2, 0\right) \frac{p_R - p_L}{f_c\left(\frac{2}{\gamma\, M_0^2} + p_R + p_L\right)}$$

$$p_u = K_u\, \mathcal{P}^{+}_{(5)}\,(M_L)\; \mathcal{P}^{-}_{(5)}\,(M_R)\; (\rho_L + \rho_R)\; f_c\, c_{\frac{1}{2}}\; (\mathbf{u}_R \cdot \mathbf{n} - \mathbf{u}_L \cdot \mathbf{n})$$

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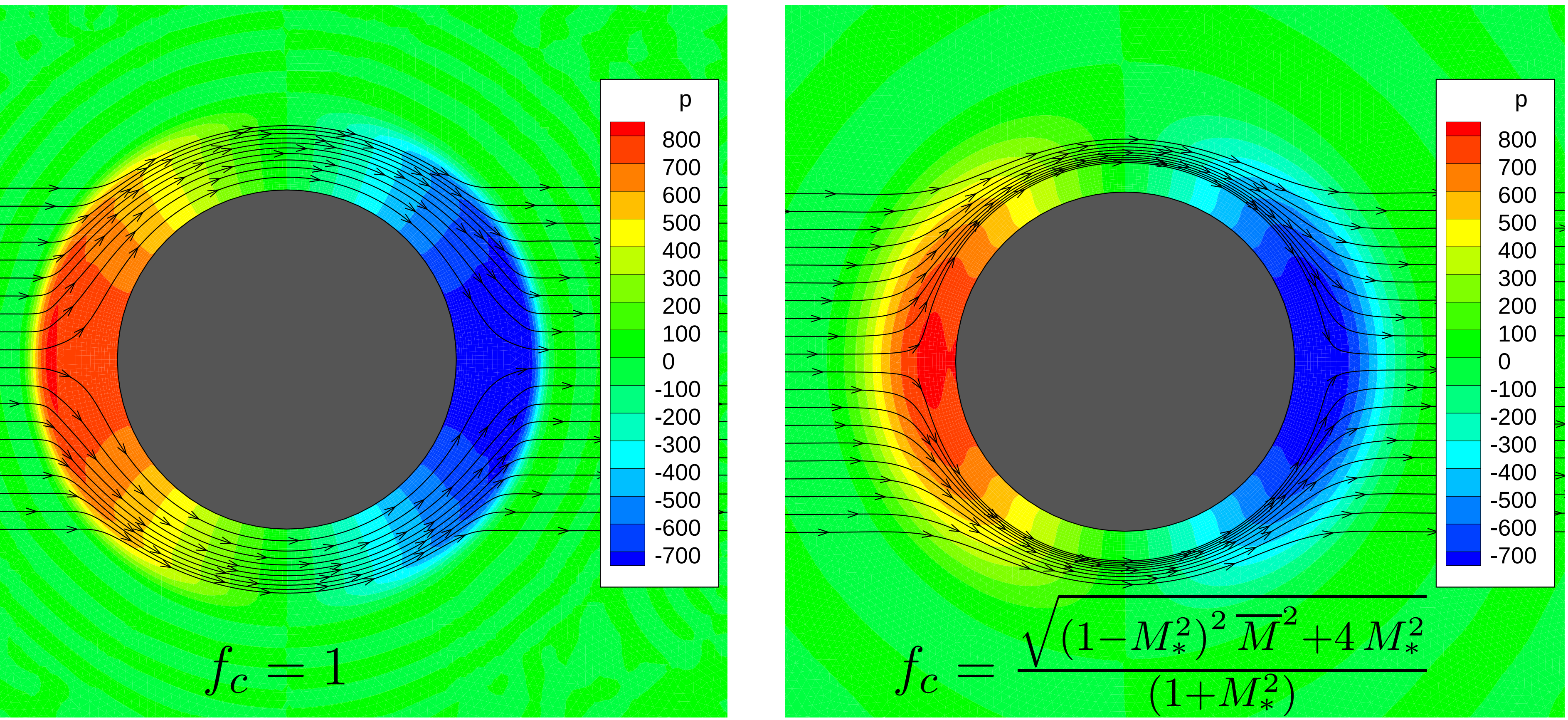
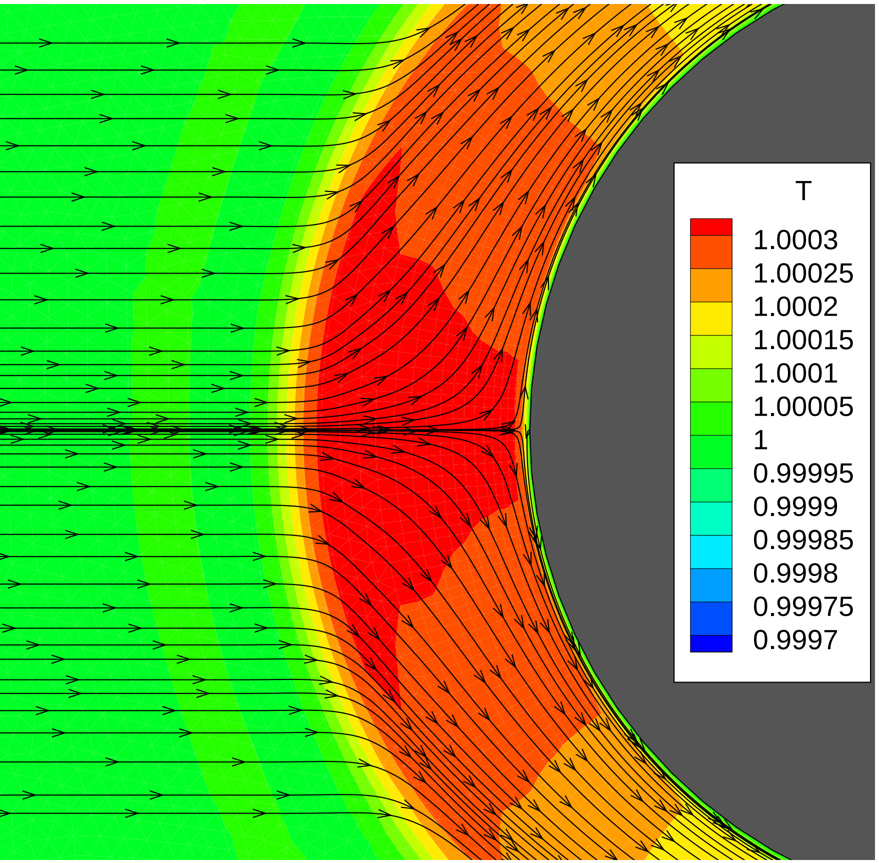
No Prec. $\rightarrow f_c = 1$ Steady Prec. $\rightarrow f_c = \dfrac{\sqrt{(1-M_*^2)^2\, \overline{M}^2 + 4\, M_*^2}}{(1+M_*^2)}$

AUSM+up scheme

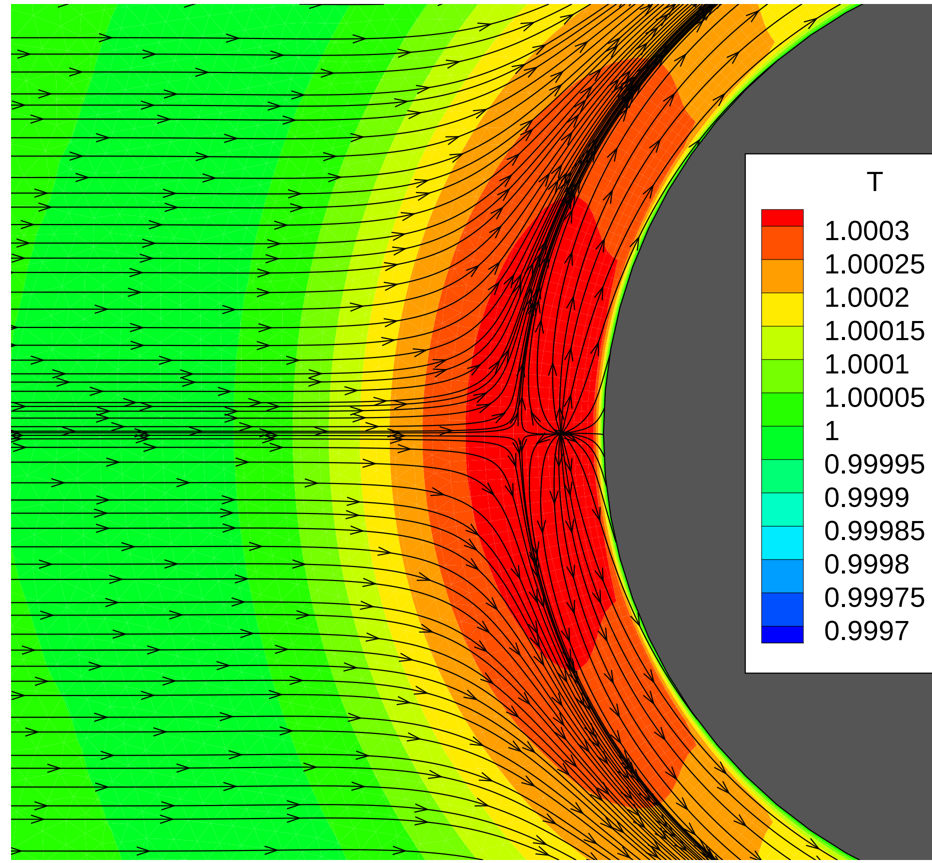
Impulsively starting cylinder at $M_0 = 10^{-3}$ and $CFL_C \sim 1$



AUSM+up scheme



$$f_c = 1$$



$$f_c = \frac{\sqrt{(1-M_*^2)^2\,\overline{M}^2 + 4\,M_*^2}}{(1+M_*^2)}$$

AUSM+up scheme

Modification of the function f_c

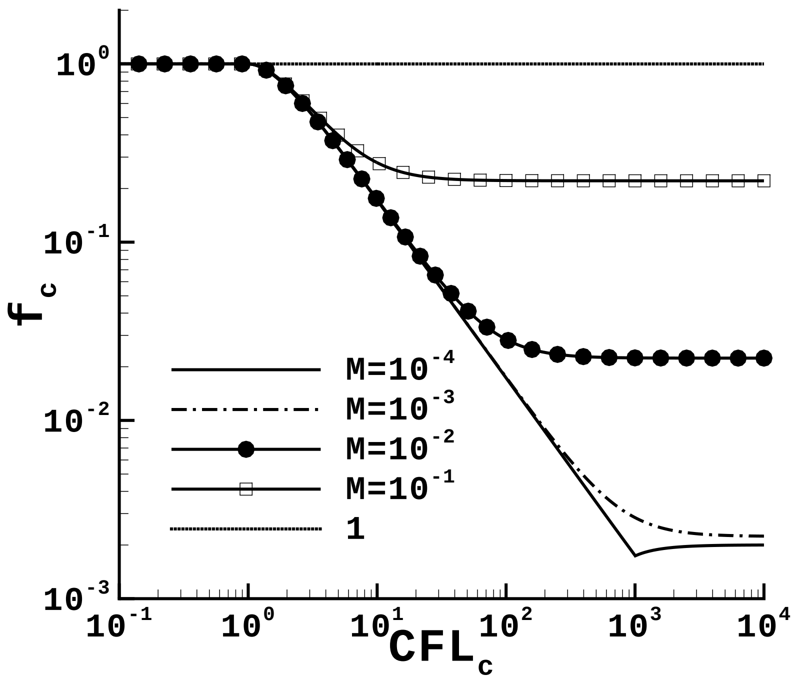
$$\lambda_{4,5} = \theta\ u_k\ \left(1 - i\, CFL_u^{-1}\right)\ T_\pm$$

$$T_\pm = \tfrac{1}{2}\ \left(1 + M_*^2\right) \pm \tfrac{1}{2}\ \sqrt{\left(1 - M_*^2\right)^2 - \frac{4\, M_*^2}{\left(i\, M + CFL_c^{-1}\right)^2}}$$

If we only look to the waves propagation speed

$$\begin{aligned}
\mathrm{Re}\,(\lambda_{4,5}) &= \theta\, u_k\ \left[\mathrm{Re}\,(T_\pm) + CFL_u^{-1}\ \mathrm{Im}\,(T_\pm)\right] \\
&= \theta\, u_k\, \frac{1}{2}\ \left(1 + M_*^2\right) \left[1 \pm \frac{1}{M}\ \frac{M\, \mathrm{Re}\left(\sqrt{*}\right) + CFL_c^{-1}\ \mathrm{Im}\left(\sqrt{*}\right)}{\left(1 + M_*^2\right)}\right] \\
&= \theta\, u_k\, \frac{1}{2}\ \left(1 + M_*^2\right) \left[1 \pm \frac{1}{M}\ f_c\right]
\end{aligned}$$

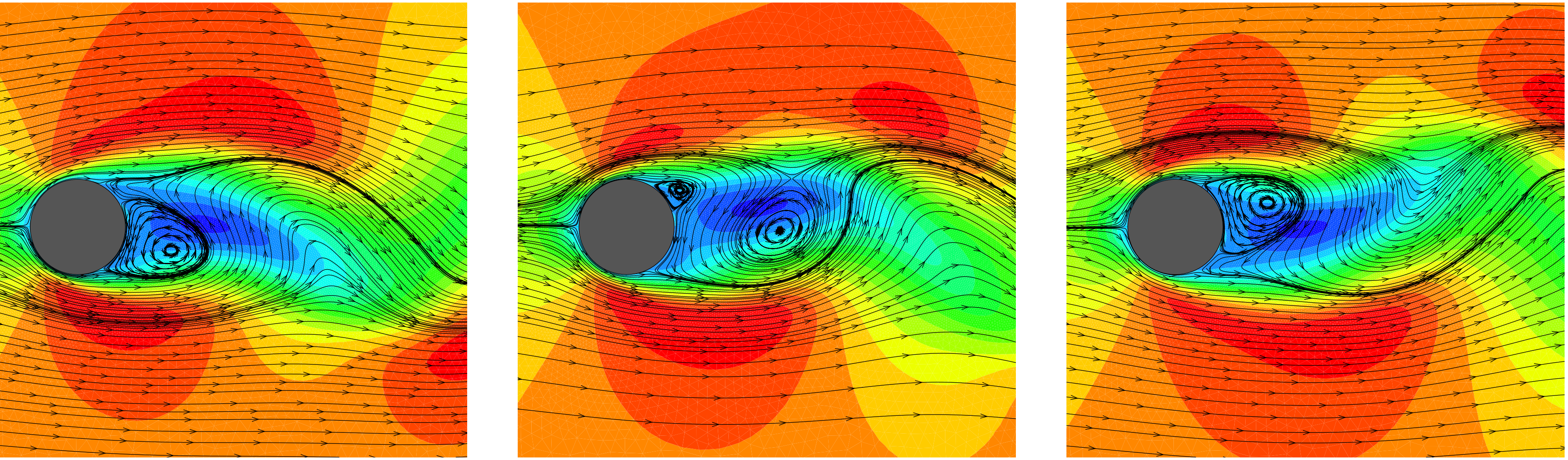
AUSM+up scheme



No Prec.	$f_c = 1$
Steady Prec.	$f_c = \dfrac{\sqrt{(1-M_*^2)^2\,\overline{M}^2 + 4\,M_*^2}}{(1+M_*^2)}$
Unsteady Prec.	$f_c = \dfrac{M\,\mathrm{Re}\left(\sqrt{*}\right) + CFL_c^{-1}\,\mathrm{Im}\left(\sqrt{*}\right)}{(1+M_*^2)}$

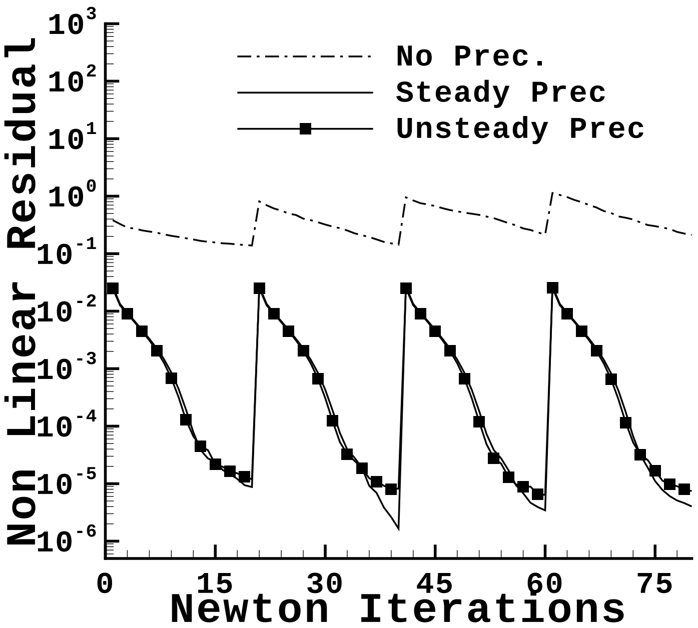
Periodic laminar flow past a cylinder

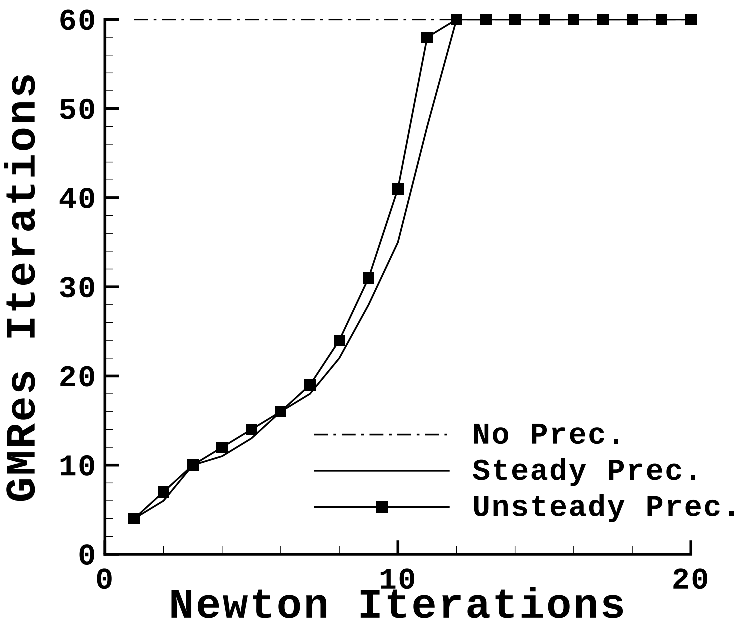
$Re = 100$, $M_0 = 10^{-3}$ and $CFL_c = 10^3$, $CFL_u = 1$



$\overline{C_D} = 1.32 \quad \Delta C_D = 0.0085 \quad \Delta C_L = 0.315 \quad Str_{\mathsf{VS}} = 0.166$

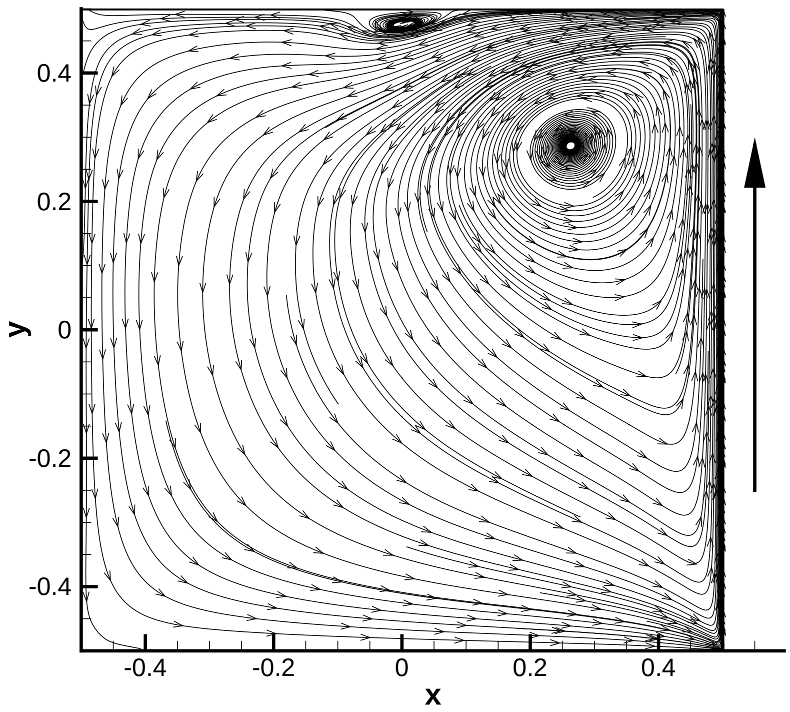
Periodic laminar flow past a cylinder





3D laminar lid driven cavity flow

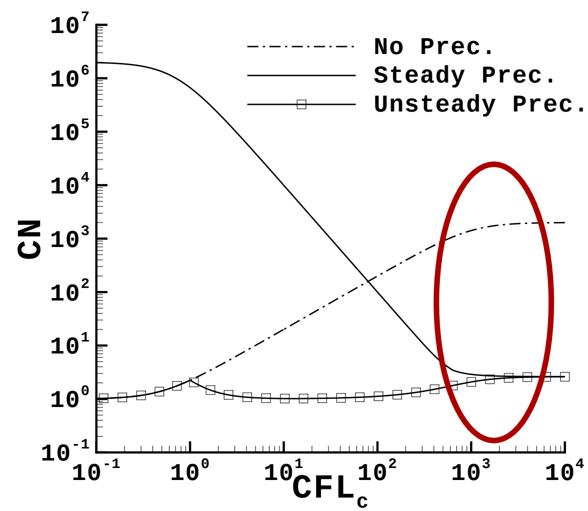
$Re = 1000$ and $M_0 = 10^{-3}$

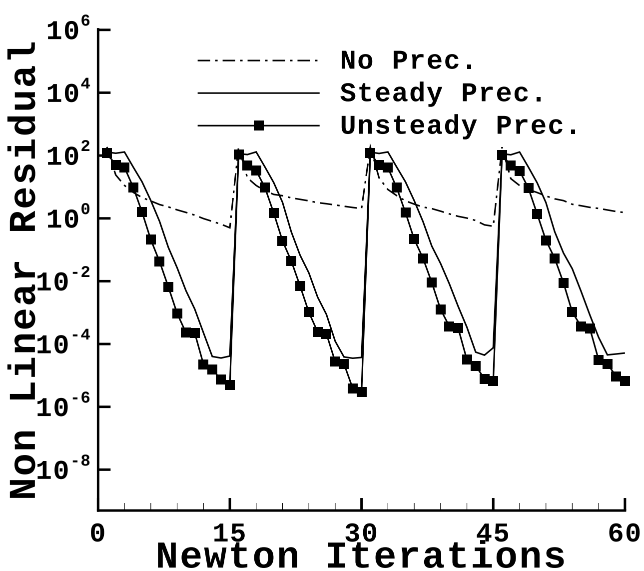


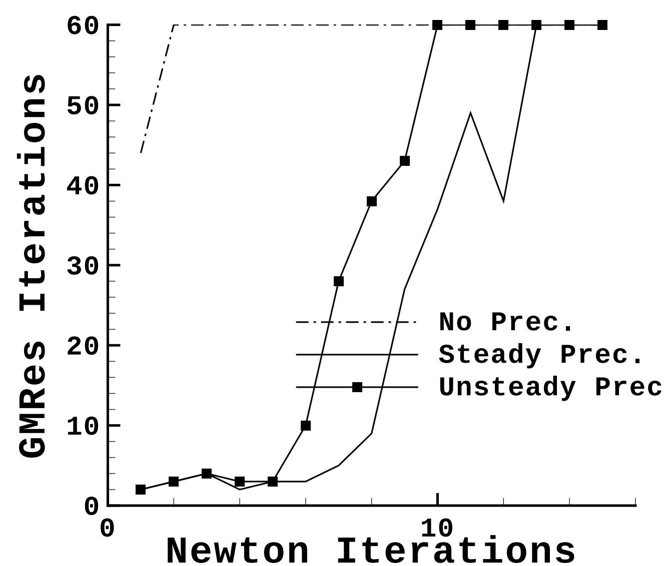
Convergence study for $CFL_c = 1,\ 10^2,\ 10^3 = M_0^{-1}$

3D laminar lid driven cavity flow

- $CFL_c = 10^3, CFL_u = 1$
- Steady and Unsteady Prec. converge
- Steady and Unsteady Prec. are equivalent
- No Prec. does not converge

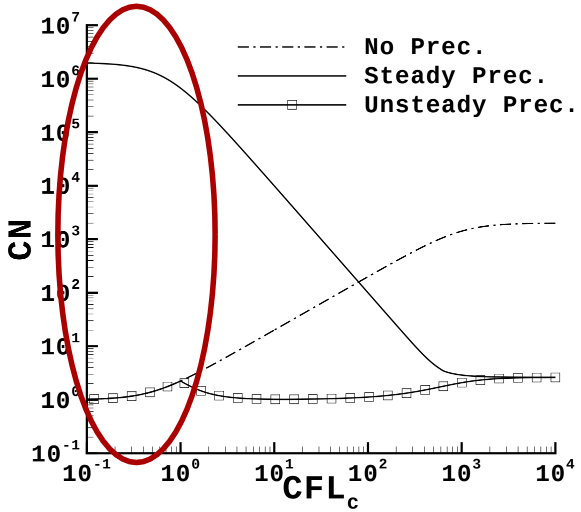


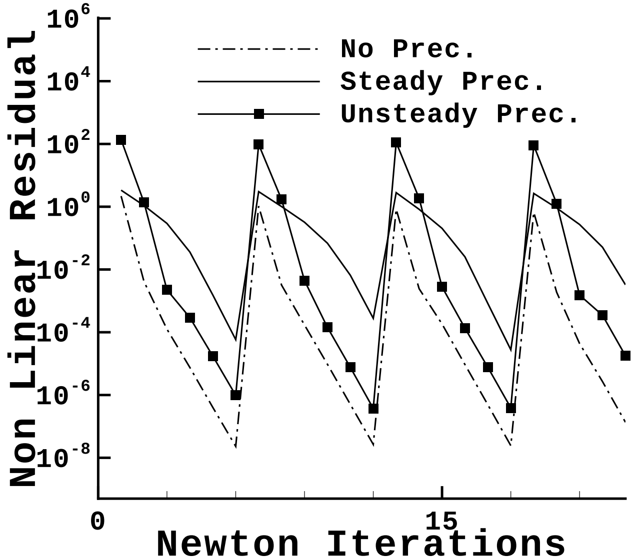


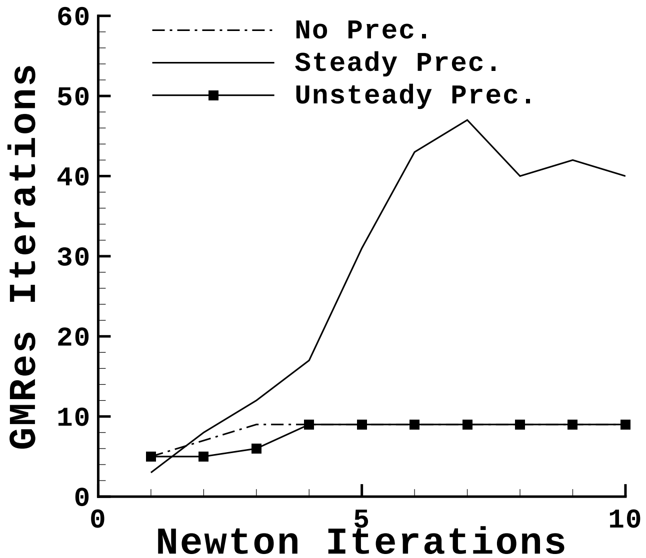


3D laminar lid driven cavity flow

- $CFL_c = 1, CFL_u = 10^{-3}$
- Unsteady and No Prec. converge
- Unsteady and No Prec. are equivalent
- Steady Prec. converges
  but lots of GMRes iterations are needed

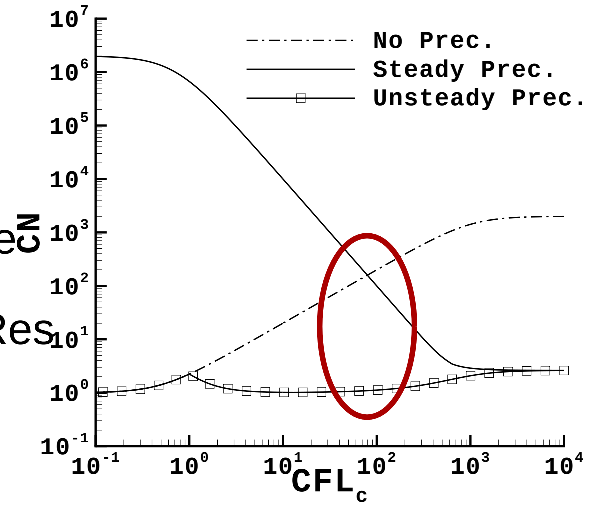


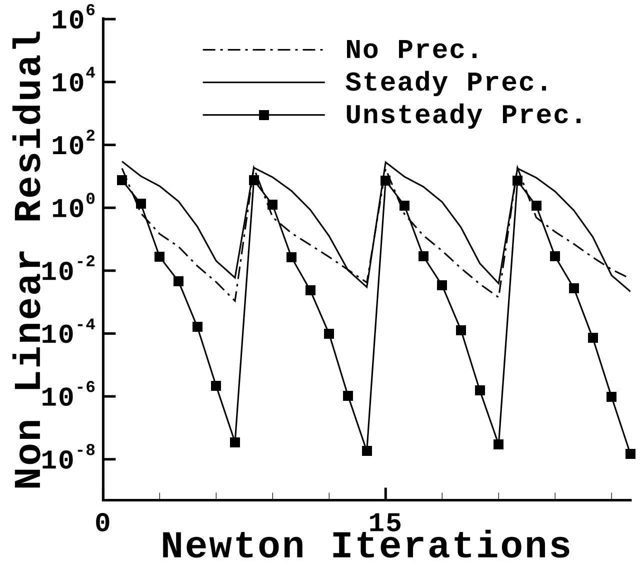


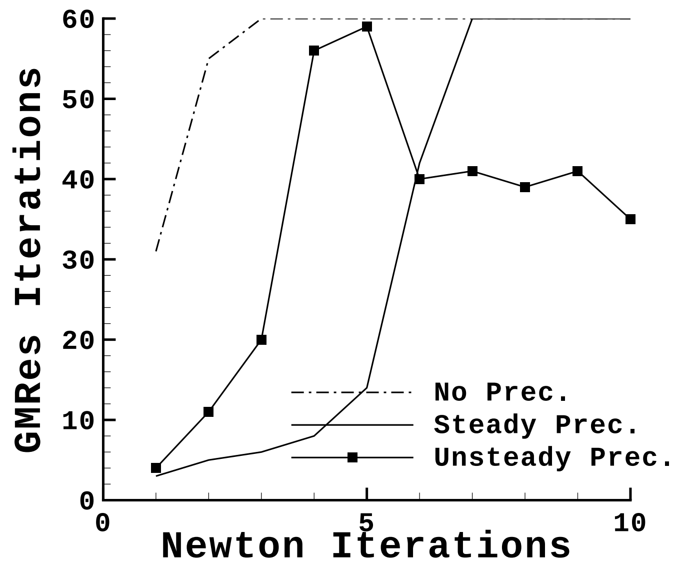


3D laminar lid driven cavity flow

- $CFL_c = 10^2, CFL_u = 10^{-1}$
- All methods seem to converge
  but Unsteady Prec. improves convergence
- Unsteady Prec. yields more efficient GMRes
- Effect of f_c can be seen for low residuals







Conclusions

- Condition number depends on CFL_c and Mach numbers
- New unsteady preconditioning improves convergence
- AUSM+up scheme has been modified for solving low Mach number flows with any CFL_c numbers
- What about the diffusion terms ?
- A better study of space and time accuracy should be done