EPJ manuscript No.
(will be inserted by the editor)

Arguments against the flatness problem in classical cosmology: a review

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Rec: 5 September 2020, Accp: 27 January 2021

Abstract. Several authors (including myself) have made claims, none of which has been convincingly rebutted, that the flatness problem, as formulated by Dicke and Peebles, is not really a problem but rather a misunderstanding. In particular, we all agree that no fine-tuning in the early Universe is needed in order to explain the fact that there is no strong departure from flatness, neither in the early Universe nor now. Nevertheless, the flatness problem is still widely perceived to be real, since it is still routinely mentioned as an outstanding (in both senses) problem in cosmology in papers and books. Most of the arguments against the idea of a flatness problem are based on the change with time of the density parameter Ω and normalized cosmological constant λ (often assumed to be zero before there was strong evidence that it has a non-negligible positive value) and, since the Hubble constant H is not considered, are independent of time scale. In addition, taking the time scale into account, it is sometimes claimed that fine-tuning is required in order to produce a Universe which neither collapsed after a short time nor expanded so quickly that no structure formation could take place. None of those claims is correct, whether or not the cosmological constant is assumed to be zero. I briefly review the literature disputing the existence of the flatness problem, which is not as well known as it should be, compare it with some similar persistent misunderstandings, and wonder about the source of confusion.

1 Introduction

Most reviews cover topics with a relatively large literature. Reviews of topics which are not covered in textbooks are often the only source of an overview of the topic, apart from creating one's own. This review is different. Not only is there relatively little literature on the flatness problem per se (though, of course, it is often *mentioned* and has been the *motivation* for much work), it is something which is mentioned in most cosmology textbooks, usually together with the claim that it is something which is not completely understood and/or somehow puzzling within the context of classical cosmology, often with the remark that inflation can solve the flatness problem. Moreover, most literature which explicitly discusses the flatness problem disputes the fact that there is a problem. The purpose of this review is thus to

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give an overview of research into the flatness problem itself within the context of classical cosmology and point out that such research contradicts what most people who are aware of the flatness problem (i.e., most cosmologists and almost all who have read cosmology textbooks) believe; such a review is thus necessary even though the literature is not that extensive.

I begin with an overview of classical cosmology, which also serves to define my notation, before discussing the traditional flatness-problem argument. After notes on reactions to the flatness problem and the role of observations, I then briefly review the important but often overlooked topic of the evolution of cosmological parameters with time, as that is crucial to some of the later arguments. Following a brief excursion as to the relevance, or lack thereof, of inflation to the discussion, I then review arguments in the literature against the flatness problem.

2 Notation, the basic idea, and standard arguments

The discussion at the beginning of this section and in Sect. 9 is somewhat briefer than but follows that by Helbig (2020); it is repeated here in order to make this paper more self-contained. In this paper, I consider only standard Friedmann–Robertson–Walker (FRW) models (see below for definition and references), because historically fine-tuning claims have been discussed within the context of those models; also, the issues remain even in more-realistic models.

For a homogeneous and isotropic ("Robertson–Walker") universe consisting of non-relativistic matter ("dust") of density ρ and the cosmological constant Λ (with dimension time^{-2}), the change in scale factor with time is described by the Friedmann equation[1]

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3} + \frac{\Lambda R^2}{3} - kc^2\,, \tag{1}$$

with the dimensionless constant k equal to -1, 0, $+1$ depending on spatial curvature (negative, vanishing, or positive, respectively – see definition below); R is the scale factor (with dimension length) of the universe, G the gravitational constant, and c the speed of light. A dot represents differentiation with respect to proper time. (If the ρ in Eq. (1) is taken to include contributions from all equations of state, it is still valid. Here, I consider only the components dust, cosmological constant, and curvature. Note that, instead of including the cosmological constant explicitly, one can use the corresponding density $\Lambda/(8\pi G)$ and include it in ρ.) It is useful to define the following terms:

$$\begin{array}{rlll} H := & \frac{\dot{R}}{R} & & \\ \lambda := & \frac{\Lambda}{3H^2} & & \\ \Omega := & \frac{\rho}{\rho_{\mathrm{crit}}} & \equiv \frac{8\pi G\rho}{3H^2} & \\ K := & \Omega + \lambda - 1 & & \\ k := & \mathrm{sign}(K) & & \\ q := & \frac{-\ddot{R}R}{\dot{R}^2} & \equiv \frac{-\ddot{R}}{RH^2} & \equiv \frac{\Omega}{2} - \lambda \end{array} .$$

The Hubble constant H has the dimension time^{-1}; all other quantities defined above are dimensionless: the normalized cosmological constant λ, the density parameter Ω, the curvature parameter K, and the deceleration parameter q. ρ_{crit} is known as the

[1] Such models are often known as Friedmann–Robertson–Walker (FRW) models (Friedmann, 1922, 1924; Robertson, 1935, 1936; Walker, 1935, 1937, the latter paper by Walker is very often incorrectly cited as having been published in 1936).

critical density.[2] See Helbig (2012) or Kayser, Helbig and Schramm (1997) for more details on this notation.

From the definitions above, it follows that

$$R = \frac{c}{H}\frac{k}{\sqrt{|\Omega + \lambda - 1|}} = \frac{c}{H}\frac{k}{\sqrt{|K|}} \tag{2}$$

for $k \neq 0$; for $k = 0$, the radius of curvature is infinite – the scale factor *at the present time* is then usually taken to be c/H_0.[3] Note that K is positive if the curvature is positive. Often, Ω_k is defined as $-K$, so that the Friedman equation is $\Omega_\mathrm{m} + \Omega_\lambda + \Omega_\mathrm{k} = 1$ ($\Omega_\mathrm{m} \equiv \Omega$, $\Omega_\lambda \equiv \lambda$).[4] (Another common form of the Friedmann equation is $K = kc^2/(H^2R^2)$.) Denoting the current epoch of observation with the suffix 0 and using the definitions above, one can write the Friedmann equation as

$$\dot{R}^2 = \dot{R}_0^2 \left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0 \right) . \tag{3}$$

From that, one can calculate the age of the universe as a function of the scale factor

$$t = \int_0^R \frac{\mathrm{d}R}{\sqrt{\dot{R}_0^2 \left(\frac{\Omega_0 R_0}{R} + \frac{\lambda_0 R^2}{R_0^2} - K_0 \right)}} . \tag{4}$$

(Note that Eq. (4) can be inverted to give the scale factor as a function of time.) Using the definition of H, Eq. (4) can be written as

$$t = \frac{1}{H_0} \int_0^R \frac{\mathrm{d}R}{\sqrt{\left(\frac{\Omega_0 R_0^3}{R} + \lambda_0 R^2 - K_0 R_0^2 \right)}} . \tag{5}$$

Alternatively, dividing the Friedmann equation, Eq. (3), by R^2 and factoring out R_0^{-2} on the right-hand side results in

$$\left(\frac{\dot{R}}{R} \right)^2 = \left(\frac{\dot{R}_0}{R_0} \right)^2 \left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2} \right) \tag{6}$$

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[2] For $\lambda = 0$ and $k = 0$, $\rho = \rho_\mathrm{crit} = (3H^2)/(8\pi G)$. That density is "critical" in the sense that, for $\lambda = 0$, a greater (lesser) density implies a positive (negative) curvature and a universe (assumed to be expanding now) which will collapse in the future (expand forever); similarly, for $k = 0$, a greater (lesser) density implies a negative (positive) cosmological constant and a universe (assumed to be expanding now) which will collapse in the future (expand forever). However, in the general case ($k \neq 0$ and $\lambda \neq 0$), ρ_crit doesn't have any special meaning, though Ω remains a useful parameter.

[3] In general, one can describe the change in time of the scale factor in relation to a fiducial, usually the current, scale factor; a is often defined as the relative scale factor R/R_0 and is thus dimensionless. Another common approach, which I use here, is to take the scale factor R to be the radius of curvature as given by Eq. (2) for $k \neq 0$. In this case, this works for any time t, not just t_0. For $k = 0$, R_0, the scale factor at a fiducial time (usually taken to be the present), is arbitrary but is often set to c/H_0. Note, however, that in general $R \neq c/H$, including the flat case where $R_0 = c/H_0$ ($R = c/H$ at all times in the special case of the relativistic equivalent of the Milne model with $\lambda = 0$ and $\Omega = 0$ and hence $k = -1$).

[4] I have long used K as defined above, as do Goliath and Ellis (1999), though Wainwright and Ellis (2005) define K with the opposite sign, using it as others use Ω_k. One also sees K used as I use k, (e.g. Ellis, Maartens and MacCallum, 2012).

or, due to the definition of H, in

$$H^2 = H_0^2 \left(\frac{\Omega_0 R_0^3}{R^3} + \lambda_0 - \frac{K_0 R_0^2}{R^2} \right) , \tag{7}$$

which expresses the Hubble constant as a function of the scale factor. From the definitions above follows the dependence of λ on H,

$$\lambda = \lambda_0 \left(\frac{H_0}{H} \right)^2 . \tag{8}$$

Since the density depends on the scale factor,

$$\rho = \rho_0 \left(\frac{R_0}{R} \right)^3 , \tag{9}$$

Ω depends on it as well as on H,

$$\Omega = \Omega_0 \left(\frac{H_0}{H} \right)^2 \left(\frac{R_0}{R} \right)^3 ; \tag{10}$$

note that H and R are related by Eq. (7). Thus, in an expanding universe, λ and Ω can increase with time only if H decreases.[5]

Dicke and Peebles (1979), building on early ideas by Dicke (1970), claimed that a universe with $\Omega \neq 1$[6] is inherently unstable. Many concluded from that that $\Omega = 1$ must hold exactly, which, if one assumes that $\Lambda = 0$ (which was common in the time after Dicke and Peebles (1979) until observations made it clear in the 1990s that $\Lambda > 0$), implies that our Universe must be the Einstein–de Sitter universe. For example, Charlton and Turner (1987) took that to mean that the Universe must be flat and investigated various smooth components in addition to the known constituents, the idea being that such components, as long as not observationally ruled out, could resolve the conflict between the "theoretical prejudice" (their words) in favour of a flat Universe and observations suggesting a lower density (which implies a non-flat Universe if one assumes that $\Lambda = 0$, as many did at the time). They discussed various cosmological tests which could be used to discriminate between the various models studied. Many other works followed that pattern, often invoking time-dependent "constants" of nature, especially in order to explain a Universe which is nearly flat but not (for all practical purposes) exactly flat (e.g. Turner et al., 1984; Vilenkin, 1984; Vittorio and Turner, 1987; Turner and White, 1997; Spergel and Pen, 1997; Barrow and Magueijo, 1999). Turner et al. (1984) describe not only their goal but one shared with the other authors as well: "Theoretical prejudices argue strongly for a flat universe; however, observations do not support this view. We point out that this apparent conflict could be resolved if . . . " They have in common taking the flatness problem as given, and thus the belief in a flat Universe and the (at the time)

[5] That is an important point. Since all non-empty big-bang models begin their evolution arbitrarily close to the Einstein-de Sitter model with $\lambda = 0$ and $\Omega = 1$, large values of those parameters can be due *only* to a low value of the Hubble constant.

[6] They assumed that $\Lambda = 0$. If one replaces Ω with $\Omega + \lambda$, then some, but not all, of their arguments still hold. For example, if $\Omega + \lambda = 1$ exactly, that holds for all time. On the other hand, the individual values of Ω and λ evolve with time (even though their sum is constant) in a flat universe (except in the cases of the Einstein–de Sitter universe with $\Lambda = 0$ and $\Omega = 1$ and the de Sitter universe with $\Lambda = 1$ and $\Omega = 0$; if $\Lambda < 0$ then both evolve to $(-)\infty$ at the time of maximum expansion (such universes always collapse in the future)).

lack of observation of at least near-flatness of our Universe. Today, the flatness of the Universe is an observational fact, to a much greater extent than in the time of Dicke and Peebles (1979) ($\Omega_0 + \lambda_0 \approx 1$ to within a per cent or better, rather than to within an order of magnitude – nevertheless, that was close enough to inspire the flatness problem in the first place); the point of this paper is to rebut the former point. Of course, such a rebuttal does not rule out such alternative cosmologies, but reduces the need for them, assuming that there is no other convincing evidence in their favour. While those papers attempted to resolve the conflict between theoretical prejudices for $\Omega_0 = 1$ and observations indicating a low-density Universe, others simply assumed the now almost standard flat Universe as a backdrop for discussing other ideas (e.g. Vittorio and Silk, 1985; Waga, 1993; Martel, 1995; Malhotra and Turner, 1995; Abdel-Rahman, 1997).

By "unstable", I mean that a slightly different initial value would lead to a vastly different later state, i.e., sensitivity to initial conditions. In this paper, I am not concerned with instability in the sense of sensitivity to random and/or external perturbations. Of course, the latter implies the former, but not vice versa. So while both the Einstein–de Sitter model and Einstein's static model, static because it is infinitely fine-tuned such that attraction due to matter and repulsion due to the cosmological constant are exactly balanced, are unstable (in both senses), in this paper I am concerned with only the first sense. In a *perfect* FRW universe, such unstable fixed points are actually stable, because there is no way to perturb them. Of course, that does not apply to a universe which corresponds to the Einstein–de Sitter model or Einstein's static model only on a large-scale average, but such instabilities are beyond the scope of this paper. See the beginning of Sect. 5 for more details.

Note that Dicke (1970) already mentioned two aspects of the traditional flatness problem (see below), namely the initial fine-tuning and the time-scale problem. It is also clear that he was not merely pointing out a feature of FRW models, but believed that there is a real fine-tuning problem: "There seems to be no fundamental theoretical reason for such a fine balance", which also implied the time-scale problem, either a runaway universe or a short-lived one: "No stars could have formed in such a universe, for it would not have existed long enough to form stars". At least at the time, he didn't see the Anthropic Principle as a possible explanation, but believed that "the universe was very nearly flat for a good physical reason" (Dicke, 1990).[7] In contrast, Collins and Hawking (1973) explicitly invoked the Anthropic Principle, and in particular *other* work by Dicke (1961) and unpublished work by Carter from 1968 which appeared later (Carter, 1974). The Anthropic Principle is largely irrelevant for research into the flatness problem because if the latter is explained by the former, then

[7] Although historically the issues of the flatness problem, the Anthropic Principle, fine-tuning, and the Multiverse have been intertwined, appearing in various combinations, in this paper about the flatness problem, I mention the Anthropic Principle here because it informs about Dicke's reasoning, and otherwise only when referring to previous work which mentioned it or when it appears to be the only possible explanation. The discussion of fine-tuning is restricted to the sense in which it is usually used in connection with the flatness problem, and the Multiverse is only shorthand for any (real or imagined) ensemble of universes. (The term "Multiverse" is usually used for an ensemble of universes which exists in some physical sense, as opposed to merely a range of mathematical possibilities.) For some history of the relationship between the flatness problem and the Anthropic Principle, see Williams (2007); for the Multiverse in general, see Carr (2007); for the Anthropic Principle in general, see Barrow and Tipler (1988); for fine-tuning, including its relation to the Multiverse, see Lewis and Barnes (2017) and Adams (2019); for various different Multiverses, including the relation of some of them to fine-tuning, see Tegmark (2014). Of course, the whole point of the Anthropic Principle is to explain things which seem to be fine-tuned, and some sort of Multiverse is usually invoked to provide the necessary ensemble.

no other explanation is needed; thus, despite the interesting historical connections, I don't dwell on the Anthropic Principle here. Note, however, that the flatness problem played an important role in the development of the Anthropic Principle (Hawking, 1990).[8]

Since most references to the flatness problem in the literature can be traced back to Dicke and Peebles (1979) and/or Dicke (1970), it is worth quoting the entire text from the latter on the flatness problem:

> Another matter [the first is the horizon problem, discussed in his preceding paragraph] is equally puzzling. The constant v_0^2 in eq. (4) is very small, so small that we are uncertain with our poor knowledge of ρ as to whether or not it is zero. But the first term on the right of eq. (4) was very much larger much earlier, at least 10^2 times as great when the galaxies first started to form and at least 10^{13} times as great when nuclear reactions were taking place in the "fireball," assuming that "fireball" story [the hot big bang] to which I shall return is correct. The puzzle here is the following: how did the initial explosion become started with such precision, the outward radial motion became [sic] so finely adjusted as to enable the various parts of the Universe to fly apart while continuously slowing in the rate of expansion?
>
> There seems to be no fundamental theoretical reason for such a fine balance. If the fireball had expanded only .1 per cent faster, the present rate of expansion would have been 3×10^3 times as great. Had the initial expansion rate been .1 per cent less and [sic] the Universe would have expanded to only 3×10^{-6} of its present radius before collapsing. At this maximum radius the density of ordinary matter would have been 10^{-12} gm/cm^3, over 10^{16} times as great as the present mass density. No stars could have formed in such a Universe, for it would not have existed long enough to form stars.

His v_0^2 is essentially the negative of the last term in Eq. (1) and his "first term on the right" is essentially the first term on the right of Eq. (1) (note also that Dicke assumes $\Lambda = 0$). I'll discuss various aspects of the flatness problem below. For now, note that Dicke sees a **fine-tuning problem** (*"how did the initial explosion become started with such precision"*) and a time-scale problem ("it would not have existed long enough to form stars"). He doesn't explicitly mention the **instability problem**, i.e., *why is $\Omega \approx 1$ today, given that it starts out at arbitrarily close to 1 but, unless it starts out at exactly 1, it will evolve away from 1.* (The instability problem is more general, as the time-scale problem requires additional assumptions – see Sect. 9 – , though the two are related; most discussion of the flatness problem has been concentrated on the more general instability problem rather than on the time-scale problem, and also on the fine-tuning problem.) Note that his gedankenexperiment consists of perturbing only the expansion rate, while keeping the time after the big bang (presumably the time "when nuclear reactions were taking place", i.e., the time of big-bang nucleosynthesis) constant.

Dicke and Peebles (1979) have two paragraphs on the flatness problem (separated by one discussing an aspect of the horizon problem, which they had also discussed in their preceding paragraph):

> The relationships of widely separated parts of the universe [i.e., the horizon problem] are not the only problem. There is a remarkable balance of mass

[8] Of course, wrong assumptions can nevertheless lead to interesting developments. An example from cosmology is Einstein's mistaken belief that the Universe is static which led him to introduce the cosmological constant (Einstein, 1917; O'Raifeartaigh et al., 2017), which turned out to be correct, though of course he could have introduced it even in a non-static Universe.

density and expansion rate. In general relativity theory with $\Lambda = 0$ the two are related by the equation

$$H^2 = \left(\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 = \frac{8}{3}\pi G\rho(t) - \frac{c^2}{R^2a^2} \tag{9.1}$$

where $a(t)$ is the expansion parameter, R is a constant, and $|R|a(t)$ is the magnitude of the space curvature (measured in a hypersurface of roughly constant galaxy proper number density, at fixed cosmic time t). The present relative value of the two terms on the right side of this equation is poorly known, because the mean mass density, ρ, is so uncertain, but it is unlikely that the first term is less than 3 per cent of the magnitude of the second. Since ρ varies as a^{-3} (or more rapidly if pressure is important), the mass term on the right-hand side dominates the curvature term when a is less than about 3 per cent of its present value. Tracing the expansion back in time, one finds that at $t \sim 1$ s, when much of the helium is thought to have been produced, the mass term is some 14 orders of magnitude larger than the curvature term. This means the expansion rate has been tuned to agree with the mass density to an accuracy better than 1 part in 10^{14}. In the limit $t \to 0$ this balance between the effective kinetic energy of expansion, measured by H^2, and the gravitational potential energy, measured by $\frac{8}{3}\pi G\rho$, is arbitrarily accurate.

Their notation is similar to mine, but their R is my R_0 and their a is my R/R_0. Note that while the characterization of fine-tuning by Dicke (1970), "how did the initial explosion become started with such precision", could perhaps be interpreted as simply asking why the Universe is described by an FRW model, Dicke and Peebles (1979) make it clear that they are puzzled by the "remarkable balance of mass density and expansion rate" which "has been tuned to agree with the mass density" and even that it is "arbitrarily accurate" in the limit $t \to 0$. Since that is a generic feature of FRW models, there is nothing remarkable if one assumes, as they do here, an FRW model, thus they seem to be puzzled by the values of the terms in the Friedmann equation, even *given the fact* that the Universe is described by an FRW model. Their discussion of the limit $t \to 0$ makes it clear that they are concerned with the fine-tuning aspect of the flatness problem, while the statement "it is unlikely that the first term is [now] less than 3 per cent of the magnitude of the second" refers to the instability problem, i.e., why is $\Omega \approx 1$ today.

Their next paragraph on the flatness problem discusses the time-scale problem, though perhaps it is not immediately obvious:

> The global H and ρ say that the present value of $c/a|R|$ is less than about $10^{-17}\mathrm{s}^{-1}$. It seems curious that such a small quantity should have been built into the universe at the big bang, and so it has often been suggested that the only "reasonable" value is $R^{-2} = 0$. This was more or less the position of Einstein and de Sitter (1932). By then it was recognized that Λ and R^{-2} both could be negative as well as positive, and that by appropriate choices of these two parameters one could arrive at quite a variety of model universes. Einstein and de Sitter suggested that Λ be dropped, and that, until the observational measures of curvature improve, it is reasonable to concentrate on the simplest case, $R^{-2} = 0$.

In other words, if the curvature term is small, it is likely to be exactly 0. Whether the first implies the second involves a discussion of fine-tuning in general which is beyond the scope of this paper. It is not a coincidence, though, that $10^{-17}\mathrm{s}^{-1}$ is approximately the reciprocal of the Hubble time $1/H_0$, thus connecting the flatness problem with the "oldness problem".[9] The idea here is that (assuming, as they do, $\Lambda = 0$) a flat universe is perfectly flat while one with $\Omega > 0$ and hence $k = +1$ would have a very large radius of curvature, i.e., R^{-2} very small, implying some sort of fine-tuning. It is not completely clear, but they might also be alluding to the fact that a flat universe will expand forever while one with $\Omega > 0$ and hence $k = +1$ will collapse after a finite time but, if that time is of the order of the age of our Universe, that implies fine-tuning to get a universe as old as our Universe. While that might be fine-tuning in some sense, if so, then there is an obvious weak-anthropic explanation, as discussed in Sect. 9. (At the same time, simply asserting infinite fine-tuning as a better explanation than a very high degree of fine-tuning is somewhat questionable in this context. In other contexts, where fine-tuning involves some sort of cancellation mechanism, then that is less questionable; one deems that some (unknown) symmetry principle or whatever is more likely than a very close near-cancellation.)

Note that their characterization of Einstein and de Sitter (1932) is somewhat questionable. The assertion that it was "more or less the position of Einstein and de Sitter" that "[i]t seems curious that such a small quantity should have been built into the universe at the big bang" and thus that R^{-2} is a "reasonable" value is at best unclear, since Einstein and de Sitter didn't give that as a reason. As is well known, Einstein thought Λ to be unnecessary once it was discovered that the Universe is expanding, and the assumption that $k = 0$ was explicitly because "[t]here is no direct observational evidence for the curvature" (Einstein and de Sitter, 1932). The latter jibes better with the statement that it was only a practical matter "that, until the observational measures of curvature improve, it is reasonable to concentrate on the simplest case, $R^{-2} = 0$", which of course is in contradiction with believing that it *must* be exactly 0. O'Raifeartaigh et al. (2015) note that Einstein was not convinced that we live in a flat universe, but rather that his interest in the Einstein–de Sitter model was to point out that a finite density of matter does not necessarily imply spatial curvature if the universe is not static, perhaps in contrast to the fact that that most of the models which had been discussed up until that time had had $k = +1$. Also, his interest was to find the simplest model which could account for observation, but allowing for the possibility that it be refined if demanded by observations; in the words of Einstein as translated by O'Raifeartaigh et al. (2015): "Of course, this does not mean that such a curvature (positive or negative) does not exist." The paper[10] analysed and translated to English by O'Raifeartaigh et al. (2015) provides more details on Einstein's thinking at the time. Nevertheless, Einstein and de Sitter (1932) do write, in their two-page paper, that "the question arises whether it is possible to represent the observed facts without introducing a curvature at all". The final sentence is also significant: "The curvature is, however, essentially determinable, and an increase in the precision of the data derived from observations will enable us in the future to fix its sign and to determine its value." In other words, the Einstein–de Sitter model was a purely practical suggestion, making things as simple as possible but not simpler,[11] and there is no indication that they favoured $k = 0$ on probability grounds. The goal of finding the simplest possible explanation (which, of course, is not in conflict with any observation) was a guiding maxim thoughout Einstein's career

[9] The term "age problem" is usually used for a conflict between the age of the oldest objects in the Universe and the age of the Universe if computed using a too high value of H_0.

[10] That paper is contained in a book (Einstein, 1933) which is a collection of three works by Einstein, translated from German to French by Maurice Solovine.

[11] An aphorism along those lines is often attributed to Einstein, but, though it is probable that he said something similar, and certainly it is a valid characterization of his thought, there is no direct written evidence for such a phrase (O'Toole, 2011).

(e.g. van Dongen, 2010). Interestingly, Einstein and de Sitter each did not think that the paper was very important, but each believed that the other did (Eddington, 1940b; Nussbaumer and Bieri, 2009).

Peebles (1993) says that the Einstein–de Sitter model is to be preferred both because it is the simplest case and because inflation implies that the Universe is flat, but then does add the caveat that part of the energy density of the Universe could be in the cosmological constant (though noting that astronomically preferred values are a problem for particle physicists) and notes that it is a good idea to try to measure all terms in the Friedmann equation, rather than simply assuming that some are zero. He then notes that the Dicke coincidences argument suggests that models in which more than one term is non-zero are problematic, and then suggests examining those in which two, but not all three, are non-zero. (Though to me it seems that if one accepts the coincidences argument then it doesn't make that much difference whether two or three terms are non-zero.) Somewhat later on, he notes that the coincidences argument suggests that a universe with a long quasi-static phase is unlikely, which is true, but does not point out that that implies that fine-tuning is necessary to *avoid* a significantly non-flat universe within the corresponding class of models ($k = +1$ and $\lambda_0 > 0$).[12]

Note that Peebles and Ratra (2003) discuss the flatness problem as a variant of the coincidence problem, e.g., if Ω is not exactly 1, then one has to explain why it is of the same order of magnitude as some other component, i.e., λ or the curvature term, due to the fact that the ratio changes with time. (Note that even in a flat universe, $\Omega + \lambda = 1$, unless one or the other is exactly 0, the ratio depends on time. However, it differs appreciably from 1 only shortly after the big bang and, if $\Lambda > 0$, also when the universe has essentially reached the asymptotic de Sitter phase, but that would be no more puzzling than living at a finite time after the big bang in a universe which lasts forever.) "This would be a most remarkable and unlikely-looking coincidence". As discussed below, that is actually not the case here.[13] The next sentence is: "The multiple coincidences required for the near vanishing of $\dot{a}$ and $\ddot{a}$ at a redshift not much larger than unity makes an even stronger case against Lemaître's coasting model, by this line of argument." The irony here is that Lemaître's coasting model (Lemaître, 1927, 1931a,b, 1933, 1934), a universe which goes through a quasi-static phase around the inflection point of $R(t)$, is, by the usual definition, a highly non-flat universe during the coasting phase, i.e., Ω and λ are $\gg 1$, but such a universe requires fine-tuning in order to exist – essentially a reverse of the traditional flatness-problem argument (which implicitly assumes $\Lambda = 0$), which claims that fine-tuning is needed in order to *avoid* $\Omega \gg 1$. One of the goals of this paper is to convince the reader that the former argument is false and the latter true.

Since R is inversely proportional to $\sqrt{\Omega + \lambda - 1}$, the term "flatness" is associated with $\Omega + \lambda = 1$. However, the actual radius of curvature is also inversely proportional

[12] I distinctly remember a similar comment in a semi-popular article (including pictures of astronomers) but attributed directly to Dicke rather than Peebles, though I don't remember if it was an actual reference, a summary of a remark he made, or just something which followed from his thinking. Any hints as to where I might find that would be highly appreciated.

[13] However, in general, *if two quantities are unrelated*, the default expectation should be that their ratio is not ≈ 1. There is some confusion here because sometimes it is claimed that "coincidences" where some ratio is ≈ 1 need an explanation, while other times it is claimed that ratios which are *not* ≈ 1, i.e., very large or very small numbers, are in need of an explanation. A little thought shows that one needs an explanation where some ratio is ≈ 1, the exception, usually in a particle-physics context, being very small numbers which come about via some sort of near-cancellation, that near-cancellation of course requiring a ratio very close to 1.

to the Hubble constant H. In an expanding universe, the radius of curvature in physical units, which is often (as in this paper) equivalent to the scale factor, of course increases with time. In discussions of the flatness problem, the discussion is always about $\Omega + \lambda$, not about the physical radius of curvature. Because Ω and λ are proportional to H^{-2}, equivalently the discussion concerns the radius of curvature in units of the Hubble length c/H. For example, Lemaître's coasting universe always has $\dot{R} > 0$, though $\dot{R} \approx 0$ in the coasting phase. (Since $z = R_0/R - 1$, during such a phase a small difference in redshift z corresponds to a large difference in many forms of distance.) In the coasting phase, however, since $\dot{R}$ and hence H become very small, Ω and λ become very large, hence such a universe is considered to be highly non-flat in that the radius of curvature is small in units of the Hubble length. (Of course, the changes – in the Hubble length and in Ω and λ – cancel so that the physical radius of curvature continues to grow throughout the expansion of the universe.) Similarly, the universe shortly after inflation is considered to be highly flat, even though the physical radius of curvature might be only a metre or so.

Peebles and Ratra (2003) claim that the coincidences argument is an extension of an argument by Bondi (1960). They note that Bondi remarks on the "remarkable property" that Ω is independent of time. The full quote from Bondi is

> Another model that deserves attention now is that of Einstein and de Sitter in which $k = 0$ and $[\Lambda] = 0$ so that $R(t) \sim T^{\frac{2}{3}}$. In addition to its outstanding simplicity, this model has the remarkable property (unique among the relativistic models) that $[\Omega]$ is constant. It may well be argued that as important a simple pure number as $[\Omega]$ should be constant during the evolution of the universe so as to provide in some sense a constant background to the application of the theory.

While Bondi's argument could be seen as a precursor to the coincidences argument, to me it seems more like the desire to have some sort of "steady state" model. (To be pedantic, models with $\Omega = 0$ also have Ω constant in time. If in addition $\lambda = 0$ or $\lambda = 1$, then λ is also constant in time, and of course $\lambda = 0$ always implies that λ is constant in time.) They also cite McCrea (1971) as presenting an early version of the argument, but all he says is

> Even this explanation may leave room for suspicion because the parameters q_0, σ_0 also depend on t_0. It may appear strange that just at our cosmic epoch these parameters should have values of the order of unity. However, it is the case that the "interesting" part of the q_0, σ_0-plane is the part in which these parameters lie between zero and a fairly small multiple of unity (in absolute value).

($\sigma_0 = \Omega_0/2$; $q_0 = \sigma_0 - \lambda_0$; t_0 is the present age of the universe.) It is not completely clear what he means by "explanation", but probably the fact that t_0, the Hubble time $1/H_0$, and $\Lambda^{-\frac{1}{2}}$ are all of the same order of magnitude (that implies that $\lambda \approx 1$), as alluded to in his preceding paragraph. It is also not clear what he means by the "interesting" part of the parameter space. Thus, he suspects that there is some reason for that (and, indeed, there is) but does not claim that some values must be exact since otherwise they would be expected not to be ≈ 1 but rather $= 1$. His reasoning would be consistent with a Dicke-style coincidences argument, but also with other explanations for the near coincidence such as those discussed in this paper.

Although writing from a historical point of view, Peebles (2020) presents the coincidence argument in essentially the same form, ignoring most of the literature on the flatness problem in the last forty years. That is probably because that topic is much less important to him than other topics (mostly in the field of "physical cosmology", in particular structure formation, big-bang nucleosynthesis, and the CMB; as far as I know, he has not revisited the topic (apart from referring to it) since Dicke and Peebles (1979)); Peebles (2020) doesn't even have an index entry for "flatness problem". He also notes that "[t]his coincidence argument may have occurred to many who did not bother to publish, because it was not at all clear what to make of it." He seems to be more agnostic about and/or less interested in it than were Dicke and Peebles (1979), and, citing Rees (1984), considers possible anthropic explanations (more on that in Sect. 9).

3 Reactions to the flatness problem

According to ADS[14] the paper by Dicke and Peebles (1979) has, at the time of writing, 34 citations. That is a surprisingly low number for such an influential paper, especially for one concerning a problem which many have dubbed the most important unsolved problem in cosmology. (Of course, ADS is not complete. However, one can compare that number to the number of citations to the seminal paper on inflation by Guth (1981), namely 7183.) There are probably several reasons for that. First, it appeared in an edited volume which, in contrast to the standard journals in the field, was not available to everyone. (Even today, their article is, as far as I know, not freely available in any form via the internet.) Second, many prefer to cite the corresponding refereed-journal paper rather than a contribution in an edited volume (whether conference proceedings, festschrift, or, as here, survey of the field), but in this case there is no corresponding refereed-journal paper! Third, as noted by Brawer (1996), the problem was not unknown before written up by Dicke and Peebles (1979) (though it was not as widely perceived to be a serious problem), so it was possible to refer to the well known flatness problem without citing a proper reference. Indeed, perhaps the only earlier possibility were the lectures by Dicke (1970), but the volume containing them is even more obscure than that in which the piece by Dicke and Peebles (1979) appeared (Hawking and Israel, 1979). Fourth, the contribution by Dicke and Peebles (1979) is written in a jaunty style, which might have put off some people from citing it.

Note that the paper by Dicke and Peebles (1979) was not cited at all before that of Guth (1981) appeared. That perhaps supports somewhat the claim by Brawer (1996) that the flatness problem was not considered to be an important issue until inflation suggested a solution to it. Since the first citation in 1981, the citation rate has remained roughly uniform (completely so within the noise). Usually, it is cited for some reason other than the flatness problem, as a reason to believe that our Universe is described by the Einstein–de Sitter model, or in connection with some mechanism beyond the scope of classical cosmology which is intended to explain why our Universe is not described by the Einstein–de Sitter model. I count only four papers, all relatively recent, which take issue with the idea that there is a flatness problem in classical cosmology and thus rebut the claim of Dicke and Peebles (1979) while also directly citing their paper: Helbig (2012), Carroll (2014), Holman (2018), and Helbig (2020). Of course, as mentioned below, earlier papers do rebut it, but do not cite it.

As discussed in Sects. 7–9, there are papers which dispute what is clearly the same claim without citing their paper. The first appears to have been by Cho and Kantowski (1994), well after the papers by Dicke and Peebles (1979) and Guth (1981). I can only speculate as to why it took so long. Perhaps it was finally becoming clear that $\Omega_0 \approx 0.3$ is a robust result, and thus our Universe is not well described by the Einstein–de Sitter model. At the time, it was unclear whether $\lambda_0 = 0$ or $\lambda_0 = 0.7$ was

[14] See the acknowledgements below. All citation metrics mentioned in this article refer to those derived from ADS at the time of writing.

a better fit. Of course, any value of λ_0 between ≈ 0 (though it could have been slightly negative) and ≈ 0.7 would not have been ruled out, but the general consensus was to have the simplest (for some subjective definition of "simplest") working model which fits the data, either $\lambda_0 = 0$ or a flat Universe. Thus, the Einstein–de Sitter model was called into question, but it was realized that inflation generically implies only a nearly flat universe, not the Einstein–de Sitter model per se, i.e., it says nothing about the value of λ (e.g. Ostriker and Steinhardt, 1995). However, in that case the individual values of Ω and λ are unstable in the same sense that Ω is for $\lambda = 0$ and $\Omega \neq 0$ even if their sum is approximately constant. That could have cast doubt on the fine-tuning argument presented by Dicke and Peebles (1979) and emphasized by Guth (1981), even though inflation could still be the explanation for a flat Universe. Perhaps that led some to question the robustness of such arguments.

Most papers disputing the flatness problem don't appear to have been motivated by any specific event, but rather a desire to point something out which the authors thought should be better known. It is also interesting that most of the authors have a background which is more general relativity than astronomical cosmology. (My impression is that the flatness problem is perceived to be real more often among astronomers and particle physicists than among those who work mainly in general relativity.) Most papers motivated by it, at least indirectly, which, it could be argued, includes all papers written on inflation, take the existence of the flatness problem as given.

The consensus, or at least the standard thinking, in a field is perhaps best judged by what appears in books, rather than papers. Using as a more or less random sample (certainly not selected with respect to the flatness problem) books that I own, ranging from (almost) popular science to technical monographs, they fall into four groups. The first group repeat the standard flatness-problem narrative (Harrison, 2000; Kragh, 2007; Jones et al., 2015; Schneider, 2015; Ryden, 2017; Pasachoff and Filippenko, 2018; Wright, 2020). Longair and Smeenk (2019) also belongs to that group but, in the same volume (Kragh and Longair, 2019), Smeenk (2019) takes a somewhat more careful view, at least hinting at recent ideas which cast the flatness problem in a different light. He does cite Holman (2018), but it is not exactly clear why and to what extent he agrees. The second group take a somewhat unclear approach. Heacox (2015) notes that essentially all universes start arbitrarily close to the Einstein–de Sitter model, but seems to accept that as a given and doesn't comment on it or mention the flatness problem. That is mentioned, in more or less the standard manner, later in the book, when inflation is discussed. Similarly, mentioning the flatness problem only briefly in a very long book, Jones (2017) notes that in the early universe, Ω is "extremely close to unity no matter what the current value of Ω_0", thus at least denying that there is something special about *our* Universe. However, he then remarks that "[one] part in 10^{20} is extraordinarily fine tuning of the high-z state." The third group first present the flatness problem in the standard form, then step back and note various caveats: Tegmark (2014), Lewis and Barnes (2017), and Barnes and Lewis (2020); the last mention explicitly several papers discussed below (Hawking and Page, 1988; Evrard and Coles, 1995; Gibbons and Turok, 2008; Carroll and Tam, 2010). The fourth group seem to more or less recognize the dubious nature of the flatness problem. Coles and Ellis (1997) are the most explicit, stating that "*there is no flatness problem* in a purely classical cosmological model" [emphasis in the original]. Ellis et al. (2012) discuss the flatness problem (though not by name), but, somewhat surprisingly, explicitly within the context of the Multiverse, citing several papers discussed below (Gibbons et al., 1987; Hawking and Page, 1988; Evrard and Coles, 1995; Kirchner and Ellis, 2003; Gibbons and Turok, 2008; Carroll and Tam, 2010). It is clear that those in the fourth group are sceptical of the flatness problem because (some of) the authors have (co-)written papers on the topic. The somewhat milder scepticism of third group is perhaps due to the fact that I have discussed the flatness problem with (some of) the authors. While most of the authors in the first two groups are cosmologists, to my knowledge none has written a technical paper on the flatness problem, and their description of it differs little if any from that of others, whether scientists or popular-science writers. That supports my view that most of the literature concerning the flatness problem has been largely ignored.

It is interesting to compare different editions of cosmology textbooks; the discussion of observations is routinely updated, but – at best – new theoretical developments, as opposed to refinements of older ideas, are mentioned. There are more newer than older books in my lists above. If anything, however, that strengthens my claim that the literature has been mostly ignored. (Most or all of the books I have read, but do not own, which mention the flatness problem fall into the first category.) Peebles (2020) obviously deserves special mention. On other topics, he spends a lot of time discussing how the community opinion changed, works which were neglected for a long time which later became popular, and so on. (But also some dead ends (referring to another topic): "The idea is elegant, of historical interest, and worth considering, but wrong.") Similar to many other authors, his stance on the flatness problem is essentially the same as that of Dicke and Peebles (1979), but he is perhaps a bit less adamant about it. Perhaps his opinion has softened somewhat, but I think it more likely that he is just recapping the history here (and the fact is that most people think that nothing much concerning that topic has happened since Dicke and Peebles (1979)), and also that it is not so important for him. The Dicke and Peebles (1979) article was perhaps mostly Dicke's work, and Peebles hasn't returned to that topic in any substantial form since then. The book doesn't even have an index entry for "flatness problem", although it is otherwise very detailed.

4 The role of observations

In its original formulation by Dicke and Peebles (1979), assuming $\Lambda = 0$, the flatness problem was often perceived as implying that our Universe must be the Einstein–de Sitter Universe, even though it is far from clear to what, if any, extent their arguments apply to a real, as opposed to ideal, universe (see the discussion by Coles and Ellis, 1997). Although it had already become clear that observations imply $\Omega_0 \lesssim 0.3$ or less (e.g. Gott et al., 1974, who favoured $\Omega_0 = 0.06 \pm 0.02$), a conclusion which has remained unchanged for almost half a century (e.g. *Planck* Collaboration, 2020; Peebles, 2020), at the time uncertainties were such that the Einstein–de Sitter model was still viable. While many observers tended to believe the observations and distrust theoretical arguments which suggested a different value for Ω_0 (e.g. Carlberg, 1998, the oral version was even more adamant that $\Omega_0 \approx 0.3$), often in direct contrast to theorists (e.g. Kolb, 1998, though the written version is much less adamant that $\Omega_0 = 1$), the Einstein–de Sitter model was still considered marginally valid as late as the Princeton debates (Turok, 1997). However, it had become far from certain, and claims to have verified it observationally were routinely rebutted. For example Kellermann (1993) claimed that the angular-size–redshift relation for small-scale radio jets in galaxies provided convincing evidence for the Einstein–de Sitter model. That claim was promptly disputed by Kayser (1995) and Stepanas and Saha (1995), in both cases on general statistical grounds (i.e., they used more-robust statistical methods), as well as by Dabrowski et al. (1995), who pointed out the bias in the sample due to relativistic beaming, also rebutting the conclusion. Nevertheless, since the case for the existence of non-baryonic dark matter on non-cosmological scales had been strong since at least the mid-1970s, and the amount of missing matter seemed to increase with scale, it didn't seem completely unreasonable that, by the time one reached cosmologically intersting scales, one would measure $\Omega \approx 1$ (which many would have taken to indicate $\Omega \equiv 1$). Peebles (2020) provides a good overview of the increasing evidence for different types of dark matter, distinguishing between "the astronomers' subluminal matter", "the particle physicists' nonbaryonic matter" and "the cosmologists' dark matter", something which is not always sufficiently appreciated.

Carlberg (1998) provides a good indication of how probably most observers viewed the cosmological parameters at a time about half-way between Dicke and Peebles (1979) and today: "Commenting on the progress being made in estimating the values of the cosmological parameters has perils not unlike those of critiquing a great artwork, perhaps an opera, while it is being staged for the first time." He notes that while there was a time when higher estimates were considered probable, they had come down again to near the value favoured by Gott et al. (1974), there being two main reasons for the higher values: "Virtually all theorists concluded that in the absence of overwhelming evidence to the contrary, $\Omega_M = 1$. A little later there was the largely unanticipated discovery of large scale flows of galaxies which appeared to be a very strong argument for $\Omega_M \simeq 1$. ... In the past few years, the Ω_M tide has reversed to generally favouring low Ω_M values again. That is, evidence for a low Ω_M that strikes many as overwhelming is gradually being accumulated. ... Much of the community would agree that Ω_M is someplace in the range 0.1–04." Already at the time of the corresponding conference, "The preliminary evidence from the CMB fluctuation measurements ... is that if Ω_M is low there must be a nonzero Ω_Λ to produce a peak in the angular power spectrum at a sufficiently large angle." (We now know the position of that peak precisely enough that it indicates that the universe is at least very close to being spatially flat.) Evidence for a non-zero cosmological constant was still rather indirect: there was a hint that without it the Universe could not be older than the oldest objects in it, but that was still rather tentative. That changed after the m–z relation for Type Ia supernovae indicated $\lambda_0 > 0$ and, moreover, an accelerating Universe (Riess et al., 1998; Perlmutter et al., 1999); combined with the CMB evidence for spatial flatness, and, later, measurements from baryon acoustic oscillations, the "concordance values" of $\Omega_0 \approx 0.3$ and $\lambda_0 \approx 0.7$, which had already been suggested years previously (e.g. Ostriker and Steinhardt, 1995), were essentially confirmed. It is still true today that "[i]t is impossible to avoid a philosophical note when contemplating the Ω values."

Thus, when Dicke and Peebles (1979) first popularized the flatness problem, and Guth (1981) emphasized that problem and his solution (inflation), there was evidence that $\Omega_0 \approx 0.3$, but it was not overwhelming, $\Lambda = 0$ was generally assumed, and the Einstein–de Sitter model was not yet completely ruled out. Many theorists saw the Einstein–de Sitter as the most likely, based on the flatness-problem argument, but that should not be overestimated because many also assumed it simply because it makes order-of-magnitude calculations easier, and those were good enough for many purposes considering the observational uncertainties at the time. Fifteen years later, theorists had already made the case for what are today known as the "concordance" values of the "standard model", $\Omega_0 \approx 0.3$ and $\lambda_0 \approx 0.7$ (e.g. Ostriker and Steinhardt, 1995; Krauss and Turner, 1995; Krauss, 1998). Another ten years later, those values had become consensus. The situation has not changed much since then, although the uncertainties have been reduced. (At the moment, the main problem in observational cosmology seems to be the "tension" between various measurements of the Hubble constant, though that is not relevant for this paper.)

Thus, apart from the possibility that it had become to appear more likely that the Einstein–de Sitter model doesn't describe our Universe, which, as mentioned above, might have inspired the first paper to question the flatness problem (Cho and Kantowski, 1994), observations do not seem to have played a role in questioning the flatness problem. Since Cho and Kantowski (1994), the observational situation has not changed much (apart from decreasing uncertainties). Rather, the motivation seems to be to question the flatness problem as expounded by Dicke and Peebles (1979). In particular, that motivation is independent of whether our Universe is actually close to being flat, as observations strongly suggest, for two reasons. First, the motivation is to point out the error in the logic of Dicke and Peebles (1979), independently of any observations. Second, the original flatness problem was the question why $\Omega_0 \approx 1$ to within, say, an order of magnitude, not to a per cent or better as current observations suggest. While interesting historically that inflation made a much stronger prediction (i.e., that the Universe is very close to being exactly flat, perhaps close enough that any curvature could never be detected) than that required to explain the flatness problem at the time, the subsequent observation of a very flat Universe does not imply that the problem perceived by Dicke and Peebles (1979) was real. In other words, the existence of a solution does not demonstrate that there must be a problem. Also, as mentioned below, especially in connection with the work of Lake (2005), it is not even clear whether even the very flat Universe indicated by observations requires inflation (or some other mechanism) which goes beyond classical cosmology.

5 Key insight: how λ and Ω change with time

It is strange that the fact that the static model of Einstein (1917) is unstable (Eddington, 1930)[15] was used as an argument against such a static model, but the same argument led to the opposite conclusion for the Einstein–de Sitter model. While the Einstein–de Sitter model is not static in space, i.e., it expands (though it asymptotically approaches a static state for $t = \infty$), it is static in the sense that its trajectory in the λ–Ω plane is a point. In that sense, it is similar to the static Einstein model, and also to the de Sitter model with $\lambda = 1$ and $\Omega = 0$ and (the relativistic equivalent of) the Milne model with $\lambda = 0$ and $\Omega = 0$. In the language of dynamical systems, both the Einstein static model and the Einstein–de Sitter model are unstable fixed points.[16] In other words, in the former case the fact that it is unstable led to it being ruled out as a model of our Universe (undoubtedly helped by the observations indicating that our Universe is expanding[17]), while in the latter case it led to the conclusion that this model must describe our Universe exactly, since any deviation would cause it to rapidly evolve away (of course, helped by the fact that no spatial curvature has been detected, and still hasn't). (Eddington preferred a model expanding away from a static Einstein universe in the infinite past. His primary motivation seems to have

[15] But see the interesting discussion by Barrow et al. (2003) for caveats in a more general context.

[16] Under the assumption that the universe is now expanding, the Einstein–de Sitter model is a repulsor and the de Sitter model an attractor. The Milne universe a saddle point. Those remarks apply to the case of dust (i.e., non-relativistic matter) and radiation; in general the type of each fixed point depends on the equations of states of the various components (e.g. Uzan and Lehoucq, 2001). The static Einstein model is approached asymptotically for expanding universes with a smaller scale factor while those with a larger scale factor, such as that favoured by Eddington, have been expanding away from it (forever in the completely unperturbed case). However, while all non-empty big-bang models start arbitrarily close to the Einstein–de Sitter model and all non-empty models which expand forever asymptotically approach the de Sitter model except that which asymptotically approaches the static Einstein model, only one infinitely fine-tuned trajectory in the λ–Ω plane includes the static Einstein model.

[17] Einstein himself seems to have been more convinced by the instability argument than by observations when he gave up his static model (Nussbaumer, 2014).

been philosophical: "Philosophically, the notion of a beginning of the present order of Nature is repugnant" (Eddington, 1931); "the most satisfactory theory would be one which made the beginning not too unaesthetically abrupt" (Eddington, 1940a, p. 58).) See McCoy (2020) for a discussion of the contrast between the perceptions of those two unstable cosmological models.

The evolution of cosmological models in the λ–Ω plane is essential for understanding the flatness problem (or lack thereof). The important point is not *that* λ and Ω in general change with time, but the *quantitative behaviour*.[18] An excellent visualization of that is provided by a Java animation by Leahy (2003). The definitive study for dust models with a cosmological constant was made by Stabell and Refsdal (1966); although they used the parameters $\sigma = \Omega/2$ and $q = \sigma - \lambda$, the discussion is easy to follow and would differ little from a corresponding discussion using λ and Ω. Their discussion covers more than the range of cosmological models allowed by current data; a similar classification allowing for more-general equations of state was given by Harrison (1967). Madsen and Ellis (1988) and Madsen et al. (1992) extended the discussion to inflationary models; although not explicitly addressing the flatness problem, they noted that the probability of observing $\Omega \approx 1$ depends on the time of observation and that the set of initial conditions that will lead to Ω deviating significantly from 1 at a given time is non-zero in at least some measures. Ehlers and Rindler (1989) expanded the discussion by Stabell and Refsdal (1966) to include radiation as well as dust and a cosmological constant, pointing out (though not developing the idea) that $(K^3)/(\Omega^2\lambda)$ is a constant of motion, i.e., an invariant quantity during the evolution of the universe. Goliath and Ellis (1999) examined homogenous models with a non-negative cosmological constant, not just FRW models, but also Bianchi and Kantowski–Sachs models, though with no discussion of the flatness problem. Uzan and Lehoucq (2001), in a pedagogical discussion based on Newtonian cosmology, generalized the discussion by Stabell and Refsdal (1966) to arbitrary equations of state; the flatness problem is mentioned, but, as is often the case, assumed to be a real problem. For more details on that topic, see the book written by Coley (2003) and the one edited by Wainwright and Ellis (2005); for a pedagogical introduction, see the paper by García-Salcedo et al. (2015).

6 Excursion: the role of inflation

By the early 1980s, the Einstein–de Sitter model had become the standard model,[19]both because of the instability-argument aspect of the flatness problem ("if Ω isn't exactly 1, then it should be very different from 1, but we observe it to be ≈ 1,

[18] As discussed below, while Lake (2005) has presented perhaps the strongest argument against the flatness problem (especially with regard to the instability problem), and one which is particularly relevant in that it applies to universes, like our Universe, which have a positive cosmological constant and will expand forever, other arguments argue against the flatness problem even with the $\Lambda = 0$ assumption in the original formulation.

[19] After it had been suggested by Einstein and de Sitter (1932), the Einstein–de Sitter model was often used as a fiducial model, because it makes calculations easy. Such a model has an age which is 2/3 the Hubble time. Whenever the age of the Universe seemed to be larger than that, or even larger than the Hubble time, a model with a positive cosmological constant and an age larger than the Hubble time was sometimes invoked, the idea going back to Lemaître (1927, 1931a,b, 1933, 1934). Also, a closed model with no cosmological constant was sometimes favoured, both because the evidence seemed to indicate it (e.g. Sandage, 1968; Rees, 1969) and because it was favoured by some on philosophical grounds, for example by Einstein (1931) (see also O'Raifeartaigh and McCann, 2014, for discussion and translation). By the late 1960s, estimates of the Hubble constant were much lower, and

and 1 is allowed within the uncertainties, therefore it is probably exactly 1") and because it was believed that inflation naturally leads to a universe extremely close to the Einstein–de Sitter universe (at least now[20]). Both of those are theoretical arguments, though the observation of the order of magnitude of Ω_0 also played a role. With regard to quantitative observations, the fact that no spatial curvature had been detected (and still hasn't) at least allowed it to remain a viable model. While many observers (e.g. Carlberg, 1998) believed the observational evidence for a low-density Universe, some (e.g. Sandage, 1995) sided with theoretical particle physicists in believing that inflation must lead to a flat Universe which, coupled with the still viable assumption that $\Lambda = 0$, implies the Einstein–de Sitter model. Arguments against the conclusion that our Universe must be (arbitrarily close to) the Einstein–de Sitter universe are discussed below. The important point is that, despite the obvious objections discussed above, the idea that our Universe is described by the Einstein–de Sitter model was uncritically believed by many up until the late 1990s or so (e.g. Sandage, 1995) (and by some even later). Of course, observations had almost always indicated $\Omega_0 \approx 0.3$ (e.g. Gott et al., 1974; *Planck* Collaboration, 2020; Peebles, 2020);[21]in this paper, I am concerned only with theoretical arguments. As long as $\Omega_0 = 1$ was not ruled out observationally, the Einstein–de Sitter model was often assumed only because it is mathematically simple. After the idea of inflation became popular (see next paragraph), that was seen as additional support, since the Einstein–de Sitter model is flat, although some advocated a flat model with a cosmological constant, combining the goals of flatness (suggested by inflation) and the observed value of $\Omega_0 \approx 0.3$ (e.g. Ostriker and Steinhardt, 1995; Krauss and Turner, 1995; Krauss, 1998). Note that such a universe is exactly flat or very close to it, and is so throughout its existence, so there is no flatness problem in the literal sense. However, in such a universe, Ω still evolves away from 1, so the flatness problem in the sense of "why is $\Omega \approx 1$ now", the instability problem, still exists, as does a fine-tuning problem with respect to Ω (or λ), though if $\lambda > 0$ then large values do not occur, as Ω evolves from 1 to 0 and λ from 0 to 1 along the line connected the Einstein–de Sitter and de Sitter models (or very close to it). Hence, while inflation can make the universe flat, that means only that $\Omega + \lambda \approx 1$, not that the universe is very close to the Einstein–de Sitter model; if one assumes $\Lambda = 0$, as was common when the flatness problem and inflation were first formulated, then the two are equivalent. That is one case where dropping the assumption that $\Lambda = 0$ modifies the discussion somewhat; another such case is discussed in Sect. 8.

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hence the Hubble time much longer, and also the fact that different observers found very different values indicated that major uncertainty was involved, thus making the age problem in such a model less serious. However, before the Einstein–de Sitter model came to the fore around 1980, there was no "standard model" in the sense that the term has been used since then, both because of a lack of a clear choice and also because there was no paradigm in the modern sense including structure formation etc. Nevertheless, the Einstein–de Sitter model was often used as a concrete example, due to its mathematical simplicity.

[20] Inflation essentially causes the universe to increase in size by several orders of magnitude, thus making it appear flat for the same reason that we normally don't notice the curvature of the Earth but do notice the curvature of a bowling ball. After inflation, however, the universe evolves as any other FRW model, hence if the universe is not exactly the Einstein–de Sitter universe, Ω (and, if non-zero, λ) will evolve away from the values of the Einstein–de Sitter model. Thus, as pointed out explicitly by Ellis and Rothman (1987), and also by Raine and Thomas (2001) in their section 8.2, the flatness problem will appear again in the future, at least if $\Lambda = 0$. In other words, even if inflation can solve the fine-tuning problem (though of course I argue here that it is not really a problem), it can't solve the instability problem, at least not for all time.

[21] On the other hand, there was no direct evidence of spatial curvature, and still isn't.

Even though the Einstein–de Sitter model has now been ruled out, the fact that our Universe is at least very close to flat (i.e., $\lambda + \Omega \approx 1$ to high precision) has perhaps led many to believe[22] that there must be a flatness problem in classical cosmology and that there is some explanation – such as inflation (e.g. Guth, 1981; Linde, 1982) – for the fact that we live in a universe that, according to the argument, would seem to be unstable (with the question still open whether an approximate, as opposed to ideal, flat universe would suffer from instability in the same way). The claim of Dicke and Peebles (1979) has thus inspired a substantial body of work in (mostly theoretical) cosmology, and is still found in its original form in modern textbooks (e.g. Ryden, 2017; Longair and Smeenk, 2019; Wright, 2020) and review articles (e.g. O'Raifeartaigh et al., 2018; Adams, 2019). Such work usually takes the existence of the flatness problem as given and suggests some solution. There are many reviews of inflation and related issues (e.g. Narlikar and Padmanabhan, 1991), but when discussing the flatness problem in classical cosmology, inflation plays a role only in that the idea of inflation has rekindled interest in classical cosmology. Even observational astronomers are familiar with the flatness problem and see inflation as an attractive solution (e.g. Schmidt, 1989; Sandage, 1995). Here, as mentioned above, I review something else, the relatively small number of claims, largely ignored, that the flatness problem does not exist.

Inflation is thus something of a red herring[23] with regard to the flatness problem in classical cosmology, since the latter is the question why Ω is within, say, at most a few orders of magnitude of 1. A more-or-less generic prediction of inflation is a highly flat universe, and indeed our Universe is observed to be highly flat. It is not completely clear whether the arguments against the classical flatness problem presented here can *also* explain the fact that $\Omega_0 + \lambda_0 \approx 1$ to, depending on various assumptions, within a per cent or so (e.g. *Planck* Collaboration, 2020). Of course, inflation could have happened even if there is no flatness problem in classical cosmology. On the other hand, if there is some other explanation for the observed flatness, then that is not a motivation for inflation, though there could be other motivations. Even if one believes that inflation solves the flatness problem, the classical flatness problem is still interesting because, as shown below, the arguments in favour of the non-existence of

[22] Interestingly, Brawer (1996) suggested that neither the horizon problem (see McCoy (2015) for an overview of the horizon problem) nor the flatness problem was considered to be an important issue until inflation suggested a solution to them; thus, inflation might have been a solution in search of a problem. In this work, I restrict myself to classical cosmology. (By "classical cosmology" I mean the study of FRW models, which implies that general relativity is exactly valid and that the universe is, except for test particles, perfectly homogeneous and isotropic.) Also, I assume that a component with one equation of state cannot change into a component with another equation of state, an essential ingredient of inflation that, as in our Universe, stops at some point. Inflation is thus out of scope. Nevertheless, the analysis of the horizon and flatness problems (as a prelude to investigating whether inflation solves them) by McCoy (2015, 2016) is relevant in that he also concludes that the flatness problem is far from being as clear-cut as most still imagine; he takes a somewhat different view later (McCoy, 2018a). See McCoy (2015) and references therein for some background on philosophical approaches to the horizon and flatness problem; those approaches "are either very limited in scope or are generally motivated by peculiar philosophical concerns distant from the kinds of concerns relevant to the practice of cosmology" and he attempts "to take a more physically motivated point of of view, and therefore takes the concerns voiced by cosmologists as a starting point". Nevertheless, his paper is much more philosophical than the present paper.

[23] It is thus in good company with the fishy arguments in favour of the flatness problem which pervade the literature.

the classical flatness problem also imply a solution to (or the non-existence of) the coincidence problem.

Of course, other non-standard models (defined here as anything going beyond FRW models) can be constructed which have no flatness problem. Perhaps more interesting is the fact that anthropic considerations rule out flat or nearly-flat models as well as models in which the cosmological constant is too close to zero in non-standard models with varying "constants" of nature – invented for entirely different reasons having nothing to do with the flatness problem – (e.g. Barrow et al., 2002), although perhaps such models are more attractive to some because it is (mistakenly) believed that they don't suffer from the flatness problem (or can easily avoid it) whereas standard models do, similar to the case for inflationary models.

7 The measure of a universe

How can one calculate the probability of a given universe, in particular of our Universe, given a (real or imagined) set of possible universes (sometimes known as the Multiverse)? Dicke and Peebles (1979) implicitly assumed that Ω is a free parameter in the early universe, e.g., with a flat probability distribution, thus implying that $\Omega = 1$ almost exactly in the early Universe must be due to some sort of fine-tuning,[24] since a slight departure would lead to values of Ω (and λ) vastly different from those observed today. (In the following, I will restrict the discussion to Ω unless the inclusion of λ leads to a qualitatively different scenario.[25]) That that argument is flawed can be seen from the fact that it holds regardless of what the value of Ω is today: for any given value of Ω today, it is always possible to find a time at which Ω was arbitrarily close to 1. That follows from the fact that all non-empty big-bang models start arbitrarily close to the Einstein–de Sitter model (e.g. Stabell and Refsdal, 1966). (Note also that $\Omega \approx 1$ today implies only that, at a given time in the early Universe, the "fine-tuning" was more precise or, equivalently, for a given amount of "fine-tuning", the corresponding time was somewhat later.) Most objections to the flatness problem, i.e., claims that it is not really a problem, are based on the claim that a flat probability distribution for Ω is not appropriate.

[24] In this paper, I use the term "fine-tuning" to mean a combination of two things: (1) changing the value by a small amount would have large consequences and (2) that value is unlikely. That is certainly the most interesting of the four possible combinations. Strictly speaking, fine-tuning refers to only the first usage, and it is a separate question how likely such a value is. However, the double meaning is used in essentially all literature on the flatness problem, so it makes sense to stick to it here.

[25] The flatness problem consists of the instability problem (why is Ω still close to 1 today?) and the fine-tuning problem (why was Ω so close to 1 in the early universe?). Dropping the assumption that Λ is 0, the two questions are still applicable, though the former could be amended to add "and λ still close to 0" and the latter to add "and λ so close to 0". The basic idea in both cases is that the Einstein–de Sitter model is an unstable fixed point, so the real issue is not closeness to flatness but closeness to the Einstein–de Sitter model. The discussion is thus qualitatively the same whether or not λ is included. Historically, $\Lambda = 0$ was a common assumption when the flatness problem was initially formulated, so much of the historical discussion is in terms of Ω only. (One does have to keep in mind, though, that some authors use Ω for what in my notation is $\Omega + \lambda$.) Λ becomes relevant in two places, both discussed below: (1) inflation can make the universe flat by driving $\Omega + \lambda$ to 1, and thus says nothing about the individual values; and (2) one solution to the flatness problem (Lake, 2005) points out that, for $\Lambda > 0$ and $k = +1$, a spatially flat universe is not unlikely, but again says nothing about the individual values.

Cho and Kantowski (1994) used the metric on the space of gravity fields given by DeWitt (1967) instead of using Ω as a coordinate and argued that that demonstrates the lack of a flatness problem, due to the fact that, when using Ω as a coordinate, the probability distribution is skewed towards the singular value $\Omega = 1$, while that is not the case for their better coordinate.[26] Their introduction gives a good description of the flatness problem as usually understood, and the rest of their paper demonstrates that the flatness problem is essentially due to using the bad coordinate Ω. In their discussion, they explicitly spell out the flaw in the standard flatness-problem argument. Their work should have cleared things up once and for all.

Evrard and Coles (1995) (see also Coles and Ellis, 1997) took a similar approach, pointing out that the flatness problem is due to using an inappropriate choice of prior (in the Bayesian sense), stating clearly "that, in the framework of a classical cosmological model, there is no flatness problem". Following Jaynes (1968), they advocated choosing a prior based on the principle of maximum information entropy, which contradicts the assumption of a constant prior for Ω. Their paper is interesting for other reasons as well. First, they point out that the evolution of the Universe can be described by "an absolute scale parameter". Although not explicitly stated, that corresponds to the mass of a spatially closed universe. Second, the singularities encountered when using Ω as a coordinate are related to the fixed points in the evolution of the Universe (for $\Lambda = 0$): the Einstein–de Sitter model with $\Omega = 1$ and the Milne model with $\Omega = 0$; other values of Ω are transitory. Those points will be discussed below. To be sure, Coule (1996) suggests a different prior[27] (Coule, 1995), but that does not change the conclusion that using Ω as a coordinate, i.e., a flat prior for its value, is misleading and that that wrong choice leads to the flatness problem. Kirchner and Ellis (2003) also used Jaynes's principle to "solve the flatness problem" (direct quotation), generalizing the results to include other equations of state. The same general idea (appropriate choice of prior) has also been used in more-general contexts, also leading to the conclusion that nearly flat cosmological models are more probable than significantly non-flat ones (e.g. Gibbons and Turok, 2008; Roukema and Blanlœil, 2010), though see Schiffrin and Wald (2012) and McCoy (2016, 2017) for some criticism of the details of such approaches. Carroll (2014), describing his work with collaborators (Carroll and Tam, 2010; Remmen and Carroll, 2013, 2014), also uses the GHS measure in a discussion of the flatness problem, noting that "flatness isn't a problem at all", "[t]he flatness problem, meanwhile, turns out to be simply a misunderstanding", "the flatness problem really isn't a problem at all; it was simply a mistake, brought about by considering an informal measure rather than one derived from the dynamics". A conclusion of Carroll and Tam (2010) is a good summary of this section: "The flatness problem, as conventionally understood, does not exist; it is an artifact of informally assuming a flat measure on the space of initial cosmological parameters" and "is not intrinsic to the standard Big Bang model". (Nevertheless, see McCoy (2016) for some criticism of the details of their approach, and also for pointing out that while their approach might solve the fine-tuning aspect of the flatness problem, it says nothing about the instability aspect.)

[26] The same point was made in a more general context by Coule (1995).
[27] That prior is the GHS measure (Gibbons et al., 1987), which implies that $\Omega \approx 1$ whether or not inflation occurred (Hawking and Page, 1988).

8 The dynamical-systems approach to cosmology illuminates the flatness problem

Most criticisms of the flatness problem address the fine-tuning problem: why was Ω very close to 1 in the early universe? Some quantitative answers are discussed in Sect. 7, though an easy way of looking at it is simply to note that all non-empty big-bang models start arbitrarily close to the Einstein–de Sitter model and hence Ω is arbitrarily close to 1 in the early universe. More interesting is perhaps the question why $\Omega \approx 1$ today, given that in general Ω evolves away from 1. In the words of Lake (2005) (his Ω is my $\Omega + \lambda$): "The flatness problem involves the explanation of [$\Omega_0 \sim 1$] given [$\Omega = \Omega(t)$]. The problem can be viewed in two ways. First, one can take the view that there is a tuning problem in the sense that at early times Ω must be finely tuned to 1 [references to Dicke and Peebles (1979) and Peacock (1999), the latter as an example of a standard argument in then current cosmological texts]. However, this argument is not entirely convincing since *all* standard models necessarily start with Ω exactly 1. More convincing is the view that except for the spatially flat case the probability that $\Omega \sim 1$ is strongly dependent on the time of observation [reference to Madsen and Ellis (1988)] and so there is an epoch problem: why should [$\Omega_0 \sim 1$] hold?" (Emphasis in original.)

Lake (2005) used the dynamical-systems approach to argue that there is no flatness problem in $k = 1$ models with a positive cosmological constant. The basic idea is that there is a constant of motion $\alpha = \mathrm{sign}(K)(27\Omega^2\lambda)/(4K^3)$ and that it makes sense to use α as a free parameter. Unless α is fine-tuned to be ≈ 1, large values of λ and Ω never occur during the evolution of the universe. In other words, one would need fine-tuning to *avoid* a nearly flat universe. Lake (2005) is quite explicit that that solves the flatness problem for the corresponding cosmological models. Note that here, as with regard to inflation, one is concerned only with spatial flatness, i.e., $\Omega + \lambda = 1$, not with being arbitarily close to the Einstein–de Sitter universe.

The idea of the universe as a dynamical system described by trajectories in the λ–Ω plane is essential to understanding (the lack of) the flatness problem. Stabell and Refsdal (1966) noted that there are trajectories in the λ–Ω plane characterized by $\alpha = 1$, which of course is constant during the evolution of the corresponding cosmological model, and indeed those models (referred to as the A1 and A2 curves, which separate models which will collapse from those which will expand forever and those with and without a big bang, respectively) play an important role in their discussion, but they did not realize that α is constant along a trajectory in general (Rolf Stabell, personal communication). (One can of course use the fact that α is constant in order to plot trajectories simply by making a contour plot.) Ehlers and Rindler (1989) actually wrote down a quantity which is the reciprocal of α (up to the constant numerical factor; that factor makes $\alpha = 1$ for the A1 and A2 curves) and mentioned that it is a constant of motion, but did not connect it to the flatness problem. Leahy (2003) might have been the first to explicitly define α (though with another notation) in that manner and point out its physical meaning for the $k = +1$ case, namely the product of the square of the mass of the universe and Λ, and also to use it to demonstrate the lack of a flatness problem.

Adler and Overduin (2005) discussed various definitions of "nearly flat", using essentially using the same parameter as α used by Lake (2005), and arriving at the same conclusion, namely that a significantly non-flat universe implies fine-tuning in α. They also presented an analogy from Newtonian physics concerning a test particle falling into a gravitational field, and showed that the corresponding quantity approaches 1 in the same manner as Ω approaches 1 the closer in time one is to the big bang, regardless of the initial velocity of the test particle. Since there is obviously no fine-tuning in the former case, neither is fine-tuning necessary in the early universe.

Helbig (2012) discussed two formulations of the flatness problem: does the fact that $\Omega = 1$ to very high precision in the early Universe (or in any FRW universe, not just in ours) imply fine-tuning? Does the fact that $\Omega \approx 1$ today, given that it is not exactly 1 (or, since $\lambda > 0$, given that $\lambda + \Omega$ is not exactly 1; only if the sum is exactly 1 does it remain constant) in our Universe imply fine-tuning? In some sense, those are of course related: were Ω vastly different from 1 today, then less fine-tuning would be required. However, since $\Omega = 1$ to very high precision at some time regardless of what the value of Ω is today, that is a generic feature of FRW models and thus there is no fine-tuning. On the other hand, if Ω evolves away from 1 if it is not exactly 1 (or, if $\Lambda \neq 0$, their sum, as well as the individual values of Ω and λ, will change with time if the sum is not exactly 1), then is it unlikely that $\Omega \approx 1$ today? There are several cases to be distinguished:

- In the case of a $k = +1$ universe which will expand forever, Lake (2005) has shown that fine-tuning is *required* to get a significantly *non*-flat universe.
- All non-big-bang models – and many big-bang models as well – are ruled out observationally, but here we are concerned with principles. All non-big-bang models have $\alpha \leq 1$ (the converse is not true, i.e., there are models with $\alpha \leq 1$ which are big-bang models; most have $\lambda < 0$ and all collapse in the future). For $\alpha \approx 1$, Lake's argument also applies to those, and of course the weak-anthropic argument mentioned below that Ω_0 is not arbitrarily small also applies to all $k = +1$ models which expand forever. Note that, for models which will expand forever, Lake's argument suggests that we should not observe large values of λ_0 and Ω_0, while for $k \leq 0$ both λ_0 and Ω_0 are always in the interval $]0, 1[$.
- The remaining cases can be divided into models which collapse (which exist for all values of k; such models exist with $\lambda_0 < 0$, $= 0$, and > 0 for $k = +1$ and with $\lambda_0 < 0$ for $k = 0$ or $k = -1$) and those which expand forever ($k \leq 0$, the case $k = +1$ having been covered by Lake; $\lambda_0 \geq 0$). If $\lambda_0 < 0$, the universe always collapses, otherwise, the universe expands forever for $k = -1$ and $k = 0$ while it does so for $k = +1$ only for non-big-bang models – after a bounce from an initial contraction – and for big-bang models with $\alpha \geq 1$; if $\alpha = 1$ the universe asymptotically approaches – after a big bang – or expands from the static Einstein model.
  - If the universe collapses, then, while the values of Ω and λ can become infinite, they are appreciably > 1 for only a relatively short (and special) time in the history of the universe (see Sect. 9).
  - In the case of eternal expansion, weak-anthropic arguments apply, i.e., asking why $\Omega \approx 1$ today, and not arbitrarily small, is equivalent to asking why we live an infinitely short time (in some sense) after the big bang if the Universe will expand forever.[28]

McCoy (2018b) discussed the (ab)use of likelihood arguments in cosmology, including a short discussion of the flatness problem. Holman (2018) discussed in detail various questionable arguments and misconceptions regarding the flatness problem as well as different varieties of it. The two formulations discussed by Helbig (2012),

[28] I always wonder when it is suggested that, in a universe which will expand forever, such as our Universe, it is somehow strange that we observe a value of Ω significantly greater than 0 and a value of λ significantly less than 1, since those are the asymptotic values. (A variant of that is the "coincidence problem", ie, why does $\Omega \approx \lambda$ hold today?) However, in a universe which lasts forever, one is in some sense always infinitely close to the beginning. Nevertheless, after some point the values of the cosmological parameters will be close to their asymptotic values, but assuming that we should expect to observe such values implicitly assumes that life is possible in an arbitarily old universe, while rather general assumptions (lifetimes of stars, etc.) indicate that that is probably not the case.

who dubbed them the "qualitative" and "quantitative" flatness problems, are called the "fine-tuning argument" and the "instability argument", respectively, by Holman. Although not a review per se, it is an excellent discussion of the flatness problem, misunderstandings of it, and arguments against it, in particular the "reverse–fine-tuning" argument of Lake (2005) and the relative–time-scale argument of Helbig (2012), as well as related issues in a wider context.

Perhaps because of the widespread belief that inflation causes the Universe to be flat and hence solves the flatness problem, for many the coincidence problem has become the new "big puzzle" in cosmology (e.g. Tegmark et al., 2005). It would take an entire paper to discuss the literature mentioning the so-called coincidence problem, i.e., why are Ω_0 and λ_0 of the same order of magnitude. However, the dynamical-systems approach to cosmology shows that that is not a problem for essentially the same reasons why the flatness problem doesn't exist. The essential quantity is $\Omega+\lambda-1$.

If $\Omega + \lambda \gg 1$ (i.e., in most of the λ–Ω plane) then the universe is highly non-flat and hence unlikely to be observed via the arguments above, including those cases where their ratio is vastly different from 1.

If $\Omega + \lambda \approx 1$ or $\Omega + \lambda = 1$ then of course the universe is (nearly) flat (essentially a line in the λ–Ω plane). For $\lambda \ll 0$, their ratio is always ≈ 1. For $\lambda \approx 1$, λ/Ω can become arbitrarily large, but only after an arbitrarily large time, so that is unlikely to be observe for weak-anthropic reasons (i.e., for the same reason that in a universe which will last forever we are still, in some sense, infinitely close to the big bang in time). The case $\lambda \approx 0$ occurs near the big bang and is also unlikely to be observed due to weak-anthropic arguments: life needs time to evolve. (Note that $\mathrm{d}\Omega/\,\mathrm{d}t$ and $\mathrm{d}\lambda/\,\mathrm{d}t$ are much larger for $\lambda \approx 0$ than for $\lambda \approx 1$.)

The case $\Omega+\lambda \approx 0$ is not significantly non-flat by the usual definition. (The usual definition of flatness is $\Omega + \lambda \approx 1$. Since a priori the sum can take any value in the interval $[-\infty, +\infty]$, $\Omega + \lambda \approx 0$ is hardly an extreme case.) It can be divided into two cases. For $\lambda \ll 0$, it is unlikely due to the relative-time-scale argument mentioned above. For $\lambda \approx 0$ (essentially a point in the λ–Ω plane), $\mathrm{d}\Omega/\,\mathrm{d}t$ is not very small as is the case for $\lambda \approx 1$, so is there a coincidence problem here? One could say no since it implies $\alpha \approx 0$ and hence a fine-tuned universe. But even ignoring that argument, there is a weak-anthropic argument why such a universe is unlikely to be observed. For $\lambda \approx \Omega \approx 0$, the age of the universe is $\approx 1/H$. If H_0 is small, then the universe is very old, and such a state is presumably reached after life has become extinct. If H_0 is large, then the universe is very young, and such a state is presumably reached before life has had a chance to evolve. Thus, in order to observe $\lambda \approx \Omega \approx 0$, H_0 would have to be fine-tuned. In other words, in such a universe, $\lambda \approx \Omega \approx 0$ holds almost always; thus, if it is observed, no fine-tuning of λ or Ω is needed, but rather of H_0.

Finally, for $\Omega + \lambda \ll 0$, the universe is highly non-flat and hence unlikely to be observed via the arguments above. In particular, all such universes collapse in the future, and thus $\Omega + \lambda \ll 0$ occurs only during a relatively short and special time.

9 Time-scale arguments

The first suggestion that the flatness problem could be avoided via a time-scale argument seems to be due to Tangherlini (1993), though not in the context of an FRW universe. Rindler (2001) pointed out that "the so-called 'flatness problem' – the alleged improbability of finding the value of Ω_0 even within a factor of 10 of unity" seems unproblematic for two reasons, first that "at the big bang ($R = 0$), Ω always starts at one and then wanders away from that value unless $k = \Lambda = 0$" (thus disputing the fine-tuning problem) and second that, in FRW models with $\lambda = 0$ and $\Omega > 1$, "$\Omega < 10 \ldots$ is true for fully 60 per cent of the entire time interval" (thus disputing the instability problem). The second point is also obvious from figure 5 in Sandage (1968) (keeping in mind that $\Omega = 2q$ for $\Lambda = 0$). Helbig (2012), extending the discussion to include $\lambda \neq 0$, noted that while Ω and λ evolve to $(-)\infty$ ($-$ applies to λ for $\lambda < 0$) in models which collapse in the future, in general the values are large only for a relatively short time near the period of maximum expansion, so a typical observer should not expect to see such large values.[29] The important point is the *relative* amount of time during which Ω and λ are $\gg 1$, i.e., the fraction of the total time.

It is sometimes claimed, following Dicke (1970), "that any deviations from flat geometry in the early universe would quickly escalate into a runaway open or closed universe, neither of which is observed" (O'Raifeartaigh et al., 2018, footnote 40, is a typical example). So even though it has been demonstrated that there is no flatness problem, i.e., a typical observer should not be puzzled by the fact that large values of Ω and λ are not observed – because the corresponding cosmological models are unlikely (Lake, 2005) or because such values occur only during a short and special time in the history of the universe (Helbig, 2012) – nor is some sort of fine-tuning necessary to explain the fact that $\Omega = 1$ to high precision in the early universe, nevertheless it is often claimed that Dicke had a valid point: even if there is no flatness problem in that sense, some sort of fine-tuning is necessary, because otherwise a short time after the big bang the Universe would have collapsed or Ω would have evolved to a value $\ll 1$. Just as the argument of the relative time scale shows why the tight-rope analogy (e.g. Coles, 2009) is misleading, the quick-escalation claim is also wrong for essentially the same reason, namely the use of an inappropriate gedankenexperiment.

The argument is usually something like this:

> Imagine, shortly after the big bang, slightly increasing the density of the Universe; that would cause it to collapse after a very short time, perhaps only a few seconds or less.

Another version replaces "density" by "density parameter", i.e., Ω. Increasing the density while keeping the Hubble constant h fixed would also increase Ω, and vice versa. However, one could also increase Ω by keeping the density constant and decreasing h. That should already hint at the resolution: the Friedmann equation is called the Friedmann equation because it is an equation; it makes no sense to imagine changing just *one* parameter. One would have to change at least two in order for the equation to remain valid. However, in general such minimal changes describe universes very different from our own, such as a closed universe with a mass of one kilogram. Yes, such a universe might collapse after a very short time, but that is irrelevant since it is not our Universe nor even a slight perturbation of it in any meaningful sense. The fact that small changes in the early universe can lead to large changes at late times is simply another aspect of the fine-tuning and instability arguments discussed above, which concern the change in the values of λ and Ω with time. However, when the age of the universe is important, the Hubble constant H obviously plays a role. Any universe described by λ and Ω can be given an arbitrary age by adjusting the value of H, though such a universe might be unlikely due to the arguments in Sect. 7. Alternatively, a universe with a value of H similar to the value of H_0 in our Universe but with a vastly different age will also differ significantly in other respects, such as having a very short lifetime or expanding so quickly that no structure can form, and is hence ruled out by weak-anthropic arguments. In other words, such gedankenexperiments fail because it is impossible to have a universe which differs from ours in only *one* respect.

[29] All models with $\lambda < 1$ collapse. If $\lambda = 0$, models collapse for $\Omega > 1$. For $\lambda > 1$, models will also collapse provided $\Omega > 1$ and λ is not too large. For some of those, Ω and λ can be large for a significant time, but those are unlikely due to the argument of Lake (2005).

As another way of looking at the argument, consider the following: In the Einstein–de Sitter model the density in kg per m^3 is given by $1/(6\pi G t^2)$ where G is the gravitational constant and t is the age of the universe in seconds. Ignoring for the moment that precise calculations would require a much more precise value of G than what is known, and assuming that $G = 6.6743 \times 10^{-11}$ $m^3/kg/s^2$ exactly, then one second after the big bang the density of the universeis about 794,864,595 kg per m^3. If we increase the density by just 1 kg per m^3, then the universe is closed and we must add a curvature term so that the Friedmann equation remains an equation. That means that the universe collapses after only about a million years. (Alternatively, decreasing the density by the same amount would lead to the universe expanding forever and becoming curvature dominated after roughly the same time, which would end the growth of structure.) Despite the smallness of the density perturbation, such a universe can hardly be said to be a small perturbation of the Einstein–de Sitter model with an age of, say, ten billion years. A small perturbation of an Einstein–de Sitter universe at an age of ten billion years would of course correspond to a much smaller perturbation at the age of one second. Of course, that is precisely the fine-tuning argument. The point is that there is, in classical cosmology, nothing special about the time one second after the big bang, nor is there anything special about any other time. However large the perturbation at any given time, and whatever the values of the cosmological parameters are at that time, one can always find a time early in the lifetime of universe when that same perturbation was as small as one likes. Since that is true of all FRW models, our Universe is not special in that sense. (One could of course argue, depending on the definition of fine-tuning, that that implies that all FRW models are fine-tuned. The point is that there is no evidence that our Universe is special and that some "mechanism" must have adjusted the cosmological parameters so that they have special values shortly after the big bang.)

It is clear that Guth (1981) believed that there was a flatness problem in that sense:

> A universe can survive $\sim 10^{10}$ years only by extreme fine tuning of the initial values of ρ and H, so that ρ is very near $\rho_{\rm cr}$. For the initial conditions taken at $T_0 = 10^{17}$ GeV, the value of H_0 must be fine tuned to an accuracty of one part in 10^{55}. In the standard model this incredibly precise initial relationship must be assumed without explanation.

$T_0 = 10^{17}$ GeV is about $T_0 = 10^{30}$ K.[30] Since the "standard model" implies an FRW model, the claim is that even *given* an FRW model some explanation for that fine-tuning is needed. However, *all* FRW models exhibit such behaviour; it is an inherent feature. While it is true that a universe which lasts significantly less time than our Universe would have less fine-tuning at a given time, that is not significant, since there is nothing special (in classical cosmology) about the time corresponding to $T_0 = 10^{17}$ GeV. (Interestingly, Guth also mentions that choosing a lower initial temperature (corresponding to a later initial time) would still imply fine-tuning, only somewhat less.) As pointed out by Kragh (2007), of the papers in the reference list of Guth (1981), only two are astronomy papers.[31]

It thus seems plausible that Guth saw a problem where none really exists. In a fascinating glimpse into the thinking at that time, Brawer (1996) claims that that neither the horizon problem nor, especially, the flatness problem was considered to be an important issue until inflation suggested a solution to them. Her thesis, containing

[30] In contrast to my notation, which is not unusual, Guth here uses the suffix 0 to refer to some "initial" time, not to the present time.

[31] Those are Rindler (1956), the standard paper on horizons and cosmology, and Dicke and Peebles (1979). Kragh counts a total of 79, but I count 86, which is strange since apparently both Kragh and I refer to the version reproduced by Bernstein and Feinberg (1986).

many direct quotations and a full interview with Guth, demonstrates the many views on those topics even then. It appears that Guth made an extra effort in his paper to convince the community that the flatness problem is, in fact, a problem (and thus that inflation offers a solution). In the appendix to his article, Guth writes "This appendix is added in hope that some skeptics can be convinced that the flatness problem is real." He does note that in *classical* cosmology, "one always finds that when the equations are extrapolated backward in time, $\Omega \to 1$ as $t \to 0$. Thus ... it is natural for Ω to be very nearly equal to 1 at early times." One might interpret that as indicating that there is no flatness problem in classical cosmology, but rather it arises if one takes into account that GR must break down due to quantum effects in the early universe. However, "for physicists who take this point of view, the flatness problem must be restated in other terms. ... the model universe must be specified by its conserved quantities.".

If one assumes, as he does, that $\Lambda = 0$, then for a spatially closed dust universe one such conserved quantity is the total mass. Guth, however, takes radiation into account, appropriate for the early universe, and notes that the quantity $\varepsilon = k/R^2T^2 < 3\times 10^{-57}$ (T is the temperature and units are such that $\hbar = c = k = 1$; in this sense, k is the Boltzmann constant, but otherwise (including the inequality) k is the curvature parameter in the Robertson–Walker metric), which is essentially a restatement of the fact that the Universe is big, arguing that dimensionless numbers need an explanation unless they are of order 1. While quantum gravity might imply that the typical universe is small, without a theory of quantum gravity that is not a robust prediction. In any case, as discussed above, the "naturalness" argument against small numbers makes sense at most when some sort of cancellation is involved, but that doesn't seem to be the case here. Perhaps inflation can offer an explanation as to why the Universe is as big as it is, but that really has nothing to do with fine-tuning per se, and is not an issue in classical, as opposed to quantum, cosmology. In any case, there are weak-anthropic reasons for why the universe must be large, interestingly pointed out by Dicke (1961), and in a Multiverse scenario no other explanation is really needed.[32] The flatness problem in connection with the age (and mass) of the Universe is discussed in more detail by Helbig (2020). Note that for a spatially closed universe ($k = +1$, assuming, as always in this paper, a trivial topology), a long-lived universe also implies a massive universe, and so there might be a weak-anthropic reason for the fact that our Universe is old and massive. That idea was suggested by Rees (1984), though he favoured "some fundamental reason", such as inflation. Which mechanism one prefers is a matter of taste, and in any case the Universe doesn't care about our preferences. However, the question whether inflation is the only possible explanation for the observed near flatness of the Universe is a question which can be answered objectively, although, whatever the outcome, other evidence must be invoked to determine whether inflation actually occurred.

I am not the first to wonder whether inflation is a solution in search of a problem. Rothman and Ellis (1987) wrote

> A peculiar situation has arisen in cosmology. Over the last five years physicists have been hard at work on a theory that set out to resolve two problems that may not exist. This theory has no evidence to support it, and the one prediction it does make appears to be incorrect.

The one prediction, a nearly flat Universe, has turned out to be true, though to be fair only because of the existence of the cosmological constant, the original ideas about

[32] Interestingly, the idea of eternal inflation, as developed by Linde (1986, 2007), is often touted as a "mechanism" for the Multiverse, though of course the Multiverse is a concept which exists independently of inflation (e.g. Trimble, 2009; Tegmark, 2014).

inflation assuming (as many did at the time) that $\Lambda = 0$. Their anti-inflation article was in *Astronomy* magazine, which I subscribed to as a child; whether it appeared there rather than in a regular journal because of the bandwagon effect as claimed by Burbidge (1988), I don't know.

10 Is it really that simple?

From Eqs. (7) and (8) it follows that λ approaches 0 as R approaches 0 (i.e., at the big bang); similarly, Ω approaches 1. Since that is true whatever the values of λ_0 and Ω_0, and hence holds for all FRW models, there does not need to be any "mechanism" which fine-tunes those values in the early Universe. In particular, there is nothing special about the observed values of λ_0 and Ω_0. Thus, there is no evidence that *our* Universe is somehow special in that sense. Since it makes no sense to say that *all* universes are special, none are. (Were λ_0 and Ω_0 vastly different from their observed values, that would mean only that a given degree of "fine-tuning" occurred at a slightly different time, or at a given time there was a different degree of "fine-tuning".) To be sure, that is a consequence of the FRW models; it is built into them. One can certainly ask the question *why* the Universe is well described by an FRW model, but that is a different question; the flatness problem, as usually formulated, claims that, even *given* that the Universe is well described by an FRW model, there is something strange about the values of λ and Ω in the early Universe in that they must be fine-tuned to deviate only very slightly from those of the Einstein–de Sitter model. In essence, regarding the limits as $R \to 0$ is the solution to the flatness problem in the sense of the "fine-tuning problem" (see Sect. 7 and references therein).

The other aspect, the "instability problem", i.e., why are Ω_0 and λ_0 still near the Einstein–de Sitter values even after such a long time, is a bit more subtle, but still simple: the values can be vastly different, but either for only a short time during the history of the universe (Helbig, 2012) or in the case of *real* fine-tuning between λ and Ω (Lake, 2005). While one needs to quantitatively examine the change in values of λ and Ω during the evolution of the universe in order to arrive at both of those conclusions, once one has done so, the conclusion is almost obvious. In other words, as Lake (2005) pointed out, the values of Ω and λ are strongly dependent on the time of observation (the "epoch problem"). However, either epochs with Ω and λ vastly different from the Einstein–de Sitter values are relatively short (models which collapse in the future), fine-tuning of Ω and λ is *necessary* in order to get a significant departure from the Einstein–de Sitter values (Lake's argument, so that in most of those models ($k = +1$ and expansion forever) there is no epoch problem), or the weak-anthropic argument applies that the Universe is not arbitrarily old (and hence Ω is not ≈ 0 and λ, if non-zero, is not ≈ 1, the values approached asymptotically).

Finally, making use of the weak-anthropic argument that a universe must be old and large enough to support observers, the allowed range of λ and Ω at the early times typically considered when discussing the flatness problem is only a small part of the infinite possible range (Helbig, 2020).

So why is the flatness problem still regarded as a real problem?[33] To a large extent it is certainly due to people repeating what they have learned and not keeping up with the literature, coupled with the fact that it sounds more interesting to discuss problems than their solutions. Similar things have happened in cosmology before. For example, the "paradox", often attributed to Olbers (1823) (English translation: Olbers, 1826), of the darkness of the night sky is even today still held to be a paradox by some, or they believe in a wrong solution, despite the solution having been

[33] There is some anecdotal evidence that that might be more the case in the astronomical community than in the general-relativity community.

known since the middle of the nineteenth century (Poe, 1848), and having been shown quantitatively later by Thomson (1901). While it might be expecting too much of cosmologists to keep up with the writings of Edgar Allan Poe, or even Lord Kelvin, the very well known cosmologist Edward Harrison[34] has chronicled the history of that topic in several articles (Harrison, 1964, 1965, 1974, 1977, 1980, 1984a,b, 1986, 1987, 1990a,b), a chapter in his excellent and popular cosmology textbook (Harrison, 2000, chap. 24), and even wrote an entire book about it (Harrison, 1987). Others have also discussed it in the leading journals (e.g. Wesson et al., 1987). Other examples where genuine confusion was cleared up long ago concern cosmological horizons (Rindler, 1956)[35] and the relation between the redshift–distance and velocity–distance laws, which again was definitively cleared up by Harrison (1993, 2000), who also complained about the persistence of a wrong idea in the literature – in one book review (Rowan-Robinson and Harrison, 1979), those who propagate wrong concepts about cosmological reshifts were even described as "the blind leading the blind". Still, even some professional astronomers still get that wrong.[36]

11 Summary and conclusions

Since its original formulation by Dicke (1970) and Dicke and Peebles (1979), especially after the idea of inflation became popular (e.g. Guth, 1981; Linde, 1982), many arguments were made, though largely ignored, which demonstrated that neither is fine-tuning in the early Universe needed in order to explain the values of λ_0 and Ω_0 observed today, whatever they might be, nor is it puzzling that we don't observe values $\gg 1$ or $\ll 1$ for them. (As stressed by Holman (2018), those are two sides of the same coin.) Such arguments are best understood via the dynamical-systems approach to cosmology, i.e., quantitatively studying how the cosmological parameters change with time during the evolution of the universe. Also, the argument that the early Universe must have been fine-tuned in order for it to last as long as it has is wrong since it is based on the impossible idea of modifying just *one* parameter in the early Universe. Of course, there are weak-anthropic reasons for why our Universe is long-lived, but (in contrast to some other weak-anthropic anthropic arguments) those are not needed to explain observed fine-tuning, since there is no fine-tuning needed in order to resolve the flatness problem.

The arguments discussed above have been made in the leading journals in the field, often by people well known and respected for other contributions, yet the argument of Dicke and Peebles (1979) is still often stated as an unquestionable fact. (As far as I know, no-one has presented any argument in favour of the existence of the flatness problem which goes beyond the arguments of Dicke and Peebles (1979).) Also, the fine-tuning problem is essentially solved merely by taking limits as $R \to 0$. Perhaps the real flatness problem is the question as to why it is still considered to be a problem

[34] For example, his paper "Fluctuations at the Threshold of Classical Cosmology" (on what later came to be known as the Harrison–Zel'dovich spectrum of primordial fluctuations) (Harrison, 1970) has, at the time of writing, according to ADS, 583 citations, and the term "Harrison-Zel'dovich spectrum" has certainly been mentioned much more often without citation.

[35] That definitive treatment was directly inspired (Rindler, 2013) by confusion in the literature (Whitrow, 1953). (See from around 01:33:00 to about 01:37:00 in Rindler (2013), though the entire video, not just Rindler's contribution, is worth watching. Interestingly, at around 01:36:05, Rindler laments that his work on cosmological horizons led to the idea of inflation.)

[36] For example, see the comment by Helbig (2017) on the book by van den Heuvel (2016), prompted by a review of the book by Trimble (2017).

by many. I can only speculate, but I think that there are several reasons. First, the authority of Dicke and Peebles, and of many of those who agree with them regarding the flatness problem, might have put some people off. Second, few people work in the type of mathematical cosmology where quantitative aspects of the flatness problem become apparent. Third, especially due to the popularity of inflation, the existence of the flatness problem became "part of the lore" and was just not questioned very often. Fourth, from about 1990, before the flatness problem became firmly implanted in the minds of many, cosmology had become a data-driven science which distracted from the armchair arguments of yore. Fifth, as mentioned above, it is more interesting to claim that there is some basic problem in the field. Sixth, many who have argued against the flatness problem are at least as strongly part of the general-relativity community as of the astronomical community; my impression is that the flatness problem is perceived to be greater in the latter. Seventh, neither Dicke and Peebles (1979) nor Dicke (1970) is easily available, thus few people probably realize how little material (all of the relevant parts are quoted above) is actually there.

Acknowledgements. I thank the referees for very useful comments and suggestions. There is some overlap between this paper and my paper discussing the time-scale argument in more detail (Helbig, 2020), particularly in Sects. 2 and 9 and, naturally, to some extent in Sect. 11; that allows each paper to be read independently; I see no reason to change the text merely for the sake of change. This research has made use of NASA's Astrophysics Data System Bibliographic Services.

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