Zeitschrift für Analysis und ihre Anwendungen
(submitted)

Some remarks on the Boyd functions related to the admissible sequences

Thomas Lamby and Samuel Nicolay

Abstract. In this paper, we discuss the existing relations between the Boyd functions and the admissible sequences, with a particular interest to the Boyd indices. We introduce several results binding these notions and propose some constructions to obtain a Boyd function from an admissible sequence. We finally discuss the notion of generalized interpolation space based on these concepts.

1. Introduction

The Besov spaces $B^s_{p,q}$ ($s \in \mathbb{R}$, $p, q \in [1,\infty]$) naturally arise through the real interpolation theory (see [2] for instance). For example, they "lie" between the Sobolev spaces H^t_p and H^u_p with $s = (1-\alpha)t + \alpha u$ and $\alpha \in (0,1)$, since we have $B^s_{p,q} = [H^t_p, H^u_p]_{\alpha,q}$. A classical generalization of the Besov spaces was introduced in [9, 19] where the function $t \mapsto t^s$ appearing in the K-method of interpolation is replaced by a Boyd function (see Section 8).

Definition 1.1. A function $\phi : (0,\infty) \to (0,\infty)$ is a Boyd function if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0,\infty)$.

In this context, the so-called Boyd indices have many applications (see for example [4, 5, 12, 14, 19]).

Definition 1.2. The lower and upper Boyd indices of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

respectively.

2020 Mathematics Subject Classification. 26A12, 46E35, 46B70.
Keywords. Boyd functions, admissible sequences, interpolation spaces.

Another approach for providing generalized spaces is proposed in [10]; it relies on the so-called admissible sequences.

Definition 1.3. A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j.

One also associates Boyd indices to such a sequence.

Definition 1.4. Given an admissible sequence σ, let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \sup_{j\in\mathbb{N}} \frac{\log \underline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j}$$

and

$$\overline{s}(\sigma) := \inf_{j\in\mathbb{N}} \frac{\log \overline{\sigma}_j}{\log 2^j} = \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j},$$

respectively.

The analogy between Definition 1.2 and Definition 1.4 is clear by remarking that we have, for a Boyd function ϕ,

$$\underline{b}(\phi) = \lim_{t\to\infty} \frac{\log \underline{\phi}(t)}{\log t},$$

where we have set $\underline{\phi}(t) := \inf_{s>0} \phi(st)/\phi(s)$. For example, as expected, the K-method of interpolation can be generalized using admissible sequences in order to obtain an interpolation method similar to the one using Boyd functions [17]. Of course, these techniques (relying on either the Boyd functions or the admissible sequences) are not limited to the Besov spaces; they have been applied to the Triebel-Lizorkin spaces, Lorentz spaces, the pointwise Hölder spaces and the T_u^p spaces of Calderón-Zygmund among others [9–11, 15, 18, 19].

It is well known that there is a connection between Boyd functions and admissible sequences. Many authors illustrate this link with the following example [1, 8, 20]; given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1}-\sigma_j}{2^j}(t-2^j)+\sigma_j & \text{if } t\in[2^j,2^{j+1}),\ j\in\mathbb{N}_0 \\ \sigma_0 & \text{if } t\in(0,1) \end{cases}, \tag{1.1}$$

with $\sigma_0 = 1$ is a Boyd function. For instance, the generalized Besov space associated to ϕ_σ is the generalized Besov space associated to σ [1]. However, we easily show that with such a construction, we necessarily have $\underline{b}(\phi) \leq 0 \leq \overline{b}(\phi)$, so that even in the simplest case where $\sigma_j = 2^{sj}$ with $s > 0$ ($j \in \mathbb{N}$), $\underline{b}(\phi) < \underline{s}(\sigma) = s$. In other words, the Boyd indices are not preserved with this construction. Of course, with this sequence, $\phi_\sigma(t)$ is comparable to the function ϕ such that $\phi(t) = t^s$ for $t \geq 1$ and $\phi(t) = 1$ for $t < 1$ (more rigorously, they are equivalent, see Definition 2.1). Starting from the fundamental Boyd function $t \mapsto t^s$, one can impose the supplementary condition $1/\phi(t) = \phi(1/t)$ to the Boyd functions in

Definition 1.1. This is done in [7], where the authors use the construction (1.1) for $t \geq 1$ but impose $\phi(t) = 1/\phi(1/t)$ for $t \in (0,1)$. To quote their own words, "somehow unexpectedly, it turns out that the lower and upper Boyd indices of any such interpolating function do coincide with the corresponding indices of the starting sequence".

Here, we investigate the relations between the Boyd functions and the admissible sequences. By doing so, we underline the fundamental differences between Definition 1.2 and Definition 1.4. We show that a Boyd function can be identified to a couple of germs, while a germ can be associated to an admissible sequence. That being said, two admissible sequences yield four Boyd indices, which means that there is no natural way to associate an admissible sequence to a Boyd function. Roughly speaking, an admissible sequence defines a Boyd function either on $(0,1]$ or $[1,\infty)$. As a consequence, the admissible sequences are best suited for pointwise spaces since only asymptotic conditions are usually needed, while the Boyd functions are more adapted to uniform spaces on open sets. In particular, there is a subtle difference in the generalized interpolation theories mentionned above, since the method relying on the admissible sequences correspond to the method based on the Boyd functions for a specific class of functions only.

In this work, we first present the basic properties of the Boyd functions to show that they can be decomposed into two parts, leading to a representation theorem in Section 3. We then give the essential properties of the admissible sequences and make some original remarks before investigating the relations between these two notions in Section 5, with a special attention to the Boyd indices. Next, starting from an admissible sequence, we propose some constructions of Boyd functions. By doing so, we are lead to improve a result of Merucci in [19]. In Section 8, as an application, we discuss the notion of generalized interpolation space through the lens of both the Boyd functions and the admissible sequences.

We will denote by $\mathcal{B}$ the set of the Boyd functions and by I the interval $(0,\infty)$. We use the letter C for a generic positive constant independent of the variable parameters whose value may be different at each occurrence.

2. Properties of the Boyd functions

We list here the basic properties of the Boyd functions. For more details, we refer to [19].

From the definition, it is easy to check that for a Boyd function ϕ, the indices $\underline{b}(\phi)$ and $\overline{b}(\phi)$ are two numbers such that $\underline{b}(\phi) \leq \overline{b}(\phi)$. Moreover, given $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that

$$C^{-1} t^{\overline{b}(\phi)+\varepsilon} \leq \phi(t) \leq C t^{\underline{b}(\phi)-\varepsilon},$$

for any $t \leq R$. In the same way, we also have

$$C^{-1} t^{\underline{b}(\phi)-\varepsilon} \leq \phi(t) \leq C t^{\overline{b}(\phi)+\varepsilon},$$

for any $t \geq R$. We have related inequalities for $\bar{\phi}$, that is $\bar{\phi}(t) \leq C t^{\underline{b}(\phi)-\varepsilon}$ for $t \leq R$ and $\bar{\phi}(t) \leq C t^{\overline{b}(\phi)+\varepsilon}$ for $t \geq R$.

Definition 2.1. Two functions $f : I \to I$ and $g : I \to I$ are said to be equivalent if there exists a constant $C > 0$ such that $C^{-1}g \le f \le Cg$ on I. If f and g satisfy such a relation, we will write $f \sim g$. Since sequences can be seen as functions, we will also use this notion of equivalence for sequences.

If ϕ_1 and ϕ_2 are two Boyd functions such that $\phi_1 \sim \phi_2$, then $\underline{b}(\phi_1) = \underline{b}(\phi_2)$ and $\overline{b}(\phi_1) = \overline{b}(\phi_2)$. Let $\mathcal{B}'$ denote the set of functions $f : I \to I$ that belong to $C^1(I)$ with $f(1) = 1$ and satisfy

$$0 < \inf_{t>0} t\frac{|f'(t)|}{f(t)} \le \sup_{t>0} t\frac{|f'(t)|}{f(t)} < \infty. \tag{2.1}$$

One can show that $\mathcal{B}'$ is a subset of $\mathcal{B}$. If $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ (resp. $\overline{b}(\phi) < 0$), then there exists a non-decreasing bijection (resp. a non-increasing bijection) $\psi \in \mathcal{B}'$ such that $\phi \sim \psi$ and $\psi^{-1} \in \mathcal{B}'$ (see also Proposition 7.2). Such a result is usually used for change-of-variable techniques (see for example [7, 19]).

Let us give an example of a Boyd function that we will use in the sequel.

Example 2.2. Given $r, s \in \mathbb{R}$, let $\phi_{r,s} : I \to I$ be defined by

$$\phi_{r,s}(t) = \begin{cases} t^r & \text{if } t \le 1 \\ t^s & \text{if } t \ge 1 \end{cases}. \tag{2.2}$$

Let us suppose that $r \le s$ (the reasoning is similar in the other case). Given $t > 0$, for $u \ge 1$, we have $\phi(tu)/\phi(u) = t^s$ if $tu \ge 1$ and $\phi(tu)/\phi(u) = t^r u^{r-s}$ otherwise. Now, if $u \in (0, 1)$, we have $\phi(tu)/\phi(u) = t^r$ if $tu \le 1$ and $\phi(tu)/\phi(u) = t^s u^{s-t}$ otherwise. As a consequence, $\bar{\phi}(t) \le \max\{t^r, t^s\}$ and $\phi_{r,s}$ is a Boyd function. Moreover, we have $t^s \le \bar{\phi}(t) \le \max\{t^r, t^s\} = t^s$ for $t \ge 1$ and $t^r \le \bar{\phi}(t) \le \max\{t^r, t^s\} = t^r$ for $t \le 1$, which implies $\underline{b}(\phi_{r,s}) = r$ and $\overline{b}(\phi_{r,s}) = s$. Of course, $\phi_{r,s}$ verifies inequalities (2.1).

In the sequel, we will design more complicated Boyd functions using admissible sequences.

3. A representation theorem

Here, we associate $\mathcal{B}$ to the space $\mathcal{B}^\infty \times \mathcal{B}^\infty$ (see Definition 3.1) to obtain a representation theorem similar to the ones presented in [3, 16]. From a conceptual point of view, this decomposition of the Boyd functions into a cartesian product will allow us to consider that an element of $\mathcal{B}$ is defined by two germs, each being associated to an element of $\mathcal{B}^\infty$.

Definition 3.1. We will denote by $\mathcal{B}^\infty$ the set of continuous functions $\phi : [1, \infty) \to I$ such that $\phi(1) = 1$ and

$$0 < \underline{\phi}(t) := \inf_{s\ge1} \frac{\phi(ts)}{\phi(s)} \le \overline{\phi}(t) := \sup_{s\ge1} \frac{\phi(ts)}{\phi(s)} < \infty, \tag{3.1}$$

for any $t \ge 1$.

Let us adopt the following conventions, by analogy with the admissible sequences. For $\phi \in \mathcal{B}^\infty$, we set

$$\underline{s}(\phi) := \lim_{t\to\infty} \frac{\log \underline{\phi}(t)}{\log t}$$

and

$$\overline{s}(\phi) := \lim_{t\to\infty} \frac{\log \bar{\phi}(t)}{\log t}.$$

If $\phi \in \mathcal{B}^\infty$, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C^{-1} t^{(\underline{s}(\phi)-\varepsilon)} \le \underline{\phi}(t) \le \frac{\phi(ts)}{\phi(s)} \le \overline{\phi}(t) \le C t^{\overline{s}(\phi)+\varepsilon}, \tag{3.2}$$

for any $t, s \ge 1$. It is easy to check that we have $\phi \in \mathcal{B}^\infty$ if and only if $\psi : t \mapsto \phi(1/t)$ is a continuous function defined on $(0,1]$ such that $\psi(1) = 1$ and

$$0 < \inf_{s\le 1} \frac{\psi(ts)}{\psi(s)} \le \sup_{s\le 1} \frac{\psi(ts)}{\psi(s)} < \infty.$$

For a short period of time, we will designate the space of such functions ψ by $\mathcal{B}^0$. In some way, for $\phi \in \mathcal{B}$, we will relate $\mathcal{B}^\infty$ to the sequences $(\phi(2^j))_{j\in\mathbb{N}}$ and $\mathcal{B}^0$ to $(\phi(2^{-j}))_{j\in\mathbb{N}}$.

Given $\phi \in \mathcal{B}$, we denote by ϕ_∞ the restriction of ϕ to $[1,\infty)$ and by ϕ_0 the restriction of ϕ to $(0,1]$. Of course, we have $\phi_\infty \in \mathcal{B}^\infty$ and $\phi_0 \in \mathcal{B}^0$. On the other hand, the converse is also true; more precisely, we have the following result.

Proposition 3.2. *The application*

$$\tau : \mathcal{B} \to \mathcal{B}^\infty \times \mathcal{B}^\infty \quad \phi \mapsto (t \mapsto \frac{1}{\phi_0(1/t)}, \phi_\infty)$$

is a bijection.

Proof. Let us consider the application that maps $(\phi_1, \phi_2) \in \mathcal{B}^\infty \times \mathcal{B}^\infty$ to

$$t \mapsto \phi(t) := \begin{cases} 1/\phi_1(1/t) & \text{if } t \in (0,1] \\ \phi_2(t) & \text{if } t \in [1,\infty) \end{cases}.$$

We have to show that ϕ belongs to $\mathcal{B}$. We need to adjust the proof depending on the order between $\underline{s}(\phi_1)$ and $\underline{s}(\phi_2)$ on one hand and $\overline{s}(\phi_2)$ and $\overline{s}(\phi_1)$ on the other hand. This thus leads to four possibilities, which are very similar to each other. Let us handle the case where

$$\underline{s}(\phi_1) \le \underline{s}(\phi_2) \le \overline{s}(\phi_2) \le \overline{s}(\phi_1).$$

The idea is to use (3.2). Let $t > 0$; we want to show that

$$\overline{\phi}(t) = \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty.$$

If we have $s \geq 1$ and $ts \geq 1$, then $\phi(ts)/\phi(s) \leq \overline{\phi_2}(t)$. If $s \leq 1$ and $ts \leq 1$, we have

$$\frac{\phi(ts)}{\phi(s)} = \Big(\frac{\phi_1(\frac{1}{ts})}{\phi_1(1/s)}\Big)^{-1} \leq 1/\underline{\phi_1}(1/t).$$

For $s \geq 1$ and $ts \leq 1$ we get

$$\frac{\phi(ts)}{\phi(s)} = \big(\phi_1(\frac{1}{ts})\,\phi_2(s)\big)^{-1} \leq C\frac{(ts)^{\underline{s}(\phi_1)-\varepsilon}}{s^{\underline{s}(\phi_2)-\varepsilon}} \leq Ct^{\underline{s}(\phi_1)-\varepsilon},$$

for some $\varepsilon > 0$. Finally, if $s \leq 1$ and $ts \geq 1$, we can write

$$\frac{\phi(ts)}{\phi(s)} = \phi_2(ts)\,\phi_1(1/s) \leq C\frac{(ts)^{\overline{s}(\phi_2)+\varepsilon}}{s^{\overline{s}(\phi_1)+\varepsilon}} \leq Ct^{\overline{s}(\phi_1)+\varepsilon},$$

for some $\varepsilon > 0$. The remaining cases can be treated in the same way. ■

To obtain a representation theorem for the Boyd functions, we first need a representation theorem for the elements of $\mathcal{B}^\infty$.

Theorem 3.3. *A function $\phi : [1,\infty) \to I$ belongs to $\mathcal{B}^\infty$ if and only if $\phi(1) = 1$ and there exist two bounded continuous functions $\eta, \xi : [1,\infty) \to I$ such that*

$$\phi(t) = e^{\eta(t)+\int_1^t \xi(s)\,\frac{ds}{s}}.$$

Proof. Suppose that $\phi \in \mathcal{B}^\infty$ and define $H(t) := \log\phi(t)$ for $t \geq 1$. Let us set $\psi = \tau^{-1}(\phi,\phi)$, where τ is defined in Proposition 3.2. There exists a constant C such that

$$H(t) = C + \int_t^{2t} \frac{H(t)-H(s)}{\log 2}\frac{ds}{s} + \int_1^t \frac{H(2s)-H(s)}{\log 2}\frac{ds}{s}.$$

Now, set

$$\eta(t) := C + \int_t^{2t} \frac{H(t)-H(s)}{\log 2}\frac{ds}{s} \quad \text{and} \quad \xi(s) := \frac{H(2s)-H(s)}{\log 2}.$$

Obviously, η and ξ are continuous functions. Let us show that η and ξ are bounded. For all $s \geq 1$, one has

$$\xi(s) = \frac{H(2s)-H(s)}{\log 2} = \frac{\log\big(\phi(2s)/\phi(s)\big)}{\log 2},$$

so that

$$\frac{\log\underline{\phi}(2)}{\log 2} \leq \xi(s) \leq \frac{\log\overline{\phi}(2)}{\log 2}.$$

For all $t \geq 1$, by using the change of variables $s = tu$, one gets

$$\begin{aligned}\int_t^{2t} \frac{H(t)-H(s)}{\log 2}\frac{ds}{s} &= \int_t^{2t} \frac{\log\big(\phi(t)/\phi(s)\big)}{\log 2}\frac{ds}{s}\\ &= \int_t^{2t} \frac{\log\big(\psi(t)/\psi(s)\big)}{\log 2}\frac{ds}{s}\\ &\leq \int_t^{2t} \frac{\log\big(\overline{\psi}(t/s)\big)}{\log 2}\frac{ds}{s} \leq C\end{aligned}$$

and

$$\int_t^{2t} \frac{H(t)-H(s)}{\log 2}\frac{ds}{s} = \int_t^{2t} \frac{\log\left(\psi(t)/\psi(s)\right)}{\log 2}\frac{ds}{s}$$
$$\geq \int_t^{2t} \frac{\log\left(1/\overline{\psi}(s/t)\right)}{\log 2}\frac{ds}{s} \geq C.$$

Now suppose that $\phi(1) = 1$ and that there exist two bounded continuous functions $\eta, \xi : [1,\infty) \to I$ such that

$$\phi(t) = e^{\eta(t)+\int_1^t \xi(s)\frac{ds}{s}}.$$

Then ϕ is continuous and if $t, s \geq 1$, we have

$$\frac{\phi(ts)}{\phi(s)} = e^{\eta(ts)-\eta(s)+\int_s^{ts} \xi(s)\frac{ds}{s}},$$

so that ϕ belongs to $\mathcal{B}^\infty$. ∎

It is now possible to derive the Boyd indices from the functions ξ in Theorem 3.3 [16].

Remark 3.4. This problem is partially addressed in [3, 16]. If we ask η and ξ to be measurable instead of continuous in Theorem 3.3 (the proof can be easily adapted), the spaces $\mathfrak{B}^\infty$ of functions ϕ satisfying (3.1) lies inbetween two spaces for which we have similar representation theorems. Let us denote by SV the set of slowly varying functions [3], for which one requires $\lim_{t\to\infty} \xi(t) = 0$ and $\lim_{t\to\infty} \eta(t) = C$ for a constant C and by R the set of the functions ϕ for which one only requires η and ξ to be bounded on $[t_0,\infty)$ for some $t_0 \geq 1$ in Theorem 3.3. We obviously have $SV \subset \mathfrak{B}^\infty \subset R$.

Corollary 3.5. *A function $\phi : I \to I$ belongs to $\mathcal{B}$ if and only if $\phi(1) = 1$ and there exist four bounded continuous functions $\eta_0, \xi_0 : (0,1] \to I$ and $\eta_\infty, \xi_\infty : [1,\infty) \to I$ such that*

$$\phi(t) = \begin{cases} e^{\eta_0(t)+\int_1^{1/t} \xi_0(s)\frac{ds}{s}} & \text{if } t \in (0,1] \\ e^{\eta_\infty(t)+\int_1^{t} \xi_\infty(s)\frac{ds}{s}} & \text{if } t \in [1,\infty) \end{cases}.$$

4. Properties of the admissible sequences

We recall here the basic properties of the admissible sequences, give a remark about alternative definitions and an obvious consequence of Theorem 3.3. For more details about admissible sequences, we refer to [10].

Concerning the admissible sequences, we have inequalities similar to (3.2): if σ is an admissible sequence, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C^{-1}2^{(\underline{s}(\sigma)-\varepsilon)j} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C2^{(\overline{s}(\sigma)+\varepsilon)j}, \tag{4.1}$$

for any $j, k \in \mathbb{N}$.

Remark 4.1. The previous inequalities are not valid for $\varepsilon = 0$. For example, if we set $\sigma_j = 2^{js}|\log 2^{-j}|$, the corresponding admissible sequence σ is such that $\overline{\sigma}_j = 2^{js}(1+j)$ and $\overline{s}(\sigma) = s$. Therefore, for any ε, there exists a constant $C > 0$ such that $\overline{\sigma}_j \leq C2^{(s+\varepsilon)j}$, but we cannot have $\overline{\sigma}_j \leq C2^{sj}$.

The following example is taken from [16] and is due to [13]. It shows that an admissible sequence has not necessarily a fixed main order and their upper and lower Boyd indices do not necessarily coincide as remarked in [20]. It thus generalises the slowly varying functions of Karamata [3].

Example 4.2. Consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define the admissible sequence σ by

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ 2^{j_{2n}} 4^{j-j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases}.$$

The sequence oscillates between $(j)_j$ and $(2^j)_j$ and we have $\underline{s}(\sigma) = 0$ and $\overline{s}(\sigma) = 1$.

From the inequalities (4.1), we know that an admissible sequence σ is oscillating between $C^{-1}2^{(\underline{s}(\sigma)-\varepsilon)j}$ and $C2^{(\overline{s}(\sigma)+\varepsilon)j}$. One can wonder if the sequence really oscillates between these 2 quantities or if it can actually vary between "smaller quantities". In fact, it can oscillate between any value as the following example shows.

Example 4.3. Let $\sigma_0 = 1$, $\alpha > 0$ and σ be defined by

$$\sigma_{j+1} := \begin{cases} \sigma_j & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ \sigma_j 2^{\alpha} & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases}.$$

We have $\underline{s}(\sigma) = 0$, $\overline{s}(\sigma) = 1$ and for all $\varepsilon > 0$, there exists $C > 0$ such that $\sigma_j \leq C\,2^{j\varepsilon}$ for all j.

Remark 4.4. There exist alternative definitions of the Boyd indices. For example, in [6], the following quantities are proposed:

$$\underline{\mathfrak{s}}(\sigma) = \liminf_j \frac{\log \sigma_j}{\log 2^j} \quad \text{and} \quad \overline{\mathfrak{s}}(\sigma) = \limsup_j \frac{\log \sigma_j}{\log 2^j}.$$

While presenting the advantage of being simpler to consider than Definition 1.4, since one does not deal with the sequences $(\underline{\sigma}_j)_{j\in\mathbb{N}}$ and $(\overline{\sigma}_j)_{j\in\mathbb{N}}$, theses definitions are less precise concerning the subtle behavior of σ. For example, if we consider the sequence given in Example 4.3, we directly get $\underline{\mathfrak{s}}(\sigma) = \overline{\mathfrak{s}}(\sigma) = 0$, which only reveals that $\sigma = o((2^j)_{j\in\mathbb{N}})$. To address this problem, other indices must be considered beside $\underline{\mathfrak{s}}(\sigma)$ and $\overline{\mathfrak{s}}(\sigma)$ and they usually involve the quantities $\underline{\sigma}_j$ and $\overline{\sigma}_j$. For example, in [6], the quantities $\liminf_j \sigma_{j+1}/\sigma_j$ and $\limsup_j \sigma_{j+1}/\sigma_j$ are also taken into account.

Of course, $\overline{s}(\sigma) < 0$ implies $\sigma_j \to 0$ and $\underline{s}(\sigma) > 0$ implies $\sigma_j \to \infty$. It is easy to check that if ϕ is a Boyd function then $(\phi(2^j))_{j\in\mathbb{N}}$ and $(\phi(2^{-j}))_{j\in\mathbb{N}}$ are both admissible sequences. On the other hand, from an admissible sequence $(\sigma_j)_{j\in\mathbb{N}}$ and up to a normalising constant, one can define a function $\phi \in \mathcal{B}^\infty$ such that $\phi(2^j) = \sigma_j$ for all $j \in \mathbb{N}$. As a consequence, Theorem 3.3 implies the following result.

Corollary 4.5. *If σ is an admissible sequence, then there exist two bounded continuous functions $\eta, \xi : [1, \infty) \to I$ such that*

$$\sigma_j = e^{\eta(2^j)+\int_1^{2^j} \xi(s)\frac{ds}{s}},$$

for all $j \in \mathbb{N}$.

5. Relations between Boyd functions and admissible sequences

As the space of the Boyd functions can be seen as $\mathcal{B}^\infty \times \mathcal{B}^\infty$ and since an admissible sequence defines an element of $\mathcal{B}^\infty$, we can build a Boyd function from a pair of admissible sequences. However, as a pair of admissible sequences involves four Boyd indices, we have to investigate how the choice of the admissible sequences determines the Boyd indices of the resulting Boyd function.

Proposition 5.1. *If $\phi \in \mathcal{B}$ and $\sigma_j = \phi(2^j)$ or $\sigma_j = 1/\phi(2^{-j})$ then we have $\underline{b}(\phi) \le \underline{s}(\sigma) \le \overline{s}(\sigma) \le \overline{b}(\phi)$.*

Proof. For any $j \in \mathbb{N}$, let $\sigma_j = \phi(2^j)$; we have

$$\overline{\sigma}_j \le \sup_{k\ge1} \frac{\phi(2^j 2^k)}{\phi(2^k)} \le \sup_{s>0} \frac{\phi(2^j s)}{\phi(s)} = \bar{\phi}(2^j)$$

and so $\overline{s}(\sigma) \le \lim_j \log\bar{\phi}(2^j)/\log 2^j \le \overline{b}(\phi)$. From there, we directly get $\underline{b}(\phi) = -\overline{b}(1/\phi) \le -\overline{s}(1/\sigma) = \underline{s}(\sigma)$. For the other case, it suffices to consider $\psi \in \mathcal{B}$ defined by $\psi(t) = 1/\phi(1/t)$. ■

As a consequence, if $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then $\underline{b}(\phi) = \underline{s}(\sigma)$ and $\overline{b}(\phi) = \overline{s}(\sigma)$ imply $\underline{s}(\sigma) \le \underline{s}(\theta)$ and $\overline{s}(\theta) \le \overline{s}(\sigma)$. In fact, we have some kind of equivalence, as stated by the following result.

Proposition 5.2. *If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then*

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Proof. Let $\varepsilon > 0$, $j \in \mathbb{N}$ and suppose that $\overline{s}(\sigma) \geq \overline{s}(\theta)$ (what follows is similar for the converse inequality). If t belongs to $[2^n, 2^{n+1})$ for some $n \in \mathbb{N}_0$, we have

$$\frac{\phi(2^j t)}{\phi(t)} \leq \phi(2^j 2^n)\bar{\phi}(t2^{-n})\frac{\bar{\phi}(2^n/t)}{\phi(2^n)} \leq C\frac{\sigma_{j+n}}{\sigma_n} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

Now, if t belongs to $(2^{-n-1}, 2^{-n}]$ for some $n \in \mathbb{N}$, since

$$\frac{\phi(2^j t)}{\phi(t)} \leq \phi(2^j 2^{-n})\bar{\phi}(t2^n)\frac{\bar{\phi}(2^{-n}/t)}{\phi(2^{-n})},$$

if $j \geq n$, we can write

$$\frac{\phi(2^j t)}{\phi(t)} \leq C\sigma_{j-n}/\phi(2^{-n}) \leq C2^{(\overline{s}(\sigma)+\varepsilon)(j-n)}2^{(\overline{s}(\theta)+\varepsilon)n} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

On the other hand, for $j \leq n$, we have

$$\frac{\phi(2^j t)}{\phi(t)} \leq C\frac{1/\phi(2^{-n})}{1/\phi(2^{-(n-j)})} \leq C2^{(\overline{s}(\theta)+\varepsilon)j} \leq C2^{j(\overline{s}(\sigma)+\varepsilon)}.$$

In any case, we get

$$\overline{b}(\phi) \leq \lim_j \frac{\log C}{\log 2^j} + \frac{\log 2^{j(\overline{s}(\sigma)+\varepsilon)}}{\log 2^j} = \overline{s}(\sigma) + \varepsilon.$$

Using Proposition 5.1, we get $\overline{b}(\phi) = \overline{s}(\sigma)$. If we now suppose that $\underline{s}(\sigma) \leq \underline{s}(\theta)$, we can obtain $\underline{b}(\theta) = \underline{s}(\sigma)$ in the same way, which ends the proof. ■

We can now bind the indices of a Boyd function to the indices of its "components" in $\mathcal{B}^\infty$. Let τ be the mapping defined in Proposition 3.2 and denote by τ_1 and τ_2 its two components in $\mathcal{B}^\infty$.

Corollary 5.3. *If ϕ belongs to $\mathcal{B}$, then we have $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ and $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.*

6. Some elementary examples of a Boyd function obtained from one admissible sequence

Let us give some methods for constructing Boyd functions starting from a given admissible sequence that preserve the Boyd indices.

Given an admissible sequence σ, we are naturally led to define a function $\phi_\sigma \in \mathcal{B}$ such that $\phi_\sigma(2^j) = \sigma_j$ and $\phi_\sigma(2^{-j}) = 1/\sigma_j$ for $j \in \mathbb{N}_0$ with $\sigma_0 = 1$ in order to preserve the Boyd indices. If, in the definition of ϕ_σ, we connect the elements σ_j on $[1, \infty)$ and $1/\sigma_j$ on $(0, 1]$ with straight lines, we obtain

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \dfrac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}],\ j \in \mathbb{N}_0 \end{cases}.$$

We obviously have $\phi_\sigma \in \mathcal{B}$: since σ is an admissible sequence, we have $\bar{\phi}_\sigma(t) \leq C(t)$. Another possibility, consists in using the construction proposed in [7], that is

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1}-\sigma_j}{2^j}(t-2^j)+\sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \\ \dfrac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases}$$

and it is now clear that the Boyd indices of σ and this function ϕ_σ do coincide. Other constructions can be obtained by setting $\phi_\sigma(2^j) = \sigma_j$ for $j \in \mathbb{N}$ and $\phi_\sigma(t) = t^s$ for $t \in (0,1]$, where s satisfies $\underline{s}(\sigma) \leq s \leq \overline{s}(\sigma)$. For example, we can define

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1}-\sigma_j}{2^j}(t-2^j)+\sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ t^s & \text{if } t \in (0,1) \end{cases} \tag{6.1}$$

On the other hand, if s is chosen so that $s < \underline{s}(\sigma)$ (resp. so that $s > \overline{s}(\sigma)$) in (6.1), then we have $\underline{b}(\phi_\sigma) = s$ (resp. $\overline{b}(\phi_\sigma) = s$). This is what happens in the construction (1.1) with $s = 0$.

Given an admissible sequence σ, it can be useful to derive a regular function $\phi \in \mathcal{B}$ such that $\phi(2^j) = \sigma_j$ for all $j \in \mathbb{N}$ and

$$0 < \inf_{t>1} t\frac{|\phi'(t)|}{\phi(t)} \quad \text{or} \quad \sup_{t>1} t\frac{|\phi'(t)|}{\phi(t)} < \infty \tag{6.2}$$

(see [19] for example). This can be done using smooth transition functions. For example, let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

to define

$$g : x \mapsto \frac{f(x)}{f(x)+f(1-x)}\chi_{[0,1]} + \chi_{(1,\infty)}. \tag{6.3}$$

We will use this function to connect the points $(2^j, \sigma_j)$ and $(2^{j+1}, \sigma_{j+1})$ $(j \in \mathbb{N})$. Remark that one can replace x^2 by $e^{-1/x}, x^n$ or $\cosh(x)$ for example in the definition of f. However, with such a construction, the resulting Boyd function has a vanishing derivative at 2^j for every $j \in \mathbb{N}$. If σ is strictly increasing, we can avoid this pitfall by applying a rotation; the angle α must be chosen sufficiently small so that the resulting curve can be associated to a function. This threshold obviously depends on both the function f and the admissible sequence σ. Let us make this construction explicitly. We will work with the axes (X, Y) such that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $j \in \mathbb{N}$, we set

$$\begin{cases} X_j = 2^j\cos\alpha + \sigma_j \sin\alpha \\ Y_j = -2^j \sin\alpha + \sigma_j\cos\alpha \end{cases},$$

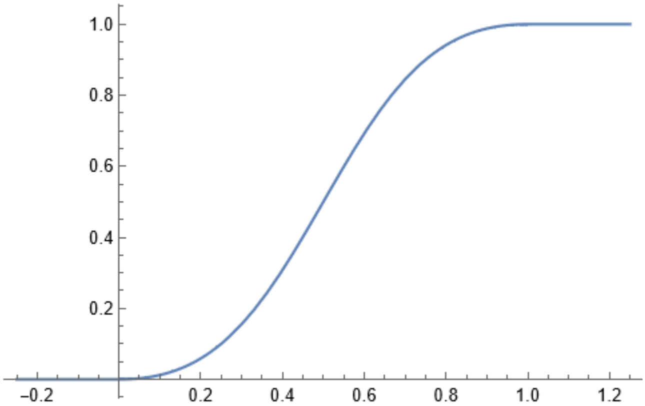

Figure 1. The function g defined by (6.3).

$$\xi^{(j)}(X) = \frac{X - X_j}{X_{j+1} - X_j}$$

and

$$\tau^{(j)}(X) = Y_j + (Y_{j+1} - Y_j)X$$

to consider the curve

$$Y = \tau^{(j)}(g(\xi^{(j)}(X)))$$

on $[X_j, X_{j+1}]$, which gives rise to

$$Y(y) = \tau^{(j)}(g(\xi^{(j)}(X(x))))$$

on the original Euclidean plane. Explicitly, we have

$$\begin{aligned} -x \sin\alpha + y\cos\alpha = {} & -2^j \sin\alpha + \sigma_j \cos\alpha \\ & + (-2^{j+1}\sin\alpha + \sigma_{j+1}\cos\alpha + 2^j \sin\alpha - \sigma_j \cos\alpha) \\ & \times \frac{1}{1 + \left(\frac{x\cos\alpha + y\sin\alpha - 2^{j+1}\cos\alpha - \sigma_{j+1}\sin\alpha}{x\cos\alpha + y\sin\alpha - 2^j\cos\alpha - \sigma_j \sin\alpha}\right)^2}, \end{aligned}$$

for $x \in [2^j, 2^{j+1}]$ with $j \in \mathbb{N}_0$ and $\sigma_0 = 1$. Let $\eta_j^{(\alpha)}$ be the function $x \mapsto y$ on $[2^j, 2^{j+1}]$. For $\alpha = 0$, we explicitly get

$$\eta_j^{(0)}(t) = \sigma_j + \frac{\sigma_{j+1} - \sigma_j}{1 + \left(\frac{t - 2^{j+1}}{t - 2^j}\right)^2}$$

on $(2^j, 2^{j+1})$. If $\alpha > 0$ is small enough, we get a function $\eta_j^{(\alpha)}$ whose explicit form is far more complicated. However, its derivative never vanish on $[2^j, 2^{j+1}]$. Now, we can

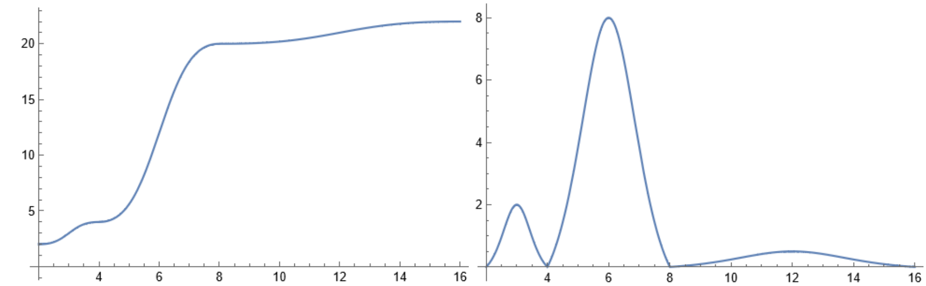

Figure 2. The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

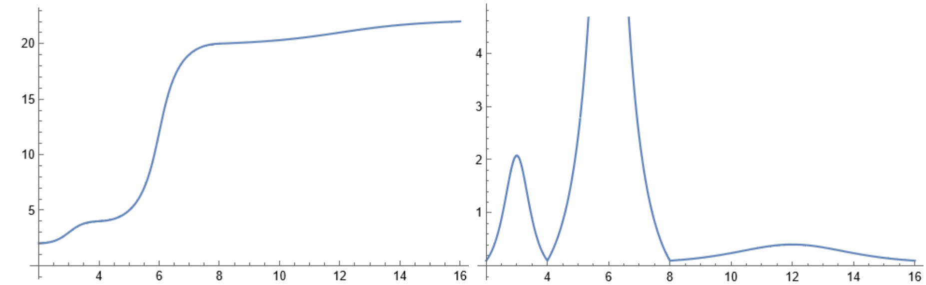

Figure 3. The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0.1$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases} .$$

It is easy to show that the first inequality (resp. the second inequality) in (6.2) is satisfied if $\sigma_j \le C2^j$ (resp. $\sigma_j \ge C\,2^j$) for every $j \in \mathbb{N}$, since in this case, $\phi(t) \le C't$ implies

$$t\frac{|\phi'(t)|}{\phi(t)} \ge \alpha/C' > 0,$$

for all $t > 1$.

7. Constructing a regular Boyd function from an admissible sequence

We can improve the previous construction of a regular Boyd function by developing some considerations given in [19]: it is possible, under some general conditions, to obtain an infinitely differentiable function that belongs to $\mathcal{B}'$. By doing so, we improve the original result where the obtained function is only continuously differentiable.

Proposition 7.1. *If σ is an admissible sequence such that either $\underline{s}(\sigma) > 0$ or $\overline{s}(\sigma) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $(\xi(2^j))_j \sim \sigma$.*

Proof. First, we construct $\phi \in \mathcal{B}$ by

$$\phi(t) = \begin{cases} \sigma_j + \frac{\sigma_{j+1}-\sigma_j}{1+\left(\frac{t-2^{j+1}}{t-2^j}\right)^2} & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases}.$$

Let $a \in \mathbb{R}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < a < +\infty$. For all $s > 0$, let us set

$$\xi_1(s) = \int_0^\infty \min\{1, \left(\frac{s}{t}\right)^a\}\, \phi(t)\, \frac{dt}{t}.$$

It then suffices to take $\xi = \xi_1/\xi_1(1)$ to conclude. ∎

As a consequence, we can improve the original result in [19], where the function ξ can be supposed differentiable. Indeed we can assume that it is infinitely differentiable.

Proposition 7.2. *If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $\xi \sim \phi$.*

8. Applications to interpolation spaces

As an application, we look at the generalized interpolation spaces defined with Boyd functions and admissible sequences.

Let us briefly recall the notion of interpolation space; for more details, see [2] for example. We consider two normed vector spaces A_0 and A_1 which are continuously embedded in a Hausdorff topological vector space H. Therefore, the spaces $A_0 \cap A_1$ and $A_0 + A_1$ are also normed vector spaces. Let us recall that the K-operator of interpolation is defined for $t > 0$ and $a \in A_0 + A_1$ by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

If $\theta \in (0,1)$ and $q \in [1,\infty]$, then a belongs to the interpolation space $[A_0, A_1]_{\theta,q}$ if $a \in A_0 + A_1$ and

$$(2^{-\theta j} K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}).$$

This last condition is equivalent to $t \mapsto t^{-\theta} K(t,a) \in L^q_*$ (where, as usual, L^q_* denotes the space of strongly measurable functions f on $(0,\infty)$, i.e. such that $(\int_0^\infty |f|^p\, dt/t)^{1/p}$ is finite). This method has been generalized in [19] as follows:

Definition 8.1. Given $\phi, \gamma \in \mathcal{B}$ and $q \in [1,\infty]$, a belongs to the generalized interpolation space $[A_0, A_1]^\gamma_{\phi,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^\gamma_{\phi,q}} := \|\phi(t)^{-1} K(\gamma(t), a)\|_{L^q_*} < \infty.$$

If γ is the identity, $[A_0, A_1]_{\phi,q}$ will stand for the space $[A_0, A_1]^{\gamma}_{\phi,q}$.

We present here several possible generalized interpolation spaces defined with admissible sequences and show their relation to the generalized interpolation spaces related to the Boyd functions. We first need some easy results.

Lemma 8.2. *If $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, then a belongs to $[A_0, A_1]^{\gamma}_{\phi,q}$ if and only if $\sum_{j\in\mathbb{Z}} \big(\frac{1}{\phi(2^j)} K(\gamma(2^j), a)\big)^q < \infty$.*

Proof. It's clear by using the properties of Boyd functions. ∎

Lemma 8.3. *Let $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$; if $\underline{b}(\gamma) > 0$, then there exists a non-decreasing bijection $\xi \in \mathcal{B}'$ such that $\xi \sim \gamma$ and*

$$[A_0, A_1]^{\gamma}_{\phi,q} = [A_0, A_1]_{\phi\circ\xi^{-1},q}.$$

Proof. Since $\underline{b}(\gamma) > 0$, there exists a strictly increasing function $\xi \in \mathcal{B}'$ such that $\xi \sim \gamma$. The change of variables $s = \xi(t)$ allows to conclude. ∎

To define a first notion of generalized interpolation space using admissible sequences, one can proceed as in [17].

Definition 8.4. Let $r, s \in \mathbb{R}$ and σ, δ be two admissible sequences such that $\underline{\delta}_1 > 1$ and

$$r < \min\{\underline{s}(\sigma)\underline{s}(\delta)^{-1}, \underline{s}(\sigma)\overline{s}(\delta)^{-1}\} \le \max\{\overline{s}(\sigma)\underline{s}(\delta)^{-1}, \overline{s}(\sigma)\overline{s}(\delta)^{-1}\} < s.$$

Then, a belongs to $[A_0, A_1]^{\sigma,\delta}_q$ whenever $(\theta_j K(\psi_j, a))_j \in l^q(\mathbb{Z})$, where

$$\theta_j := \begin{cases} \delta^{-r}_{-j}\,\sigma_{-j} & \text{if } j \in -\mathbb{N}_0 \\ \delta^{r}_{j}\,\sigma^{-1}_{j} & \text{if } j \in \mathbb{N} \end{cases}$$

and

$$\psi_j := \begin{cases} \delta^{-(s-r)}_{-j} & \text{if } j \in -\mathbb{N}_0, \\ \delta^{s-r}_{j} & \text{if } j \in \mathbb{N}. \end{cases}.$$

Let us relate this notion to Definition 8.1. For $t > 0$, set

$$\phi_1(t) = \frac{\theta_{j+1} - \theta_j}{2^j}(t - 2^j) + \sigma_j \text{ if } t \in [2^j, 2^{j+1}), (j \in \mathbb{Z})$$

and

$$\phi_2(t) = \frac{\psi_{j+1} - \psi_j}{2^j}(t - 2^j) + \psi_j \text{ if } t \in [2^j, 2^{j+1}), (j \in \mathbb{Z}).$$

It is clear that $\phi_1, \phi_2 \in \mathcal{B}$. By lemma 8.2,

$$a \in [A_0, A_1]^{\sigma,\delta}_q \Leftrightarrow a \in [A_0, A_1]^{\phi_2}_{K,\phi_1,q}.$$

Since $\underline{\delta}_1 > 1$, we have $\underline{b}(\phi_2) > 0$ and we can use Lemma 8.3 to get

$$a \in [A_0, A_1]^{\sigma,\delta}_q \Leftrightarrow a \in [A_0, A_1]_{\phi_1\circ\xi^{-1},q},$$

with $\xi \sim \phi_2$.

Remark 8.5. An intuitive generalization of interpolation spaces with a unique admissible sequence consists in taking $\delta = (2^j)_j$ in Definition 8.4 with $r = 0$ and $s = 1$.

Remark 8.6. Since $\theta_j = 1/\theta_{-j}$ for all $j \in \mathbb{N}$, this method corresponds to the interpolation spaces defined with Boyd functions for specific Boyd functions only.

Another point of view for defining interpolation spaces from admissible sequences consists in considering two sequences to characterize the two components in $\mathcal{B}^\infty$ associated to a Boyd function (see Proposition 3.2).

Definition 8.7. Let σ be an admissible sequence and $q \in [1, \infty]$; a belongs to the upper generalized interpolation space $[A_0, A_1]^\wedge_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^\wedge_{\sigma,q}} := \sum_{j=1}^{\infty} \frac{1}{\sigma_j} K(2^j, a) < \infty.$$

In the same way, a belongs to the lower generalized interpolation space $[A_0, A_1]^\vee_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^\vee_{K,\sigma,q}} := \sum_{j=1}^{\infty} \sigma_j K(2^{-j}, a) < \infty.$$

Proposition 8.8. *If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then*

$$[A_0, A_1]_{\phi,q} = [A_0, A_1]^\vee_{\delta,q} \cap [A_0, A_1]^\wedge_{\sigma,q}.$$

Proof. This is obvious, using the previous lemma. ■

To conclude this short investigation on the interpolation spaces with an evident remark, if we want to have the equivalence between the continuous and the discrete case, one needs to deal with admissible sequences on $\mathbb{Z}$. That being said, it is not clear to what extend such considerations would be useful when dealing with classical spaces.

Funding. Thomas Lamby is supported by a grant of the FNRS.

References

[1] A. Almeida. Wavelet bases in generalized Besov spaces. *J. Math. Anal. Appl.*, 304:198–211, 2005.

[2] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1976.

[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*. Cambridge University Press, 1987.

[4] D. W. Boyd. The Hilbert Transform on Rearrangement-Invariant Spaces. *Canadian J. Math.*, 19, 1967.

[5] D. W. Boyd. Indices of function spaces and their relationship to interpolation. *Canadian J. Math.*, 21:1245–1254, 1969.

[6] M. Bricchi. *Tailored function spaces and h-sets*. PhD thesis, University of Jena, 2002.

[7] M. Bricchi and S. D. Moura. Complements on growth envelopes of spaces with generalized smoothness in the sub-critical case. *Z. fur Anal. ihre Anwend.*, 22:383–398, 2003.

[8] A. M. Caetano and W. Farkas. Local growth envelopes of Besov spaces of generalized smoothness. *Z. fur Anal. ihre Anwend.*, 25:265–298, 2006.

[9] F. Cobos and D. L. Fernández. Hardy-Sobolev spaces and Besov-spaces with a function parameter. *Lecture Notes in Math.*, 1302:158–170, 1988.

[10] W. Farkas and H.-G. Leopold. Characterisations of function spaces of generalised smoothness. *Ann. Mat. Pura Appl.*, 185:1–62, 2006.

[11] D. Haroske and S. Moura. Continuity envelopes and sharp embeddings inspaces of generalized smoothness. *J. Funct. Anal.*, 254, 2008.

[12] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri. *Symmetric structures in Banach spaces*. Amer. Math. Soc., memoirs edition, 1979.

[13] G. A. Kaljabin. Characterization of spaces of generalized Liouville differentiation. *Mathematics of The USSR-Sbornik*, 33, 1977.

[14] S. G. Krein, Ju. I. Petunin, and E. M. Semenov. *Interpolation of Linear Operators*, volume 54. Amer. Math. Soc., translations of mathematical monographs edition, 1982.

[15] D. Kreit and S. Nicolay. Generalized pointwise Hölder spaces defined via admissible sequences. *J. Funct. Spaces*, 2018.

[16] T. Kühn, H. Leopold, W. Sickel, and L. Skrzypczak. Entropy numbers of embeddings of weighted Besov spaces. II. *Proc. Edinburgh Math*, 49, 2006.

[17] L. Loosveldt and S. Nicolay. Some equivalent definitions of Besov spaces of generalized smoothness. *Math. Nach.*, 292:2262–2282, 2019.

[18] L. Loosveldt and S. Nicolay. Generalized T_u^p spaces: On the trail of Calderón and Zygmund. *Diss. Math.*, 554, 2020.

[19] C. Merucci. Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces. In M. Cwikel and J. Peetre, editors, *Interpolation Spaces and Allied Topics in Analysis*, pages 183–201. Springer, 1984.

[20] S. D. Moura, J. S. Neves, and C. Schneider. Spaces of generalized smoothness in the critical case: Optimal embeddings, continuity envelopes and approximation numbers. *J. Approx. Theory*, 187:82–117, 2004.

Thomas Lamby
Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège.; thomas.lamby@uliege.be

Samuel Nicolay
Université de Liège, Département de mathématique – zone Polytech 1, 12 allée de la Découverte, Bât. B37, B-4000 Liège.; S.Nicolay@uliege.be