Real interpolation with a function parameter

Thomas Lamby (joint work with S. Nicolay)

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Some conventions

- Two topological vector spaces A_0 and A_1 are *compatible* if there is a Hausdorff topological vector space H such that A_0 and A_1 are sub-spaces of H.
- $\mathscr{N}$ denotes the category of all normed vector spaces (a sub-category of all topological vector spaces).
- $\mathscr{C}$ denotes any sub-category of the category $\mathscr{N}$ that is closed under the operations sum and intersection
- $\mathscr{C}_1$ denotes the category of all compatible couples $\overline{A} = (A_0, A_1)$ of spaces in $\mathscr{C}$.

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More conventions

- The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathscr{C}_1$ are all bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that

$$T_{A_0} : A_0 \to B_0, \quad T_{A_1} : A_1 \to B_1$$

  are morphisms in $\mathscr{C}$.
- Two basic functors from $\mathscr{C}_1$ to $\mathscr{C}$: $\Sigma(T) = \Delta(T) = T$ and

$$\Delta(\overline{A}) = A_0 \cap A_1,$$

$$\Sigma(\overline{A}) = A_0 + A_1.$$

- Let $\overline{A} = (A_0, A_1)$ be a given couple in $\mathscr{C}_1$. Then a space A in $\mathscr{C}$ will be called an *intermediate space* between A_0 and A_1 (or with respect to $\overline{A}$) if

$$\Delta(\overline{A}) \subset A \subset \Sigma(\overline{A}),$$

with continuous inclusions.

- The space A is called an *interpolation space* between A_0 and A_I (or with respect to $\overline{A}$) if in addition

$$T : \overline{A} \to \overline{A} \text{ implies } T : A \to A.$$

- More generally, let $\overline{A}$ and $\overline{B}$ be two couples in $\mathscr{C}_1$. Then we say that two spaces A and B in $\mathscr{C}$ are *interpolation spaces* with respect to $\overline{A}$ and $\overline{B}$ if A and B are intermediate spaces with respect to $\overline{A}$ and $\overline{B}$ respectively, and if

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Interpolation spaces

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Interpolation spaces

- If $\|T\|_{A,B} \leq \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$ holds, we shall say that A and B are *exact* interpolation spaces.
- If $\|T\|_{A,B} \leq C \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$ holds, we shall say that A and B are *uniform* interpolation spaces.
- The interpolation spaces A and B are of *exponent* $\theta \in [0,1]$ if

$$\|T\|_{A,B} \leq C \, \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}.$$

If $C = 1$, we say that A and B are *exact of exponent* θ.

Interpolation functor

- An *interpolation functor* on $\mathscr{C}$ is a functor F from $\mathscr{C}_1$ into $\mathscr{C}$ such that if $\overline{A}$ and $\overline{B}$ are couples in $\mathscr{C}_1$, then $F(\overline{A})$ and $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$ and

$$F(T) = T \text{ for all } T : \overline{A} \to \overline{B}.$$

- F is a *uniform* (*exact*) interpolation functor if $F(\overline{A})$ and $F(\overline{B})$ are uniform (exact) interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Similarly, F is (*exact*) of *exponent* θ if $F(\overline{A})$ and $F(\overline{B})$ are (*exact*) of exponent θ.

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Aronszajn-Gagliardo Theorem

Theorem

Consider the category $\mathscr{B}$ of all Banach spaces. Let A be an interpolation space with respect to the couple $\overline{A}$. Then there exists a minimal exact interpolation functor F_0 on $\mathscr{B}$ such that $F_0(\overline{A}) = A$.

Let $\overline{X} = (X_0, X_1)$ be a given couple in $\mathscr{B}_1$. Then $X = F_0(\overline{X})$ consists of those $x \in \Sigma(\overline{X})$, which admit a representation

$$x = \sum_j T_j a_j \quad \text{(convergence in } \Sigma(\overline{X})\text{)},$$

where $T_j : \overline{A} \to \overline{X}$, $a_j \in A$. Set

$$N_X(x) = \sum_j \max(\|T_j\|_{A_0,X_0}, \|T_j\|_{A_1,X_1})\|a_j\|_A.$$

The norm in X is the infimum of $N_X(x)$ over all admissible representations of x.

Quid for interpolation functor of exponent θ ?

Theorem

Let $\theta \in [0,1]$, consider the category $\mathscr{B}$ of all Banach spaces. Let A be an interpolation space of exponent θ with respect to the couple $\overline{A}$. Then there exists a minimal interpolation functor F_θ, which is exact and of exponent θ, such that $F_\theta(\overline{A}) = A$.

Let $\overline{X} = (X_0, X_1)$ be a given couple in $\mathscr{B}_1$. Then $X = F_\theta(\overline{X})$ consists of those $x \in \Sigma(\overline{X})$, which admit a representation

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where $T_j : \overline{A} \to \overline{X}$, $a_j \in A$. Set

$$N_\theta(x) = \sum_j \|T_j\|_{A_0,X_0}^{1-\theta} \|T_j\|_{A_1,X_1}^{\theta} \|a_j\|_A.$$

The norm in X is the infimum of $N_\theta(x)$ over all admissible representations of x.

The K-method

The K-operator of interpolation is defined for $t > 0$ and $a \in \Sigma(\overline{A})$ by

$$K(t,a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

If $\theta \in (0,1)$ and $q \in [1,\infty]$, then a belongs to the interpolation space $K_{\theta,q}(A_0, A_1)$ if $a \in \Sigma(\overline{A})$ and

$$(2^{-\theta j} K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}).$$

This last condition is equivalent to $t \mapsto t^{-\theta} K(t,a) \in L^q_*$.

For example, $B^s_{p,q} = K_{\alpha,q}(H^t_p, H^u_p)$ for $s = (1-\alpha)t + \alpha u$.

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Boyd functions

A function $\phi : (0, \infty) \to (0, \infty)$ is a *Boyd function* if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0, \infty)$. The *lower* and *upper Boyd indices* of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

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A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j$ for all j. Let $\underline{\sigma}_j := \inf_{k\geq 1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\geq 1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

$$\underline{s}(\sigma) := \lim_j \frac{\log \underline{\sigma}_j}{\log 2^j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_j \frac{\log \overline{\sigma}_j}{\log 2^j}.$$

Given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0 & \text{if } t \in (0,1) \end{cases},$$

with $\sigma_0 = 1$ is a Boyd function.

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \textit{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

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Boyd function and admissible sequence

Some elementary examples :

$$\phi_\sigma(t) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{2^j}(t-2^j)+\sigma_j & \text{if } t \in [2^j, 2^{j+1}), \\ \frac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t-2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}]. \end{cases}$$

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where s satisfies $\underline{s}(\sigma) \leq s \leq \overline{s}(\sigma)$.

Constructing a regular Boyd function from an admissible sequence

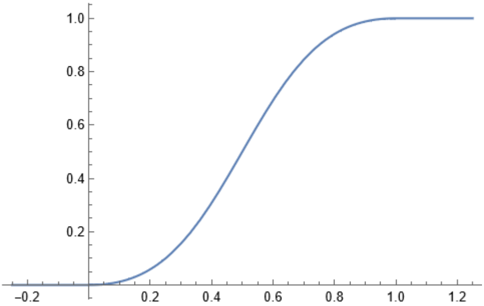
Let

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

to define

$$g : x \mapsto \frac{f(x)}{f(x) + f(1-x)}$$

on $[0, 1]$.



Constructing a regular Boyd function from an admissible sequence

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $j \in \mathbb{N}$, we set

$$\begin{cases} X_j = 2^j \cos\alpha + \sigma_j \sin\alpha \\ Y_j = -2^j \sin\alpha + \sigma_j \cos\alpha \end{cases},$$

$$\xi^{(j)}(X) = \frac{X - X_j}{X_{j+1} - X_j}$$

and

$$\tau^{(j)}(X) = Y_j + (Y_{j+1} - Y_j)X$$

to consider the curve

$$Y = \tau^{(j)}(g(\xi^{(j)}(X)))$$

on $[X_j, X_{j+1}]$.

Constructing a regular Boyd function from an admissible sequence

It gives rise to

$$Y(y) = \tau^{(j)}(g(\xi^{(j)}(X(x))))$$

on the original Euclidean plane.

Let $\eta_j^{(\alpha)}$ be the function $x \mapsto y$ on $[2^j, 2^{j+1}]$.

We can construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases} .$$

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Constructing a regular Boyd function from an admissible sequence

For $\alpha = 0$, we explicitly get

$$\eta_j^{(0)}(t) = \sigma_j + \frac{\sigma_{j+1} - \sigma_j}{1 + \left(\frac{t-2^{j+1}}{t-2^j}\right)^2}.$$

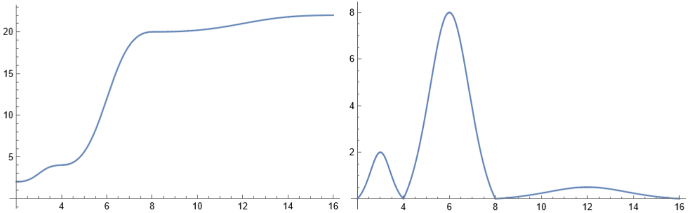

Figure: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

Constructing a regular Boyd function from an admissible sequence

If $\alpha > 0$ is small enough, we get a function $\eta_j^{(\alpha)}$ whose explicit form is far more complicated.

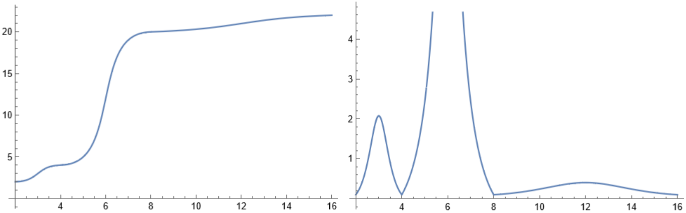

Figure: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0.1$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

Constructing a regular Boyd function from an admissible sequence

Let $\mathcal{B}'$ denote the set of functions $f : I \to I$ that belong to $C^1(I)$ with $f(1) = 1$ and satisfy

$$0 < \inf_{t>0} t \frac{|f'(t)|}{f(t)} \leq \sup_{t>0} t \frac{|f'(t)|}{f(t)} < \infty.$$

One can show that $\mathcal{B}'$ is a subset of $\mathcal{B}$. If $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ (resp. $\overline{b}(\phi) < 0$), then there exists a non-decreasing bijection (resp. a non-increasing bijection) $\psi \in \mathcal{B}'$ such that $\phi \sim \psi$ and $\psi^{-1} \in \mathcal{B}'$

Proposition

If $\phi \in \mathcal{B}$ is such that $\underline{b}(\phi) > 0$ or $\overline{b}(\phi) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $\xi \sim \phi$.

Generalized Interpolation

- The interpolation spaces A and B are of *exponent* $\phi \in \mathcal{B}$ if

$$\|T\|_{A,B} \leq C\,\overline{\psi}(\|T\|_{A_0,X_0})\overline{\phi}(\|T\|_{A_1,X_1}),$$

  where $\psi(t) = t/\phi(t)$ for all $t > 0$.
  If $C = 1$, we say that A and B are *exact of exponent* ϕ.
- F is an (*exact*) interpolation functor of *exponent* $\phi \in \mathcal{B}$ if $F(\overline{A})$ and $F(\overline{B})$ are (*exact*) of exponent ϕ.
- Let $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we let $K_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{\phi,q,K} := \int_0^\infty \left(\frac{1}{\phi(t)}K(t,a)\right)^q \frac{dt}{t} < \infty$$

  holds.

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holds.

$K_{\phi,q}$ is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$.

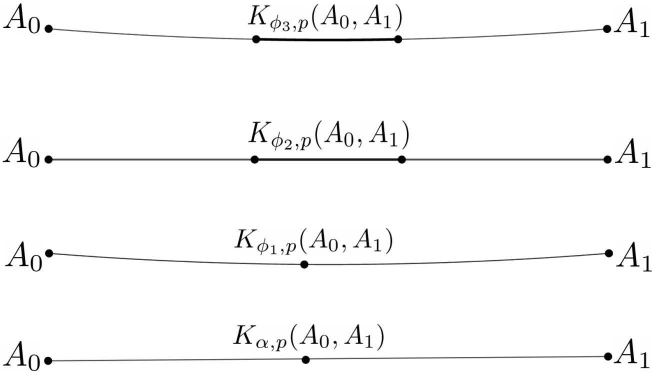

Figure: Differents interpolation spaces where for example $\phi_1(t) = t^\alpha \log(1/t)$, $\phi_2(t) = t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}$ and $\phi_3(t) = (t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}) \log(1/t)$.

The K-method

Given $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, a belongs to the generalized interpolation space $[A_0, A_1]^{\gamma}_{\phi,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\gamma}_{\phi,q}} := \|\phi(t)^{-1} K(\gamma(t), a)\|_{L^q_*} < \infty.$$

Proposition

If $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, then a belongs to $[A_0, A_1]^{\gamma}_{\phi,q}$ if and only if $\sum_{j \in \mathbb{Z}} \left(\frac{1}{\phi(2^j)} K(\gamma(2^j), a)\right)^q < \infty$.

Proposition

Let $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$; if $\underline{b}(\gamma) > 0$, then there exists $\xi \in \mathcal{B}'_+$ such that $\xi \sim \gamma$ and

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Theorem

Let $\phi \in \mathcal{B}$ such that $0 \leq \underline{b}(\phi) \leq \overline{b}(\phi) \leq 1$, consider the category $\mathscr{B}$ of all Banach spaces. Let A be an interpolation space of exponent ϕ with respect to the couple $\overline{A}$. Then there exists a minimal interpolation functor F_ϕ, which is exact and of exponent ϕ, such that $F_\phi(\overline{A}) = A$.

Let $\overline{X} = (X_0, X_1)$ be a given couple in $\mathscr{B}_1$. Then $X = F_\phi(\overline{X})$ consists of those $x \in \Sigma(\overline{X})$, which admit a representation

$$x = \sum_j T_j a_j \quad \text{(convergence in } \Sigma(\overline{X})\text{)},$$

where $T_j : \overline{A} \to \overline{X}$, $a_j \in A$. Set

$$N_\phi(x) = \sum_j \overline{\psi}(\|T_j\|_{A_0,X_0})\overline{\phi}(\|T_j\|_{A_1,X_1})\|a_j\|_A.$$

The norm in X is the infimum of the $N_\phi(x)$.

Thank you for your attention !