Switching from primary to subharmonic resonances in nonlinear systems

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Summary. This work proposes a simple method to switch between the primary and subharmonic resonances of nonlinear systems. For a primary resonance harmonically excited at a specific forcing amplitude, there exists another forcing amplitude at an integer multiple of the fundamental frequency allowing for the excitation of the corresponding subharmonic resonance having the same amplitude and frequency. Using an energy analysis, it is possible to determine the necessary forcing amplitude change to switch to the targeted subharmonic resonance. The proposed method is numerically illustrated with a transition from a 1:1 to a 1:3 resonance of a Duffing oscillator.

Introduction

Nonlinear systems feature rich dynamics, such as multistability, modal interactions, isolated responses, quasiperiodic oscillations under harmonic forcing, and chaos. Furthermore, the nonlinearities of the system generate harmonics of the forcing frequency. This leads to the appearance of different families of resonances in addition to the primary one, such as superharmonic, subharmonic, ultra-subharmonic and combination resonances [1]. Among theses, subharmonic resonances are unique in that they occur when the excitation frequency is an integer multiple of a resonance frequency.
Subharmonic resonances have attracted attention in various physics and engineering research areas, such as energy harvesting [2, 3], micro-electromechanical systems [4] and metamaterials [5]. An inherent difficulty associated with their characterization is that they usually appear as isolated branches with respect to the main nonlinear frequency response [1, 6]. Exciting them thus generally requires time-consuming stochastic approaches to bring the state of the system under test into the basin of attraction of the sought resonance [3].
This work proposes a method to excite subharmonic resonances when the corresponding primary resonance is known. If the system is excited at one of its primary resonances, and we wish to seamlessly transition to a subharmonic resonance while maintaining the same motion in terms of amplitude, frequency and phase, then an explicit relation between the forcing amplitudes in these two cases can be derived using an energy method. By performing an adequate change in excitation frequency and amplitude, it is thus possible to change a primary resonance into a subharmonic one.

Energy analysis of resonances

The following equations of motion are considered:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = f_\nu \mathbf{f}_{ext} \sin(\nu\omega t), \qquad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are structural mass, damping and stiffness matrices, respectively, $\mathbf{f}_{nl}$ is the vector of nonlinear forces, and f_ν and $\mathbf{f}_{ext}$ are the external forcing amplitude and spatial distribution, respectively. It is assumed that the structure responds periodically with an angular frequency ω to the external forcing at angular frequency $\nu\omega$ (with ν a strictly positive integer number). The response can thus be expressed by the Fourier series

$$\mathbf{x}(t) = \frac{\mathbf{x}_{c0}}{\sqrt{2}} + \sum_{n=1}^{\infty} \mathbf{x}_{sn} \sin(n\omega t) + \mathbf{x}_{cn} \cos(n\omega t). \qquad (2)$$

We now look for the conditions under which a primary resonance $\mathbf{x}_{1:1}(t)$ can be switched to a subharmonic one $\mathbf{x}_{1:\nu}(t)$, where $\mathbf{x}_{1:1}(t)$ and $\mathbf{x}_{1:\nu}(t)$ represent the solution of Equation (1) for $\nu = 1$ and $\nu \neq 1$, respectively. It is thus assumed that $\mathbf{x}(t) := \mathbf{x}_{1:1}(t) \approx \mathbf{x}_{1:\nu}(t)$. A justification of this hypothesis can be found in [6] if lightly-damped systems are considered, because primary and subharmonic resonances can both be seen as perturbations of the same periodic orbit of the underlying conservative system. The premultiplication of Equation (1) by $\dot{\mathbf{x}}^T(t)$ (where superscript T denotes transposition) and subsequent integration over one period of motion T feature the dissipation $\mathcal{D}$ and the work done by the external forcing over one period $\mathcal{W}_{ext,\nu}$

$$\mathcal{D} = \int_0^T \dot{\mathbf{x}}^T(t) \left(\mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{f}_{nl}(\mathbf{x}(t), \dot{\mathbf{x}}(t))\right) dt = \int_0^T \dot{\mathbf{x}}^T(t) f_\nu \mathbf{f}_{ext} \sin(\nu\omega t)\, dt = \mathcal{W}_{ext,\nu} = -\nu\pi \mathbf{x}_{c\nu}^T \mathbf{f}_{ext} f_\nu, \qquad (3)$$

where the last equality is obtained thanks to Equation (2). $\mathcal{D}$ depends only on the motion $\mathbf{x}(t)$ and is therefore identical for the primary and subharmonic resonances by assumption. It then follows that $\mathcal{W}_{ext,1} = \mathcal{W}_{ext,\nu}$, and hence from Equation (3) the forcing amplitudes of the primary (f_1) and subharmonic (f_ν, $\nu \neq 1$) resonances must satisfy a relation given by

$$f_\nu = \frac{\mathbf{x}_{c1}^T \mathbf{f}_{ext}}{\nu \mathbf{x}_{c\nu}^T \mathbf{f}_{ext}} f_1. \qquad (4)$$

This suggests that a subharmonic resonance may simply be excited by initiating the structure to its primary resonance, and by subsequently performing a multiplication of the forcing frequency by ν and a change in amplitude according to Equation (4). The factor in this equation can readily be computed either numerically or experimentally, as it only requires to monitor the first and ν^{th} cosine Fourier coefficients of the response at the forcing location ($\mathbf{x}_{c1}^T \mathbf{f}_{ext}$ and $\mathbf{x}_{c\nu}^T \mathbf{f}_{ext}$).

Illustration with a Duffing oscillator

To illustrate the proposed approach, we consider a Duffing oscillator governed by the equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_{nl}x^3(t) = f(t), \quad (5)$$

with parameters $m = 1$ kg, $c = 0.01$ kg/s, $k = 1$ N/m and $k_{nl} = 1$ N/m^3.
The structure is excited at its primary (phase) resonance with a harmonic forcing of amplitude 0.047 N at frequency $\omega = 2.14$ rad/s. A jump to the 1:3 subharmonic resonance is initiated by multiplying the forcing frequency by 3 and changing its amplitude to 0.49 N, this new amplitude being computed from Equation (4). The time simulation results (obtained with a Runge-Kutta time integration scheme) are depicted in Figure 1. In spite of the strong change in forcing featured in Figure 1a, nearly no transient is observed in the motion, and the forcing is effectively able to sustain a motion with threefold period. Figure 1b indicates that this regime is sustained after the transients died out, which confirms that this subharmonic motion is a stable attractor of the system.

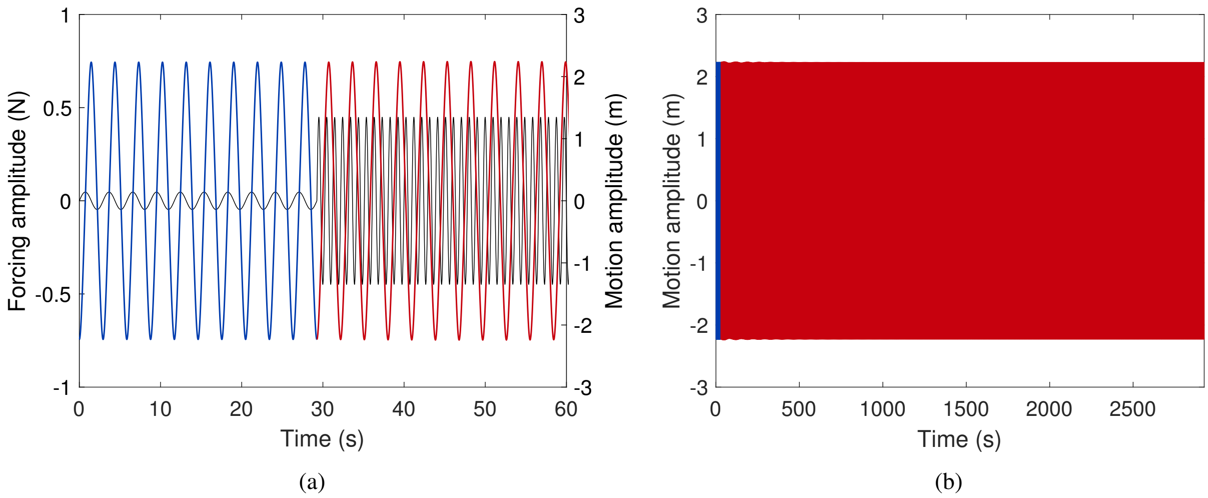

Figure 1: Time simulation of a Duffing oscillator undergoing a transition from a 1:1 to a 1:3 resonance: forcing $f(t)$ (—) and motion $x(t)$ before (—) and after (—) the change in forcing; close-up on the transition time (a) and full time simulation (b).

Conclusion

An approach to excite subharmonic resonances of nonlinear systems was proposed in this work. Based on the assumption that the resonant motions and the work provided by the external forcings are both identical for the primary and subharmonic resonances, an explicit relation was derived between the two forcing amplitudes. This allows for the transition from a primary to a subharmonic resonance through a suitable change in forcing amplitude and frequency. The proposed method was numerically demonstrated with a Duffing oscillator. This method is simple and holds great promises to find and predict subharmonic resonances of more complex nonlinear systems from the knowledge of their associated primary resonance, both numerically and experimentally.

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