Appendix 1: Development of the equations for the behaviour model

The developments presented in this appendix follow the mathematical developments proposed by Ahmed[1]

1 DISCRETIZATION OF THE EQUATIONS

The equations of the model are written in their discretized form in the table below, where subscripts k and $k+1$ stand for the beginning and the end of the time step, respectively, and Δ symbolizes the variation of any variable over the time step. For instance, the variation of the stress tensor is written using the Voigt notation as:

$$\Delta\underline{\sigma} = \underline{\sigma}_{k+1} - \underline{\sigma}_k$$

Feature	Equation	
Strain partition	$\Delta\underline{\varepsilon} = \Delta\underline{\varepsilon}^{th} + \Delta\underline{\varepsilon}^{e} + \Delta\underline{\varepsilon}^{p}$	(1)
Hooke's law	$\Delta\underline{\sigma} = \Delta\underline{\underline{E}}:\underline{\varepsilon}^{e} + \underline{\underline{E}}:\Delta\underline{\varepsilon}^{e}$	(2)
Yield surface (von Mises)	$f_{k+1} = J(\underline{\sigma}_{k+1} - \underline{X}_{k+1}) - R_{k+1} - \sigma_{y,k+1} \leq 0$	(3)
Viscoplasticity	$\frac{\Delta p}{\Delta t} = \left\langle \frac{f_{k+1}}{K} \right\rangle^{n}$ for $f_{k+1} > 0$	(4)
Isotropic hardening	$R_{k+1} = Q(1 - e^{-bp_{k+1}})$	(5)
Kinematic hardening ($i = 1{:}n_{AF}$) • Static recovery • Thermal variation • Mean stress evolution	$\frac{\Delta \underline{X}_i}{\Delta t} = \frac{2}{3} C_i \frac{\Delta \underline{\varepsilon}^{p}}{\Delta t} - \gamma_i(\underline{X}_{i,k+1} - \underline{Y}_{i,k+1})\frac{\Delta p}{\Delta t} - b_i J(\underline{X}_{i,k+1})^{r_i - 1}\underline{X}_{i,k+1} + \frac{1}{C_i}\frac{dC_i}{dT}\frac{\Delta T}{\Delta t}\underline{X}_{i,k+1}$	(6)
	With: $\frac{\Delta \underline{Y}_i}{\Delta t} = -\alpha_{b,i}\left(\frac{3}{2} Y_{st,i} \frac{\underline{X}_{i,k+1}}{J(\underline{X}_{i,k+1})} + \underline{Y}_{i,k+1}\right) J(\underline{X}_{i,k+1})^{r_i}$	(7)

$J(\underline{X})$ denotes the equivalent von Mises stress, defined as:

[1] R. Ahmed, "Constitutive Modeling for Very High Temperature Thermo- Mechanical Fatigue Responses," PhD thesis, North Carolina State University, 2013.

$$J(\underline{X}) = \sqrt{\frac{3}{2}\underline{\hat{X}}:\underline{\hat{X}}}$$

Isotropic hardening is computed using the closed form expression of $R(p)$ given by Eq. (2.13) in Section 1.1.3 of Chapter 2.

Parameters γ_i and D_{γ_i} for cyclic hardening can evolve if the option is activated in the input file. In this case, these parameters are calculated using the beginning of step values of the plastic multiplier p and the plastic strain memory surface radius q.

$$\gamma_i = \gamma_i^f - (\gamma_i^f - \gamma_i)e^{-D_{\gamma_i}p_k}$$

With:

$$\gamma_i^f = a_{\gamma_i} + b_{\gamma_i}e^{-c_{\gamma_i}q_k}$$

If parameters D_{γ_i} and E are made dependent on the maximum temperature in the loading history T_{max}, they are also computed explicitly:

$$D_{\gamma_i}(T_{k+1}, t_{k+1}) = D_{\gamma_i}(T_{max}, 0) - \left(D_{\gamma_i}(T_{max}, 0) - D_{\gamma_i}(T_{k+1}, 0)\right)e^{-b_{D_\gamma}p_k}$$

$$E(T_{k+1}, t_{k+1}) = f_E E(T_{k+1}, 0) + (1 - f_E)E(T_{max}, 0)$$

With: $f_E = f_E^S - (f_E^S - 1)e^{-b_E p_k}$

Equations (1) to (6) are rearranged to limit the total number of equations to solve.

Firstly, Eq. (1) and Eq. (2) can be combined into Eq. (8a), or Eq. (8b) in which the unknown $\Delta\underline{\varepsilon}^p$ is put in a distinct term (all the variables in the two first terms of (8b) are known):

$$\Delta\underline{\sigma} = \Delta\underline{\underline{E}}:\left(\underline{\varepsilon}_{k+1} - \underline{\varepsilon}_{k+1}^p - \underline{\varepsilon}_{k+1}^{th}\right) + \underline{\underline{E}}:\left(\Delta\underline{\varepsilon} - \Delta\underline{\varepsilon}^p - \Delta\underline{\varepsilon}^{th}\right) \quad (8a)$$

$$\Delta\underline{\sigma} = \Delta\underline{\underline{E}}:\left(\underline{\varepsilon}_{k+1} - \underline{\varepsilon}_k^p - \underline{\varepsilon}_{k+1}^{th}\right) + \underline{\underline{E}}:\left(\Delta\underline{\varepsilon} - \Delta\underline{\varepsilon}^{th}\right) - \left(\underline{\underline{E}} + \Delta\underline{\underline{E}}\right):\Delta\underline{\varepsilon}^p \quad (8b)$$

Using the hypothesis of plastic strain incompressibility ($tr\left(\underline{\varepsilon}^p\right) = 0$), the third term of Eq. (8b) can be rewritten as $2(G + \Delta G)\Delta\underline{\varepsilon}^p$, where G is the shear modulus.

2 ELASTIC PREDICTOR

As explained in Chapter 2, the equations are solved using the radial return mapping algorithm, which consists in computing a first elastic step where $\Delta p = 0$ and correcting this step with a plastic corrector if the values found in the elastic predictor do not verify the von Mises criterion.

In the elastic step, the increment of plastic strain is considered equal to zero. A trial stress $\underline{\sigma}_{k+1}^{tr}$ is computed using this hypothesis. Using Eq. (8b), the expression for $\underline{\sigma}_{k+1}^{tr}$ is:

$$\underline{\sigma}_{k+1}^{tr} = \underline{\sigma}_k + \Delta\underline{\underline{E}}:\left(\underline{\varepsilon}_{k+1} - \underline{\varepsilon}_k^p - \underline{\varepsilon}_{k+1}^{th}\right) + \underline{\underline{E}}:\left(\Delta\underline{\varepsilon} - \Delta\underline{\varepsilon}^{th}\right) \tag{9}$$

$\underline{\sigma}_{k+1}^{tr}$ can be computed directly since the total strain, thermal strain, and previous plastic strain are known.

To compute the von Mises yield surface, the value of the back-stresses must be updated. With the hypothesis of $\Delta p = 0$, Eq. (6) can be rewritten as Eq. (10):

$$\frac{\Delta\underline{X}_i}{\Delta t} = -b_i J\left(\underline{X}_{i,k+1}\right)^{r_i-1}\underline{X}_{i,k+1} + \frac{1}{C_i}\frac{dC_i}{dT}\frac{\Delta T}{\Delta t}\underline{X}_{i,k+1} \tag{10}$$

From Eq. (10), the updated back-stresses can be expressed as:

$$\underline{X}_{i,k+1} = \omega_i \underline{X}_{i,k} \tag{11a}$$

$$\omega_i = \frac{1}{1 + b_i \Delta t J\left(\underline{X}_{i,k+1}\right)^{r_i-1} - \frac{1}{C_i}\frac{dC_i}{dT}\Delta T} \tag{11b}$$

Eq. (11a) can be put into scalar form by using the equivalent von Mises stress:

$$g_i = J\left(\underline{X}_{i,k+1}\right) - \omega_i J\left(\underline{X}_{i,k}\right) = 0 \tag{12}$$

This nonlinear equation is solved using the Newton-Raphson method, by computing the consecutive iterations $J\left(\underline{X}_{i,k+1}\right)^m$ given by Eq. (13), where superscript m denotes the Newton iteration number, until convergence.

$$J\left(X_{i,k+1}\right)^{m+1} = J\left(X_{i,k+1}\right)^m - \frac{g_i^m}{\left(\frac{\partial g_i}{\partial J\left(\underline{X}_{i,k+1}\right)}\right)^m} \tag{13a}$$

$$\frac{\partial g_i}{\partial J\left(\underline{X}_{i,k+1}\right)} = 1 - \frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)}\underline{X}_{i,k} \tag{13b}$$

$$\frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)} = -\omega_i^2 b_i \Delta t (r_i - 1) J\left(\underline{X}_{i,k+1}\right)^{r_i-2} \tag{13c}$$

Once $J\left(\underline{X}_{i,k+1}\right)$ is obtained, the value of the back-stresses can be updated using Eq. (11a).

The yield criterion is then checked using the value of $\underline{\sigma}_{k+1}^{tr}$ and the updated back-stresses. If the value of $f_{k+1}^{tr} = J(\underline{\sigma}_{k+1}^{tr} - \underline{X}_{k+1}) - R_{k+1} - \sigma_{y,k+1}$ is less than 0, then the tensors $\underline{Y}_i$ must be updated before going to the next time step.

Eq. (7) can be rewritten to give a closed-form expression of tensor $\underline{Y}_{i,k+1}$:

$$\underline{Y}_{i,k+1} = \varphi_i(\underline{Y}_{i,k} - \rho_i \underline{X}_{i,k}) \quad (14a)$$

$$\varphi_i = \frac{1}{1 + \Delta t \alpha_{b_i} J(\underline{X}_{i,k+1})^{r_i}} \quad (14b)$$

$$\rho_i = \frac{3}{2} \Delta t \alpha_{b_i} Y_{st,i} J(\underline{X}_{i,k+1})^{r_i - 1} \quad (14c)$$

3 PLASTIC CORRECTOR

3.1 REDUCTION OF THE SYSTEM OF EQUATIONS

If $f^{tr} > 0$, the hypothesis of $\Delta p = 0$ is invalid and the plastic strain increment $\Delta \underline{\varepsilon}^p$ must be computed using Eq. (1) to (7). To avoid solving all these equations at once as a large system of equations, some combinations are done to obtain:

- n_{AF} nonlinear scalar equations for each $J(\underline{X}_{i,k+1})$
- 1 nonlinear scalar equation for Δp

The plastic strain increment $\Delta \underline{\varepsilon}^p$ can be written as a function of the plastic multiplier increment Δp and the normal to the yield surface $\underline{n}$:

$$\Delta \underline{\varepsilon}^p = \Delta p \underline{n} \quad (15a)$$

$$\underline{n} = \frac{3}{2} \frac{\hat{\underline{\sigma}} - \hat{\underline{X}}}{J(\underline{\sigma} - \underline{X})} \quad (15b)$$

Similarly to what was done for the elastic predictor, Eq. (6) is rewritten to obtain a new expression of $\underline{X}_{i,k+1}$:

$$\underline{X}_{i,k+1} = \omega_i \left(\underline{X}_{i,k} + \frac{2}{3} C_i \Delta p \underline{n} + \gamma_i \varphi_i \Delta p \underline{Y}_{i,k} \right) \quad (16a)$$

$$\omega_i = \frac{1}{1 + \gamma_i(1 + \varphi_i \rho_i)\Delta p + b_i \Delta t J(\underline{X}_{i,k+1})^{r_i - 1} - \frac{1}{C_i} \frac{dC_i}{dT} \Delta T} \quad (16b)$$

Where φ_i and ρ_i are defined by Eq. (14b) and (14c), respectively.

The effect of hydrostatic pressure is neglected in metal plasticity, therefore, the deviatoric stress $\hat{\underline{\sigma}}$ is used instead of the total stress. Using the deviatoric trial stress $\hat{\underline{\sigma}}_{k+1}^{tr}$ and Eq. (15a), Eq. (8b) can be rewritten as:

$$\hat{\underline{\sigma}}_{\mathrm{k+1}} = \hat{\underline{\sigma}}_{k+1}^{tr} - 2(G + \Delta G)\Delta p \underline{n} \quad (17)$$

Combining this equation together with Eq. (16a) and with some mathematical developments, the following equations can be obtained (note: the back-stress is deviatoric by nature, therefore, $\underline{X}_{k+1}$ is equivalent to $\underline{\hat{X}}_{k+1}$):

$$\underline{\hat{\sigma}}_{k+1} - \underline{X}_{k+1} = \underline{\hat{\sigma}}_{k+1}^{tr} - 2(G + \Delta G)\Delta p \underline{n} - \sum_{i=1}^{n_{AF}} \omega_i \left(\underline{X}_{i,k} + \frac{2}{3} C_i \Delta p \underline{n} + \gamma_i \varphi_i \Delta p \underline{Y}_{i,k} \right) = \underline{Z}/\Omega \tag{18a}$$

$$\underline{Z} = \underline{\hat{\sigma}}_{k+1}^{tr} - \sum_{i=1}^{n_{AF}} \omega_i \underline{X}_{i,k} - \Delta p \sum_{i=1}^{n_{AF}} \omega_i \gamma_i \varphi_i \underline{Y}_{i,k} \tag{18b}$$

$$\Omega = 1 + \frac{\Delta p}{J(\underline{\hat{\sigma}}_{k+1} - \underline{X}_{k+1})} \left(3(G + \Delta G) + \sum_{i=1}^{n_{AF}} \omega_i C_i \right) \tag{18c}$$

Combining equations (3), (4), and (5) gives an expression of $J(\underline{\hat{\sigma}}_{k+1} - \underline{X}_{k+1})$:

$$J(\underline{\hat{\sigma}}_{k+1} - \underline{X}_{k+1}) = K \left(\frac{\Delta p}{\Delta t} \right)^{\frac{1}{n}} + Q(1 - e^{-bp_{k+1}}) + \sigma_y \tag{19}$$

Combining Eq. (18a), (18c), and Eq. (19) gives:

$$\Omega J(\underline{\hat{\sigma}}_{k+1} - \underline{X}_{k+1}) - J(\underline{Z}) = 0 \tag{20a}$$

$$\boxed{\left(K \left(\frac{\Delta p}{\Delta t} \right)^{\frac{1}{n}} + Q(1 - e^{-bp_{k+1}}) + \sigma_y \right) + \Delta p \left(3(G + \Delta G) + \sum_{i=1}^{n_{AF}} \omega_i C_i \right) - J(\underline{Z}) = 0} \tag{20b}$$

Eq. (20b) is the equation for unknown Δp. The n_{AF} equations for the back-stresses can be obtained by combining Eq. (16a) with the definition of $J(\underline{X})$:

$$J(\underline{X}_{i,k+1}) = \sqrt{\frac{3}{2} \omega_i^2 \left(\underline{X}_{i,k} + \frac{2}{3} C_i \Delta p \underline{n} + \gamma_i \varphi_i \Delta p \underline{Y}_{i,k} \right) : \left(\underline{X}_{i,k} + \frac{2}{3} C_i \Delta p \underline{n} + \gamma_i \varphi_i \Delta p \underline{Y}_{i,k} \right)} \tag{21a}$$

$$J(\underline{X}_{i,k+1}) = \omega_i \sqrt{\begin{array}{l} \frac{3}{2} \underline{X}_{i,k} : \underline{X}_{i,k} + \frac{2}{3} C_i^2 \Delta p^2 \underline{n} : \underline{n} + \frac{3}{2} \gamma_i^2 \varphi_i^2 \Delta p^2 \underline{Y}_{i,k} : \underline{Y}_{i,k} + \\ 2 C_i \Delta p \underline{X}_{i,k} : \underline{n} + 3 \gamma_i \varphi_i \Delta p \underline{X}_{i,k} : \underline{Y}_{i,k} + 2 C_i \gamma_i \varphi_i \Delta p^2 \underline{n} : \underline{Y}_{i,k} \end{array}} \tag{21b}$$

From Eq. (15b), we get $\underline{n} : \underline{n} = \frac{3}{2}$. Similarly, the dot product of $\underline{X}_{i,k}$ and $\underline{X}_{i,k}$ can be rewritten as: $\frac{3}{2} \underline{X}_{i,k} : \underline{X}_{i,k} = J(\underline{X}_{i,k})^2$. The n_{AF} equations for the back-stresses can be written as:

$$g_i = J(\underline{X}_{i,k+1}) - \omega_i M_i = 0 \tag{22a}$$

$$M_i = \omega_i \sqrt{\begin{matrix} J(\underline{X}_{i,k})^2 + C_i^2 \Delta p^2 + \gamma_i^2 \varphi_i^2 \Delta p^2 J(\underline{Y}_{i,k})^2 + \\ 2C_i \Delta p \underline{X}_{i,k} : \underline{n} + 3\gamma_i \varphi_i \Delta p \underline{X}_{i,k} : \underline{Y}_{i,k} + 2C_i \gamma_i \varphi_i \Delta p^2 \underline{n} : \underline{Y}_{i,k} \end{matrix}} \tag{22b}$$

3.2 SOLUTION USING THE NEWTON METHOD

Equations (20b) and (22) are the $n_{AF} + 1$ nonlinear equations to solve for the plastic corrector step. They are solved iteratively using the Newton method:

- Step 1: Eq. (20b) is solved assuming $\underline{X}_{i,k+1} = \underline{X}_{i,k}$. If the solution given by the Newton algorithm is negative (i.e., $\Delta p < 0$), the new value of Δp is set to $\Delta p^0/10$, where Δp^0 is the initial guess for the Newton method.
- Step 2: Eq. (22) are solved together for $i = 1{:}\,n_{AF}$ using the previously found value of Δp.

If the Newton algorithm does not converge in Step 2, Steps 1 and 2 are repeated using updated values of $\underline{X}_{i,k+1}$ found at Step 2.

3.2.1 Solution of the Δp equation

Let L be the left-hand side of Eq. (20b). The Newton method consists in computing the consecutive values Δp^m using the following equation (superscript m denotes the Newton iteration number):

$$\Delta p^{m+1} = \Delta p^m - \frac{L^m}{\left(\frac{\partial L}{\partial \Delta p}\right)^m} \tag{23}$$

$\frac{\partial L}{\partial \Delta p}$ can be derived from Eq. (20b):

$$\frac{\partial L}{\partial \Delta p} = \frac{K}{n\Delta t}\left(\frac{\Delta p}{\Delta t}\right)^{\frac{1}{n}-1} + Qbe^{-b(p_k+\Delta p)} + 3(G + \Delta G) + \sum_{i=1}^{n_{AF}} \omega_i C_i + \Delta p \sum_{i=1}^{n_{AF}} \frac{\partial \omega_i}{\partial \Delta p} C_i - \frac{3}{2J(\underline{Z})}\left(\underline{Z} : \frac{\partial \underline{Z}}{\partial \Delta p}\right) \tag{24}$$

With:

$$\frac{\partial \omega_i}{\partial \Delta p} = -\omega_i^2 \gamma_i (1 + \varphi_i \rho_i) \tag{25a}$$

$$\frac{\partial \underline{Z}}{\partial \Delta p} = -\sum_{i=1}^{n_{AF}} \left(\frac{\partial \omega_i}{\partial \Delta p} \underline{X}_{i,k} \right) - \Delta p \sum_{i=1}^{n_{AFY}} \left(\frac{\partial \omega_i}{\partial \Delta p} \gamma_i \varphi_i \underline{Y}_{i,k} \right) - \sum_{i=1}^{n_{AFY}} \left(\omega_i \gamma_i \varphi_i \underline{Y}_{i,k} \right) \tag{25b}$$

3.2.2 Solution of the $J(\underline{X}_i)$ equations

The Newton method consists in finding the consecutive values of $J(X_{i,k+1})^m$ using the following equations:

$$J(X_{i,k+1})^{m+1} = J(X_{i,k+1})^m - \frac{g_i^m}{\left(\frac{\partial g_i}{\partial J(\underline{X}_{i,k+1})} \right)^m} \tag{26a}$$

$$\frac{\partial g_i}{\partial J(\underline{X}_{i,k+1})} = 1 - \omega_i \frac{\partial M_i}{\partial J(\underline{X}_{i,k+1})} - \frac{\partial \omega_i}{\partial J(\underline{X}_{i,k+1})} M_i \tag{26b}$$

The expression of $\frac{\partial \omega_i}{\partial J(\underline{X}_{i,k+1})}$ is given by:

$$\frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)} = -\omega_i^2 \left[\Delta p \gamma_i \left(\varphi_i \frac{\partial \rho_i}{\partial J\left(\underline{X}_{i,k+1}\right)} + \frac{\partial \varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \rho_i \right) + b_i \Delta t (r_i - 1) J\left(\underline{X}_{i,k+1}\right)^{r_i - 2} \right] \tag{27a}$$

$$\frac{\partial \rho_i}{\partial J(\underline{X}_{i,k+1})} = \frac{3}{2} \Delta t \alpha_{b,i} Y_{st,i} (r_i - 1) J(\underline{X}_{i,k+1})^{r_i - 2} \tag{27b}$$

$$\frac{\partial \varphi_i}{\partial J(\underline{X}_{i,k+1})} = -\varphi_i^2 \Delta t \alpha_{b,i} r_i J(\underline{X}_{i,k+1})^{r_i - 1} \tag{27c}$$

The expression of $\frac{\partial M_i}{\partial J(\underline{X}_{i,k+1})}$ is given by:

$$\begin{aligned} \frac{\partial M_i}{\partial J\left(\underline{X}_{i,k+1}\right)} = \frac{\Delta p}{M_i} \Bigg[& C_i \left(\underline{X}_{i,k} : \frac{\partial \underline{n}}{\partial J\left(\underline{X}_{i,k+1}\right)} \right) + \frac{3}{2} \gamma_i \frac{\partial \varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \left(\underline{X}_{i,k} : \underline{Y}_{i,k} \right) \\ & + \Delta p C_i \gamma_i \left\{ \frac{\partial \varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \left(\underline{n} : \underline{Y}_{i,k} \right) + \varphi_i \left(\frac{\partial \underline{n}}{\partial J\left(\underline{X}_{i,k+1}\right)} : \underline{Y}_{i,k} \right) \right\} \\ & + \frac{3}{2} \Delta p \gamma_i^2 \varphi_i \frac{\partial \varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \left(\underline{Y}_{i,k} : \underline{Y}_{i,k} \right) \Bigg] \end{aligned} \tag{28a}$$

$$\frac{\partial \underline{n}}{\partial J\left(\underline{X}_{i,k+1}\right)} = \frac{3}{2J(\underline{Z})^2} \left(J(\underline{Z}) \frac{\partial \underline{Z}}{\partial J\left(\underline{X}_{i,k+1}\right)} - \frac{\partial J(\underline{Z})}{\partial J\left(\underline{X}_{i,k+1}\right)} \underline{Z} \right) \tag{28b}$$

$$\frac{\partial \underline{Z}}{\partial J\left(\underline{X}_{i,k+1}\right)} = -\frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \left(\underline{X}_{i,k} + \Delta p \gamma_i \varphi_i \underline{Y}_{i,k} \right) - \Delta p \omega_i \gamma_i \frac{\partial \varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)} \underline{Y}_{i,k} \tag{28c}$$

$$\frac{\partial J(\underline{Z})}{\partial J\left(\underline{X}_{i,k+1}\right)} = \frac{3}{2J(\underline{Z})}\left(\underline{Z}:\frac{\partial \underline{Z}}{\partial J\left(\underline{X}_{i,k+1}\right)}\right) \tag{28d}$$

3.2.3 Updating the variables

Once the equations for Δp and the n_{AF} $J(\underline{X}_{i,k+1})$ have been solved, the rest of the state variables must be updated.

The plastic strain is updated using Eq. (15a), the stress is updated using Eq. (8a), the back-stresses can be computed using Eq. (16a).

4 CONSISTENT TANGENT MATRIX

The consistent tangent matrix is necessary for the finite-element model, as it is used for the solution of the global equilibrium equations.

The consistent tangent matrix $\underline{\underline{C}}$ is defined as:

$$\underline{\underline{C}} = \frac{d\underline{\sigma}}{d\underline{\varepsilon}} \tag{29}$$

Using Eq. (17), the expression of $\underline{\underline{C}}$ becomes Eq. (30), where $\otimes$ is the tensorial product:

$$\underline{\underline{C}} = \frac{d\underline{\sigma}^{tr}}{d\underline{\varepsilon}} - 2(G + \Delta G)\left[\left(\underline{n} \otimes \frac{d\Delta p}{d\underline{\varepsilon}}\right) + \Delta p\,\frac{d\underline{n}}{d\underline{\varepsilon}}\right] \tag{30}$$

From Eq. (9), $\frac{d\underline{\sigma}^{tr}}{d\underline{\varepsilon}}$ can be computed directly:

$$\frac{d\underline{\sigma}^{tr}}{d\underline{\varepsilon}} = \Delta\underline{\underline{E}} + \underline{\underline{E}} \tag{31}$$

4.1 COMPUTATION OF $\frac{d\Delta p}{d\underline{\varepsilon}}$

The derivative $\frac{d\Delta p}{d\underline{\varepsilon}}$ cannot be computed directly. To compute this derivative, Eq. (20b) and Eq. (22) for the n_{AF} back-stresses are used to form a system of equations of size $n_{AF} + 1$ with a matrix of unknowns $\underline{\underline{x}}_{der}$ that contains the derivative $\frac{d\Delta p}{d\underline{\varepsilon}}$:

$$\underbrace{\begin{bmatrix} \frac{\partial L}{\partial \Delta p} & \frac{\partial L}{\partial J\left(\underline{X}_{1,k+1}\right)} & \cdots & \frac{\partial L}{\partial J\left(\underline{X}_{n_{AF},k+1}\right)} \\ \frac{\partial g_1}{\partial \Delta p} & \frac{\partial g_1}{\partial J\left(\underline{X}_{1,k+1}\right)} & \cdots & \frac{\partial g_1}{\partial J\left(\underline{X}_{n_{AF},k+1}\right)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_{n_{AF}}}{\partial \Delta p} & \frac{\partial g_{n_{AF}}}{\partial J\left(\underline{X}_{1,k+1}\right)} & \cdots & \frac{\partial g_{n_{AF}}}{\partial J\left(\underline{X}_{n_{AF},k+1}\right)} \end{bmatrix}}_{\underline{\underline{A}}_{der}[(n_{AF}+1)\times(n_{AF}+1)]} \underbrace{\begin{bmatrix} \frac{d\Delta p}{d\underline{\varepsilon}} \\ \frac{dJ\left(\underline{X}_{1,k+1}\right)}{d\underline{\varepsilon}} \\ \vdots \\ \frac{dJ\left(\underline{X}_{n_{AF},k+1}\right)}{d\underline{\varepsilon}} \end{bmatrix}}_{\underline{\underline{x}}_{der}[(n_{AF}+1)\times 6]} = \underbrace{\begin{bmatrix} -\frac{\partial L}{\partial \underline{\hat{\sigma}}^{tr}} \frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} \\ -\frac{\partial g_1}{\partial \underline{\hat{\sigma}}^{tr}} \frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} \\ \vdots \\ -\frac{\partial g_{n_{AF}}}{\partial \underline{\hat{\sigma}}^{tr}} \frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} \end{bmatrix}}_{\underline{\underline{b}}_{der}[(n_{AF}+1)\times 6]} \tag{32}$$

The terms on the diagonal of $\underline{\underline{A}}_{der}$ are computed according to equations (24) and (26b). The rest of the terms are given hereafter.

From Eq. (20b), the terms $\frac{\partial L}{\partial J\left(\underline{X}_{i,k+1}\right)}$ are derived as:

$$\frac{\partial L}{\partial J\left(\underline{X}_{i,k+1}\right)} = \Delta p \frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)} C_i - \frac{\partial J\left(\underline{Z}\right)}{\partial J\left(\underline{X}_{i,k+1}\right)} \tag{33}$$

$\frac{\partial \omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)}$ and $\frac{\partial J\left(\underline{Z}\right)}{\partial J\left(\underline{X}_{i,k+1}\right)}$ are given by Eq. (27a) and (28d), respectively.

The terms $\frac{\partial g_i}{\partial J\left(\underline{X}_{j,k+1}\right)}$ (for $i \neq j$) are derived from Eq. (22a) and (22b):

$$\frac{\partial g_i}{\partial J\left(\underline{X}_{j,k+1}\right)} = -\omega_i \frac{\partial M_i}{\partial J\left(\underline{X}_{j,k+1}\right)} \tag{34a}$$

$$\frac{\partial M_i}{\partial J\left(\underline{X}_{j,k+1}\right)} = \frac{\Delta p}{M_i}\left[C_i\left(\underline{X}_{i,k} : \frac{\partial \underline{n}}{\partial J\left(\underline{X}_{j,k+1}\right)}\right) + \Delta p C_i \gamma_i \varphi_i \left(\frac{\partial \underline{n}}{\partial J\left(\underline{X}_{j,k+1}\right)} : \underline{Y}_{i,k}\right)\right] \tag{34b}$$

$\frac{\partial \underline{n}}{\partial J\left(\underline{X}_{j,k+1}\right)}$ can be calculated using equations (28b) to (28d).

Similarly, $\frac{\partial g_i}{\partial \Delta p}$ can be derived:

$$\frac{\partial g_i}{\partial \Delta p} = -\omega_i \frac{\partial M_i}{\partial \Delta p} - \frac{\partial \omega_i}{\partial \Delta p} M_i \tag{35a}$$

$$\begin{aligned} \frac{\partial M_i}{\partial \Delta p} = \frac{1}{M_i}\Bigg[& C_i \Delta p \left(\underline{X}_{i,k} : \frac{\partial \underline{n}}{\partial \Delta p}\right) + C_i\left(\underline{X}_{i,k} : \underline{n}\right) + \frac{3}{2}\gamma_i \varphi_i \left(\underline{X}_{i,k} : \underline{Y}_{i,k}\right) + C_i^2 \Delta p \\ & + 2\Delta p C_i \gamma_i \varphi_i \left(\underline{n} : \underline{Y}_{i,k}\right) + \Delta p^2 C_i \gamma_i \varphi_i \left(\frac{\partial \underline{n}}{\partial \Delta p} : \underline{Y}_{i,k}\right) \\ & + \frac{3}{2}\Delta p \gamma_i^2 \varphi_i^2 \left(\underline{Y}_{i,k} : \underline{Y}_{i,k}\right)\Bigg] \end{aligned} \tag{35b}$$

$$\frac{\partial \underline{n}}{\partial \Delta p} = \frac{3}{2}\left[\frac{1}{J(\underline{Z})}\frac{\partial \underline{Z}}{\partial \Delta p} - \frac{3}{2J(\underline{Z})^2}\left(\underline{Z}:\frac{\partial \underline{Z}}{\partial \Delta p}\right)\underline{Z}\right] \tag{35c}$$

The components of $\underline{\underline{b}}_{der}$ are detailed hereafter.

All these components require the value of $\frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}}$, which is given by Eq. (36):

$$\frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} = \left(\underline{\underline{E}} + \Delta\underline{\underline{E}}\right)^{dev} = (G + \Delta G)\begin{bmatrix} \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

The first component is given by Eq. (38) and requires the calculation of $\frac{\partial L}{\partial \underline{\hat{\sigma}}^{tr}}$ given by Eq. (37):

$$\frac{\partial L}{\partial \underline{\hat{\sigma}}^{tr}} = -\frac{3}{2J(\underline{Z})}\left(\underline{Z}:\frac{\partial \underline{Z}}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}}\right) \tag{37a}$$

$$\frac{\partial \underline{Z}}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}} = \underline{\underline{I}} \tag{37b}$$

$$\frac{\partial L}{\partial \underline{\hat{\sigma}}^{tr}} = -\frac{3\underline{Z}}{2J(\underline{Z})} \tag{37c}$$

$$\frac{\partial L}{\partial \underline{\hat{\sigma}}^{tr}}\frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} = 3(G + \Delta G)\frac{\underline{Z}}{J(\underline{Z})} \tag{38}$$

The other components of $\underline{\underline{b}}_{der}$ are given by Eq. (39a) to (39d)

$$\frac{\partial g_i}{\partial \underline{\hat{\sigma}}^{tr}} = -\frac{\Delta p \omega_i C_i}{M_i}\left[\left(\underline{X}_{i,k}:\frac{\partial \underline{n}}{\partial \underline{\hat{\sigma}}^{tr}}\right) + \Delta p \gamma_i \varphi_i\left(\frac{\partial \underline{n}}{\partial \underline{\hat{\sigma}}^{tr}}:\underline{Y}_{i,k}\right)\right] \tag{39a}$$

$$\frac{\partial \underline{n}}{\partial \underline{\hat{\sigma}}^{tr}} = \frac{3}{2}\frac{1}{J(\underline{Z})^2}\left(J(\underline{Z})\frac{\partial \underline{Z}}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}} - \frac{\partial J(\underline{Z})}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}}\otimes\underline{Z}\right) \tag{39b}$$

$$\frac{\partial J(\underline{Z})}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}} = \frac{3}{2J(\underline{Z})}\underline{Z}:\frac{\partial \underline{Z}}{\partial \underline{\hat{\sigma}}_{k+1}^{tr}} = \frac{3\underline{Z}}{2J(\underline{Z})} \tag{39c}$$

$$\frac{\partial \underline{n}}{\partial \underline{\hat{\sigma}}^{tr}} = \frac{3}{2}\frac{1}{J(\underline{Z})}\left(\underline{\underline{I}} - \frac{3}{2J(\underline{Z})^2}\underline{Z}\otimes\underline{Z}\right) \tag{39d}$$

4.2 COMPUTATION OF $\frac{d\underline{n}}{d\underline{\varepsilon}}$

The derivative of $\frac{d\underline{n}}{d\underline{\varepsilon}}$ must be calculated to compute Eq. (30):

$$\frac{d\underline{n}}{d\underline{\varepsilon}} = \frac{3}{2}\frac{1}{J(\underline{Z})^2}\left(J(\underline{Z})\frac{d\underline{Z}}{d\underline{\varepsilon}} - \underline{Z}\otimes\frac{dJ(\underline{Z})}{d\underline{\varepsilon}}\right) \tag{40}$$

Equations (41a) to (41c) give the expression of $\frac{d\underline{Z}}{d\underline{\varepsilon}}$:

$$\begin{aligned}\frac{d\underline{Z}}{d\underline{\varepsilon}} = \frac{d\underline{\hat{\sigma}}^{tr}}{d\underline{\varepsilon}} - \sum_{i=1}^{n_{AF}}\left(\underline{X}_{i,k}\otimes\frac{d\omega_i}{d\underline{\varepsilon}}\right) - \Delta p\sum_{i=1}^{n_{AFY}}\gamma_i\varphi_i\left(\underline{Y}_{i,k}\otimes\frac{d\omega_i}{d\underline{\varepsilon}}\right)\\ - \Delta p\sum_{i=1}^{n_{AFY}}\gamma_i\omega_i\left(\underline{Y}_{i,k}\otimes\frac{d\varphi_i}{d\underline{\varepsilon}}\right) - \left(\sum_{i=1}^{n_{AFY}}\omega_i\gamma_i\varphi_i\underline{Y}_{i,k}\right)\otimes\frac{d\Delta p}{d\underline{\varepsilon}}\end{aligned} \tag{41a}$$

$$\frac{d\omega_i}{d\underline{\varepsilon}} = \frac{\partial\omega_i}{\partial\Delta p}\frac{d\Delta p}{d\underline{\varepsilon}} + \frac{\partial\omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)}\frac{dJ\left(\underline{X}_{i,k+1}\right)}{d\underline{\varepsilon}} \tag{41b}$$

$$\frac{d\varphi_i}{d\underline{\varepsilon}} = \frac{\partial\omega_i}{\partial J\left(\underline{X}_{i,k+1}\right)}\frac{dJ\left(\underline{X}_{i,k+1}\right)}{d\underline{\varepsilon}} \tag{41c}$$

Equations (42a) to (42c) give the expression of $\frac{dJ(\underline{Z})}{d\underline{\varepsilon}}$:

$$\frac{dJ(\underline{Z})}{d\underline{\varepsilon}} = \frac{\partial J(\underline{Z})}{\partial\Delta p}\frac{d\Delta p}{d\underline{\varepsilon}} + \sum_{i=1}^{n_{AF}}\left(\frac{\partial J(\underline{Z})}{\partial J\left(\underline{X}_{i,k+1}\right)}\frac{dJ\left(\underline{X}_{i,k+1}\right)}{d\underline{\varepsilon}}\right) \tag{42a}$$

$$\begin{aligned}\frac{\partial J(\underline{Z})}{\partial\Delta p} = \frac{K}{n\Delta t}\left(\frac{\Delta p}{\Delta \mathrm{t}}\right)^{\frac{1}{n}-1} + Qbe^{-b(p+\Delta p)} + 3(G+\Delta G) + \sum_{i=1}^{n_{AF}}(\omega_i C_i)\\ - \Delta p\sum_{i=1}^{n_{AF}}\left(\omega_i^2\gamma_i C_i\left(1+\varphi_i\rho_i\right)\right)\end{aligned} \tag{42b}$$

$$\begin{aligned}\frac{\partial J(\underline{Z})}{\partial J\left(\underline{X}_{i,k+1}\right)} = -\Delta p\omega_i^2 C_i\Bigg[\Delta p\gamma_i\left(\varphi_i\frac{\partial\rho_i}{\partial J\left(\underline{X}_{i,k+1}\right)} + \frac{\partial\varphi_i}{\partial J\left(\underline{X}_{i,k+1}\right)}\rho_i\right)\\ + b_i\Delta t(r_i-1)J\left(\underline{X}_{i,k+1}\right)^{r_i-2}\Bigg]\end{aligned} \tag{42c}$$

Appendix 2: Equations for the damage model

1 DAMAGE MODEL

Damage is modelled by two equations: Eq. (43) for fatigue damage and Eq. (44) for creep damage. To simplify the notation, variables at the end of the time step are written with no index (e.g., $p = p_{k+1}$).

$$F_f = D_f - D_{f,k} - \left(\frac{Y(\underline{\sigma}, D)}{S_f}\right)^{s_f} (p - p_k) = 0 \tag{43}$$

$$F_c = D_c - D_{c,k} - \left(\frac{Y(\underline{\sigma}, D)}{S_c}\right)^{s_c} \frac{1}{(1-D)^{k_c}} = 0 \tag{44}$$

The expression of $Y(\underline{\sigma}, D)$ depends on the value on the parameter h. If $h = 1$, the effect of microdefects closure is not taken into account, and Y is computed according to Eq. (45). If, however, $h < 1$, Y is computed using Eq. (46) .

$$h = 1 \Rightarrow Y = \frac{1+\nu}{2E} \frac{\underline{\sigma} : \underline{\sigma}}{(1-D)^2} - \frac{\nu}{2E} \left(\frac{tr(\sigma)}{1-D}\right)^2 \tag{45}$$

$$h < 1 \Rightarrow Y = \frac{1+\nu}{2E} \left[\frac{\langle\sigma\rangle_{ij}^{+} \langle\sigma\rangle_{ij}^{+}}{(1-D)^2} + h \frac{\langle\sigma\rangle_{ij}^{-} \langle\sigma\rangle_{ij}^{-}}{(1-hD)^2}\right] - \frac{\nu}{2E} \left[\frac{\langle\sigma_{kk}\rangle^2}{(1-D)^2} + h \frac{\langle -\sigma_{kk}\rangle^2}{(1-hD)^2}\right] \tag{46}$$

2 SOLUTION TO THE DAMAGE MODEL

The solution of Eq. (43) and (44) with the Newton-Raphson method requires computing the derivative of F_f with respect to D_f and of F_c with respect to D_c:

$$\frac{\partial F_f}{\partial D_f} = 1 - \frac{s_f}{S_f} \left(\frac{Y}{S_f}\right)^{s_f - 1} \frac{\partial Y}{\partial D_f} (p - p_k) \tag{47}$$

$$\frac{\partial F_c}{\partial D_c} = 1 - \frac{s_c}{S_c} \left(\frac{Y}{S_c}\right)^{s_c - 1} \frac{\partial Y}{\partial D_c} \frac{1}{(1-D)^{k_c}} - \frac{k_c}{(1-D)^{k_c+1}} \left(\frac{Y}{S_c}\right)^{s_c} \tag{48}$$

Considering Y is a function of the total damage D and $D = D_f + D_c$, the derivative of Y with respect to D_f or D_c is the same as $\frac{\partial Y}{\partial D}$, given by Eq. (49) in the case with no microdefects closure and by Eq. (50) for the case with microdefects closure.

$$\frac{\partial Y}{\partial D} = \frac{1+\nu}{E}\frac{\underline{\sigma}:\underline{\sigma}}{(1-D)^3} - \frac{\nu}{E}\frac{\left(tr(\sigma)\right)^2}{(1-D)^3} \tag{49}$$

$$\frac{\partial Y}{\partial D} = \frac{1+\nu}{E}\left[\frac{\langle\sigma\rangle_{ij}^{+}\langle\sigma\rangle_{ij}^{+}}{(1-D)^3} + h^2\frac{\langle\sigma\rangle_{ij}^{-}\langle\sigma\rangle_{ij}^{-}}{(1-hD)^3}\right] - \frac{\nu}{E}\left[\frac{\langle\sigma_{kk}\rangle^2}{(1-D)^3} + h^2\frac{\langle-\sigma_{kk}\rangle^2}{(1-hD)^3}\right] \tag{50}$$

Appendix 3: Jacobian matrix for the coupled model

The coupled model consists of the following system of equations:

$$\begin{cases} \underline{\mathcal{R}}_{\underline{\varepsilon}^e} = \underline{\dot{\varepsilon}} - \underline{\dot{\varepsilon}}^{th} - \underline{\dot{\varepsilon}}^e - \dot{r}\underline{n} = 0 \\ \mathcal{R}_r = J\left(\frac{\underline{\sigma}}{1-D} - \underline{X}\right) - R - \sigma_y - \sigma_v = 0 \\ \underline{\mathcal{R}}_{\underline{X}_i} = \underline{\dot{X}}_i - \frac{2}{3}C_i(1-D)\underline{\dot{\varepsilon}}^p + \gamma_i\left(\underline{X}_i - \underline{Y}_i\right)\dot{r} + b_i J\left(\underline{X}_i\right)^{r_i-1}\underline{X}_i - \frac{1}{C_i}\frac{dC_i}{dT}\dot{T}\underline{X}_i = 0 \; (i = 1{:}\, n_{AF}) \\ \underline{\mathcal{R}}_{\sigma} = \underline{\varepsilon}^{\,e} - \underline{\underline{E}}^{-1}{:}\, \underline{\tilde{\sigma}} = 0 \end{cases}$$

The Jacobian matrix necessarily includes the derivatives of each of the local residual with respect to the variables $\Delta\underline{\varepsilon}^e$, Δr, $\Delta\underline{X}_i$, and $\Delta\underline{\sigma}$:

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial\underline{\mathcal{R}}_{\underline{\varepsilon}^e}}{\partial\Delta\underline{\varepsilon}^e} & \frac{\partial\underline{\mathcal{R}}_{\underline{\varepsilon}^e}}{\partial\Delta r} & \frac{\partial\underline{\mathcal{R}}_{\underline{\varepsilon}^e}}{\partial\Delta\underline{X}_i} & \frac{\partial\underline{\mathcal{R}}_{\underline{\varepsilon}^e}}{\partial\Delta\underline{\sigma}} \\ \frac{\partial\mathcal{R}_r}{\partial\Delta\underline{\varepsilon}^e} & \frac{\partial\mathcal{R}_r}{\partial\Delta r} & \frac{\partial\mathcal{R}_r}{\partial\Delta\underline{X}_i} & \frac{\partial\mathcal{R}_r}{\partial\Delta\underline{\sigma}} \\ \frac{\partial\underline{\mathcal{R}}_{\underline{X}_i}}{\partial\Delta\underline{\varepsilon}^e} & \frac{\partial\underline{\mathcal{R}}_{\underline{X}_i}}{\partial\Delta r} & \frac{\partial\underline{\mathcal{R}}_{\underline{X}_i}}{\partial\Delta\underline{X}_i} & \frac{\partial\underline{\mathcal{R}}_{\underline{X}_i}}{\partial\Delta\underline{\sigma}} \\ \frac{\partial\underline{\mathcal{R}}_{\sigma}}{\partial\Delta\underline{\varepsilon}^e} & \frac{\partial\underline{\mathcal{R}}_{\sigma}}{\partial\Delta r} & \frac{\partial\underline{\mathcal{R}}_{\sigma}}{\partial\Delta\underline{X}_i} & \frac{\partial\underline{\mathcal{R}}_{\sigma}}{\partial\Delta\underline{\sigma}} \end{bmatrix}$$

First, some useful variables are defined:

$$\underline{s} = \frac{\underline{\hat{\sigma}}}{1-D}$$

Where $\underline{\hat{\sigma}}$ is the deviatoric stress.

$$\underline{n}^X = (1-D)\underline{n} = \frac{3}{2}\frac{\underline{s} - \underline{X}}{J\left(\underline{s} - \underline{X}\right)}$$

The derivatives of $\underline{n}^X$ with respect to $\underline{s}$ and $\underline{X}$ are used in the analytical expression of the Jacobian matrix:

$$\frac{\partial\underline{n}^X}{\partial\underline{s}} = -\frac{\partial\underline{n}^X}{\partial\underline{X}} = \frac{1}{J\left(\underline{s} - \underline{X}\right)}\left[\frac{3}{2}\underline{\underline{I}} - \frac{1}{2}\underline{1}\otimes\underline{1} - \underline{n}^X\otimes\underline{n}^X\right]$$

Where $\underline{\underline{I}}$ is the 6-by-6 identity matrix, $\underline{1}$ is the identity tensor in the Voigt notation, and $\otimes$ is the tensorial product:

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underline{1} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \otimes \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{Bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & a_1b_5 & a_1b_6 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & a_2b_5 & a_2b_6 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & a_3b_5 & a_3b_6 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & a_4b_5 & a_4b_6 \\ a_5b_1 & a_5b_2 & a_5b_3 & a_5b_4 & a_5b_5 & a_5b_6 \\ a_6b_1 & a_6b_2 & a_6b_3 & a_6b_4 & a_6b_5 & a_6b_6 \end{bmatrix}$$

Derivatives of $\underline{\mathcal{R}}_{\varepsilon^e}$

$$\frac{\partial \underline{\mathcal{R}}_{\varepsilon^e}}{\partial \Delta \underline{\varepsilon}^e} = \underline{\underline{I}}$$

$$\frac{\partial \underline{\mathcal{R}}_{\varepsilon^e}}{\partial \Delta r} = \underline{n}$$

$$\frac{\partial \underline{\mathcal{R}}_{\varepsilon^e}}{\partial \Delta \underline{X}_i} = \Delta r \frac{\partial \underline{n}}{\partial \underline{X}_i} = \frac{\Delta r}{1-D} \frac{\partial \underline{n}^X}{\partial \underline{X}}$$

$$\frac{\partial \underline{\mathcal{R}}_{\varepsilon^e}}{\partial \Delta \underline{\sigma}} = \frac{\Delta r}{1-D} \frac{\partial \underline{n}^X}{\partial \underline{\sigma}} = \frac{\Delta r}{(1-D)^2} \frac{\partial \underline{n}^X}{\partial \underline{s}}$$

Derivatives of $\mathcal{R}_r$

Plastic case: $\mathcal{R}_r = f = J(\underline{s} - \underline{X}) - R(r) - \sigma_y$

$$\frac{\partial \mathcal{R}_r}{\partial \Delta \underline{\varepsilon}^e} = \{0\ 0\ 0\ 0\ 0\ 0\}$$

$$\frac{\partial \mathcal{R}_r}{\partial \Delta r} = -\frac{\partial R(r)}{\partial r} = -bQe^{-br}$$

$$\frac{\partial \mathcal{R}_r}{\partial \Delta \underline{X}_i} = \frac{\partial J(\underline{s} - \underline{X})}{\partial \Delta \underline{X}_i} = -\underline{n}^X$$

$$\frac{\partial \mathcal{R}_r}{\partial \Delta \underline{\sigma}} = \frac{\partial J(\underline{s} - \underline{X})}{\partial \underline{s}} : \frac{\partial \underline{s}}{\partial \underline{\sigma}} = \frac{1}{1-D} \underline{n}^X = \underline{n}$$

Viscoplastic case: $\mathcal{R}_r^v = \Delta r - (1-D)\Delta t \left\langle \frac{f}{K} \right\rangle^n$

$$\frac{\partial \mathcal{R}_r^v}{\partial \Delta \underline{\varepsilon}^e} = \{0\ 0\ 0\ 0\ 0\ 0\}$$

$$\frac{\partial \mathcal{R}_r^v}{\partial \Delta r} = 1 + \frac{(1-D)\Delta t n \langle f \rangle^{n-1}}{K^n} \frac{\partial \mathcal{R}_r}{\partial \Delta r}$$

$$\frac{\partial \mathcal{R}_r^v}{\partial \Delta \underline{X}_i} = \frac{(1-\mathrm{D})\Delta t n \langle f \rangle^{n-1}}{K^n} \underline{n}^X$$

$$\frac{\partial \mathcal{R}_r^v}{\partial \Delta \underline{\sigma}} = -\frac{\Delta t n \langle f \rangle^{n-1}}{K^n} \underline{n}^X$$

Derivatives of $\underline{\mathcal{R}}_{\underline{X}_i}$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{X}_i}}{\partial \Delta \underline{\varepsilon}^e} = \underline{\underline{0}}$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{X}_i}}{\partial \Delta r} = -\frac{2}{3} C_i \underline{n}^X + \gamma_i (\underline{X}_i - \underline{Y}_i)$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{X}_i}}{\partial \Delta \underline{X}_i} = \left[1 - \frac{C_i \Delta r}{J(s-X)} + \gamma_i \Delta r + b_i \Delta t J\left(\underline{X}_i\right)^{r_i - 1} - \frac{1}{C_i}\frac{\partial C_i}{\partial T}\Delta T\right] \underline{\underline{I}} + \frac{2}{3}\frac{C_i \Delta r}{J(s-X)} \underline{n}^X \otimes \underline{n}^X - \gamma_i \Delta r \frac{\partial \underline{Y}_i}{\partial \underline{X}_i} + \frac{3}{2} b_i \Delta t (r_i - 1) J\left(\underline{X}_i\right)^{r_i - 3} \underline{X}_i \otimes \underline{X}_i$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{X}_i}}{\partial \Delta \underline{\sigma}} = -\frac{2}{3}\frac{C_i \Delta r}{1-D}\frac{\partial \underline{n}^X}{\partial \underline{s}}$$

Derivatives of $\underline{\mathcal{R}}_{\underline{\sigma}}$

Case without microdefects closure:

$$\frac{\partial \underline{\mathcal{R}}_{\underline{\sigma}}}{\partial \Delta \underline{\varepsilon}^e} = \underline{\underline{I}}$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{\sigma}}}{\partial \Delta r} = \underline{0}$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{\sigma}}}{\partial \Delta \underline{X}_i} = \underline{\underline{0}}$$

$$\frac{\partial \underline{\mathcal{R}}_{\underline{\sigma}}}{\partial \Delta \underline{\sigma}} = \frac{-\underline{\underline{E}}^{-1}}{1-D}$$

Case with microdefects closure:

$\frac{\partial \underline{\mathcal{R}}_\sigma}{\partial \Delta \underline{\varepsilon}^e}$, $\frac{\partial \underline{\mathcal{R}}_\sigma}{\partial \Delta r}$, and $\frac{\partial \underline{\mathcal{R}}_\sigma}{\partial \Delta \underline{X}_i}$ are the same as for the case without microdefects closure. Due to the partition of the stress in its negative and positive parts, there is no exact analytical expression of $\frac{\partial \underline{\mathcal{R}}_\sigma}{\partial \Delta \underline{\sigma}}$. The following expression is therefore an approximation:

$$\frac{\partial \underline{\mathcal{R}}_\sigma}{\partial \Delta \underline{\sigma}} = -\frac{1+\nu}{E}\left[\frac{\underline{\mathcal{H}}_\sigma^+}{1-D} + \frac{\underline{\mathcal{H}}_\sigma^-}{1-hD}\right] + \frac{\nu}{E}\left[\frac{\mathcal{H}\left(tr\underline{\sigma}\right)}{1-D} + \frac{\mathcal{H}\left(-tr\underline{\sigma}\right)}{1-hD}\right]\underline{1}\otimes\underline{1}$$

Where $\mathcal{H}(x)$ is the Heaviside step function.