CAN'T GET THERE FROM HERE? CURIOUS LOGIC IN THE FAMOUS PAPER BY EINSTEIN AND DE SITTER

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The Einstein–de Sitter cosmological model, a spatially flat Friedmann–Robertson–Walker (FRW) cosmological model containing ordinary matter ('dust') but no radiation or cosmological constant, has been studied extensively, both because of its mathematical simplicity and, partially related to that, because it was often used as a fiducial model. It is thus interesting that Einstein and de Sitter performed their main calculation, that of the density based on the Hubble constant, in a very roundabout and confusing way. After providing some historical background, I describe their calculation and explain why their method is technically correct even though it is misleading.

Introduction

In one of his lectures at the 1993 Saas-Fee course 'The Deep Universe', Allan Sandage[1] recommended that those attending* read the cosmological literature of the 1920s and 1930s. While unfamiliar notation (not to mention unfamiliar languages) might put off some modern readers, it is a rewarding exercise, and it is possible to read it all in a reasonable time (until about 1940, Feynman read even the entire *Physical Review*). The history of the field is perhaps more important in cosmology than elsewhere, so that in itself is a reason to study the literature, but sometimes curiosities are also found; one of those is the topic of this paper.

The Einstein–de Sitter cosmological model

The Einstein–de Sitter model[2] is a mathematically simple cosmological model; the scale factor R is proportional to time as $t^{2/3}$. The age of the universe is 2/3 the Hubble time ($R/\dot{R}$, the inverse of the Hubble constant). The density in kg per m^3 is given by $1/(6\pi G t^2)$ where G is the gravitational constant and t is the age of the universe in seconds. As such, it has often been used as a fiducial model for back-of-the-envelope calculations, and for a time it was the 'standard model' of the Universe, due to several factors: no curvature had been detected

*The majority were much younger than he was; Sandage pointed out that when he had started out, computers were female and worked in the cellar.

(and still hasn't), dislike of the cosmological constant by many, belief that inflation produces a Universe very close to flat, cosmological parameters then not ruled out by observations, and so on. No doubt its mathematical simplicity also contributed to its popularity, both for esthetic and practical reasons. See ref. 3 for more on the historical importance of the Einstein–de Sitter model.

One can take Hubble's value[4] of 500 km/s/Mpc as a typical estimate of the Hubble constant at the time, which in the Einstein–de Sitter model corresponds to an age of about 1.3 billion years and a density of $\approx 4.7 \times 10^{-25}$ kg/m^3 ($\approx 4.7 \times 10^{-28}$ g/cm^3).

The short paper by Einstein and de Sitter

Despite some claims to the contrary (*e.g.,* ref. 5) the goal of their two-page paper[2] is straightforward[3]: since there was no observational determination of the value or even the sign of the curvature,* and since the cosmological constant is not necessary in an expanding universe, one can construct a simple model which nevertheless is compatible with the observational data. Now known as the Einstein–de Sitter model, such an FRW universe is spatially flat, has no cosmological constant, and otherwise contains only homogeneously distributed matter ('dust'), *i.e.,* no pressure or radiation. At least with regard to the lack of curvature, the simplicity was clearly meant to be provisional, as the last sentence indicates: "The curvature is, however, essentially determinable, and an increase in the precision of the data derived from observations will enable us in the future to fix its sign and to determine its value." It is thus in line with Einstein's philosophy to make things as simple as possible but not simpler† Interestingly, Einstein and de Sitter each did not think that the paper was very important, but each believed that the other did.[7,8]

Historical background

Although Friedmann[9,10] had explored the full range of FRW models, discussion often concentrated on fiducial models, no doubt due at least in part to the difficulty of calculations in the general case and because observations were not good enough to rule out all simple models. Early examples were the static Einstein universe[11] and the de Sitter universe[12,13], dubbed 'solution A' and 'solution B' by de Sitter[14]. The discovery of the expansion of the Universe ruled out the former‡, while the latter, containing no matter, was clearly at best an approximation. Lemaître[17] presented a model which fitted all the data (and still does, with respect to its general characteristics is not the values of the parameters), his famous *Univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques*§,

*That is still true today.

†An aphorism along those lines is often attributed to Einstein, but, though it is probable that he said something similar, and is certainly a valid characterization of his thought, there is no direct written evidence for such a phrase.[6]

‡Einstein himself seems to have been more convinced by the instability argument (*e.g.,* ref. 15) than by observations when he gave up his static model.[16]

§A translation of the paper was published as 'A Homogeneous Universe of Constant Mass and Increasing Radius accounting for the Radial Velocity of Extra-galactic Nebulæ'[18] which, though with positive curvature and a positive cosmological constant, calculations were not straightforward. After he had dropped the cosmological constant because he deemed it unnecessary in an expanding universe, Einstein preferred positive-curvature (and hence spatially closed and thus finite) cosmological models (*e.g.,* ref. 20). Thus, except for de Sitter's obviously incorrect model, preference had been for models with positive curvature. The Einstein–de Sitter model departs from that, and also drops the cosmological constant, an unusual move at the time: Eddington (*e.g.,* ref. 21) and Lemaître (*e.g.,* refs. 17,18,22–26) took the cosmological constant essentially as a given; Eddington[27] even wrote "I would as soon think of reverting to Newtonian theory as of dropping the cosmical constant."

Their calculation

I quote (apart from their footnotes) the portion of their paper containing equations in full:

> It we suppose the curvature to be zero, the line-element is
>
> $$ds^2 = -R^2(dx^2 + dy^2 + dz^2) + c^2dt^2, \tag{1}$$
>
> where R is a function of t only, and c is the velocity of light. If, for the sake of simplicity, we neglect the pressure p, the field equations without λ lead to two differential equations, of which we need only one, which in the case of zero curvature reduces to:
>
> $$\frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 = \frac{1}{3}\kappa\rho. \tag{2}$$
>
> The observations give the coefficient of expansion and the mean density:
>
> $$\frac{1}{R}\frac{dr}{cdt} = h = \frac{1}{R_B}; \quad \rho = \frac{2}{\kappa R_A^2}. \tag*{[2′]}$$
>
> Therefore we have, from (2), the theoretical relation
>
> $$h^2 = \frac{1}{3}\kappa\rho \tag{3}$$
>
> or
>
> $$\frac{R_A^2}{R_B^2} = \frac{2}{3}. \tag*{(3')}$$
>
> Taking for the coefficient of expansion
>
> $$h = 500\text{km./sec. per } 10^6 \text{ parsecs}, \tag{4}$$
>
> or
>
> $$R_B = 2 \times 10^{27}\text{cm.},$$

interestingly, left out the calculation of what is now known as the Hubble constant[19].

> we find
>
> $$R_A = 1.63 \times 10^{27}\text{cm.},$$
>
> or
>
> $$\rho = 4 \times 10^{-28} \text{ gr. cm.}^{-3}, \tag{5}$$
>
> which happens to coincide exactly with the upper limit for the density adopted by one of us.

It doesn't actually "coincide exactly": for a density of 4×10^{-28} g/cm^3 the Hubble constant in the Einstein–de Sitter universe is ≈ 465 (and the age about 1.4 billion years). However, considering the low accuracy and precision of the three quantities — the average density of the Universe, its age, and the Hubble constant — , it is a strong indication that our Universe is at least approximately described by the Einstein–de Sitter model.

Apparently they assumed that their readers would be familiar with their notation: R is the scale factor (usually taken to be the radius of curvature of the universe or, in the flat-space case, c/H_0), $\kappa = (8\pi G\rho)/(c^2)$, R_B and R_A refer to the solutions A and B mentioned above, h is the Hubble constant (usually written H today; as with other parameters, a subscript 0 refers to the value at present), and ρ the matter density. Referring to the unnumbered equation after equation (2) as (2'), it is clear from equations (2), [2'], and (3) that h has the dimension of inverse length (*i.e.,* the Hubble constant divided by the speed of light) while in equation (4) it has the usual unit of inverse time.

As stated in the title, their point is "the relation between the expansion and the mean density of the universe". That could have been calculated directly from equation (3), making their point that those two observational quantities result in a valid equation, thus providing some observational evidence that their model might be at least a rough approximation. But that is not what they did.

Rather, they introduced $h = 1/R_B$, which is essentially (the reciprocal of) the Hubble length, often adopted as the scale factor for spatially flat universes, where the infinite radius of curvature would not be useful. However, 'B' suggests 'solution B', the de Sitter universe, in which the Hubble length is not only constant in time, but is also the distance to the event horizon (in general, neither is the case). Even more strange is the introduction of R_A, the radius of curvature of the static Einstein universe, which is very different from the Einstein–de Sitter universe; the former is static, has positive spatial curvature and thus finite, and a positive cosmological constant, while the latter is expanding, spatially flat and thus infinite, and has no cosmological constant.

The relation $\rho = 2/(\kappa R_A^2)$ holds for the static Einstein universe, but doesn't hold in general. (Expressed differently, in the Einstein universe, $R^{-2} = \lambda = (4\pi G\rho)/c^2$. Note that here, λ has the dimension of inverse squared length; one also sees inverse squared time, but the former follows the notation in the Einstein & de Sitter[2] paper. Also, their λ is usually written Λ today, while $\lambda = \Lambda/(3H^2)$.) There seems to be no reason to introduce R_A, as the quantity has no special meaning in general. Using the definitions above, essentially $R_A^2 =$

$2/(\kappa\rho)$ and $R_B^2 = 3/(\kappa\rho)$, they trivially calculated the quantity $R_A^2/R_B^2 = 2/3$.* The measured value of h (equation (4)) is then used to find R_B, then the ratio used to find R_A, then the definition of R_A is used to find ρ (which could have been calculated directly from equation (3)).

Note that the actual definition of R_A cancels out. They could just as easily have used $R_A^2 = 24324.37793/(\kappa\rho)$, performed an analogous calculation, and achieved the same result. Again, they could have calculated the density ρ directly from equation (3). Why they didn't, I don't know.

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*Note that in the de Sitter universe, $R_B^2 = 3c^2/\lambda$; in the static Einstein universe, $R_A^2 = 2c^2/\lambda$. Thus, the ratio between those two is 2/3 as well.

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