On Cartan's A & B theorems

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Abstract

In this paper, we show that sheaf theory may be used to give a short proof of Cartan's A & B theorems for a Stein manifold without dealing with any particular resolution of the sheaf of holomorphic functions.

1 Introduction

Cartan's A & B theorems have a too long history to be detailed here. Let us only recall that they have been studied from two essentially different points of view.

On one hand, the original approach used by H. Cartan in the 1951–52 ENS seminar is based on a reduction of the problem to the case of a closed interval in $\mathbb{C}^n$ using both cohomological and approximation methods. This local case is then solved by using explicit calculations with the Alexander-Spanier resolution.

On the other hand, L. Hörmander shows first that the global Dolbeault complex associated to a holomorphic vector bundle on a Stein manifold is exact. To get this result, he uses ingenious L^2 methods after endowing the base manifold with a suitable but non canonical hermitian metric.

In this paper, we follow H. Cartan's approach and show how it can be simplified by making use of modern sheaf theoretic methods. In particular, we avoid the use of any specific resolution of the sheaf $\mathcal{O}$ to solve the local case (cf. Propositions **??** and **??**) and get the global result for a Stein manifold by a direct application of a general sheaf theoretic Mittag-Leffler lemma given in the appendix.

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2 General Facts

Let us recall that a coherent analytic sheaf on a closed subset F of a complex analytic manifold X is the restriction to F of a coherent $\mathcal{O}$-module defined on an open neighbourhood of F.

For such an F, Cartan's theorems are the following assertions:

Theorem A for F. *Every coherent analytic sheaf $\mathcal{F}$ on F is generated on F by its global sections.*

Theorem B for F. *Every coherent analytic sheaf $\mathcal{F}$ on F is acyclic (i.e. $H^k(F,\mathcal{F}) = 0$ if $k > 0$).*

Remark that if F is compact, theorem A means in fact that for every coherent analytic sheaf $\mathcal{F}$ on F, there is a natural number p and an epimorphism

$$\mathcal{O}^p_{X|F} \twoheadrightarrow \mathcal{F}.$$

Remark also that if theorem B is true for a closed subset F, then the long exact sequence of cohomology shows that every epimorphism

$$\mathcal{F} \twoheadrightarrow \mathcal{G}$$

between coherent analytic sheaves on F gives an epimoprhism

$$\mathcal{F}(F) \twoheadrightarrow \mathcal{G}(F)$$

at the level of global sections since the kernel of a morphism of coherent analytic sheaves is coherent.

Cartan's theorems are not independent, in fact

Proposition 2.1 *On every closed subset F of a complex analytic manifold X, theorem A is a consequence of theorem B.*

Proof. Let $x \in F$ and let $\mathcal{F}$ be a coherent analytic sheaf on F. Denote by $\mathcal{I}_x$ the sheaf of holomorphic functions vanishing at x. First, since $\mathcal{I}_x$ is a coherent ideal of $\mathcal{O}_X$, one gets the epimorphism

$$\mathcal{F} \twoheadrightarrow \mathcal{F}/\mathcal{I}_x\mathcal{F},$$

where the sheaf $\mathcal{F}/\mathcal{I}_x\mathcal{F}$ is coherent and vanishes outside $\{x\}$. Then, by taking global sections, theorem B gives us the epimorphism

$$\mathcal{F}(F) \twoheadrightarrow \mathcal{F}_x/\mathcal{I}_{x,x}\mathcal{F}_x.$$

Now, $\mathcal{F}_x$ is clearly a finite type $\mathcal{O}_{X,x}$-module. Thus $\mathcal{F}_x/\mathcal{I}_{x,x}\mathcal{F}_x$ is a finite dimensional $\mathbb{C}$-vector space and is generated by a finite number of sections $s^1, \ldots, s^q \in \mathcal{F}(F)$. Since $\mathcal{O}_{X,x}$ is a local ring, Nakayama's lemma shows that $\mathcal{F}_x$ is generated by $s^1_x, \ldots, s^q_x$. Thus, theorem A is true for F.$\square$

Our aim in this paper is to prove theorem A and B for a *Stein manifold*, i.e. a complex analytic manifold X, countable at infinity and such that:

S1 every point $x \in X$ has a local coordinate system consisting of holomorphic functions on X,

S2 the points of X are separated by holomorphic functions on X

$$\forall x_1 \neq x_2 \in X \quad \exists f \in \mathcal{O}_X(X) \quad (f(x_1) \neq f(x_2)),$$

S3 the holomorphic hull

$$\hat{K} = \{z \in X : f \in \mathcal{O}_X(X) \Rightarrow |f(z)| \leq \sup_{u \in K} |f(u)|\}$$

of a compact subset K of X is compact.

Moreover, we shall also prove Cartan's theorems for every *holomorphically convex* compact subset K of X, i.e. for every compact subset K such that $\hat{K} = K$.

Let us first deal with a simple case.

3 The case of a block

Definition 3.1 Let $n \in \mathbb{N}_0$. A *block* of $\mathbb{C}^n$ is a set of the form

$$[a_1, b_1] \times \cdots \times [a_{2n}, b_{2n}]$$

where the a_j, b_j are real numbers such that $a_j \leq b_j$. The *dimension* of such a block is the number of indices j such that $a_j < b_j$.

Let us fix a block P of $\mathbb{C}^n$, an index $j \in \{1, \ldots, 2n\}$ and a real number $r \in [a_j, b_j]$ and let us define the blocks P_-, P_0, P_+ by setting

$$\begin{aligned} P_- &= P \cap \{x \in \mathbb{R}^{2n} : x_j \leq r\}, \\ P_+ &= P \cap \{x \in \mathbb{R}^{2n} : x_j \geq r\}, \\ P_0 &= P \cap \{x \in \mathbb{R}^{2n} : x_j = r\}. \end{aligned}$$

Then, it is easy to deduce from the Cauchy representation formula that:

Lemma 3.2 (Cousin) *For any* $f_0 \in \mathcal{O}_{\mathbb{C}^n}(P_0)$*, there are* $f_+ \in \mathcal{O}_{\mathbb{C}^n}(P_+)$ *and* $f_- \in \mathcal{O}_{\mathbb{C}^n}(P_-)$ *such that*

$$f_0 = f_{+|P_0} - f_{-|P_0}$$

One has also the following classical result (cf. [**?**]):

Lemma 3.3 (Cartan) *For every non singular holomorphic matrix* A_0 *on* P_0*, there are non singular holomorphic matrices* A_+, A_- *on* P_+, P_- *such that*

$$A_0 = A_{+|P_0} A^{-1}_{-|P_0}.$$

From this result, one can deduce the

Lemma 3.4 (Glueing of generators) *Let* $\mathcal{F}$ *denote a coherent analytic sheaf on the block* P*. If* g_+ *(resp.* g_-*) is a system of generators of* $\mathcal{F}$ *on* P_+ *(resp.* P_-*) such that* $g_{+|P_0}$ *(resp.* $g_{-|P_0}$*) generates* $\mathcal{F}(P_0)$*, then there is a system of generators* g *of* $\mathcal{F}$ *on* P *and holomorphic matrices* A_+ *(resp.* A_-*) on* P_+ *(resp.* P_-*) such that*

$$g_{|P_+} = A_+ g_+ \qquad g_{|P_-} = A_- g_-.$$

Proof. The hypothesis gives us holomorphic matrices α, β on P_0 such that

$$g_{+|P_0} = \alpha g_{-|P_0} \quad g_{-|P_0} = \beta g_{+|P_0}.$$

Let us denote by M the block matrix defined by

$$M = \begin{pmatrix} I & \beta \\ -\alpha & I - \alpha\beta \end{pmatrix}.$$

It is clear that M is holomorphic on P_0 and that

$$M \begin{pmatrix} 0 \\ g_{+|P_0} \end{pmatrix} = \begin{pmatrix} g_{-|P_0} \\ 0 \end{pmatrix}.$$

Since we also have

$$M = \begin{pmatrix} I & 0 \\ -\alpha & I \end{pmatrix} \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix}$$

the matrix M is non singular and Cartan's lemma shows that

$$M = M^{-1}_{-|P_0} M_{+|P_0}$$

where M_+ (resp. M_-) is a non singular holomorphic matrix on P_+ (resp. P_-). Thus, one has

$$\left[M_+\begin{pmatrix}0\\g_+\end{pmatrix}\right]_{|P_0}=\left[M_-\begin{pmatrix}g_-\\0\end{pmatrix}\right]_{|P_0}$$

and there is a system of sections g of $\mathcal{F}$ on P such that

$$g_{|P_+}=M_+\begin{pmatrix}0\\g_+\end{pmatrix}\quad g_{|P_-}=M_-\begin{pmatrix}g_-\\0\end{pmatrix}.$$

Since the matrix M_+ (resp. M_-) is non singular on P_+ (resp. P_-), it is clear that g generates $\mathcal{F}$ on P; the conclusion follows. □

Now, let us prove theorem B for the sheaf $\mathcal{O}_{\mathbb{C}^n}$.

Proposition 3.5 *If Pis a block of* $\mathbb{C}^n$*, then one has*

$$H^k(P,\mathcal{O}_{\mathbb{C}^n})=0$$

for every $k>0$*.*

Proof. We shall proceed by recurrence on the dimension m of the block P. If $m=0$, the result is clear since in this case P is a point. So, we can assume that $m>0$ and that the result is true if $\dim P<m$ and we have to prove that it holds for $\dim P=m$.

If it were not the case, there would be P of dimension m, $k>0$ and $c\in H^k(P,\mathcal{O}_{\mathbb{C}^n})$ such that $c\neq 0$. Consider the set

$$E=\{J\subset P: J \text{ block of } \mathbb{C}^n, c_{|J}\neq 0\}$$

ordered by inclusion. Every totally ordered subset $\mathcal{J}$ of E is bounded below. As a matter of fact,the compactness of P shows that $\cap\mathcal{J}$ is a non empty block of $\mathbb{C}^n$ and that

$$H^k(\cap\mathcal{J},\ \mathcal{O}_{\mathbb{C}^n})=\varinjlim_{J\in\mathcal{J}}H^k(J,\mathcal{O}_{\mathbb{C}^n}).$$

Thus, $c_{|\cap\mathcal{J}}\neq 0$ and $\cap\mathcal{J}$ is a lower bound of $\mathcal{J}$ in E. Applying Zorn's lemma, we get a minimal member J of E. Since $J\in E$,

$$H^k(J,\mathcal{O}_{\mathbb{C}^n})\neq 0$$

and the recurrence hypothesis shows that $\dim J = m$. Thus, we can find an index j and a real number r such that the blocks

$$J_+ = J \cap \{x \in \mathbb{R}^{2n} : x_j \geq r\}$$
$$J_- = J \cap \{x \in \mathbb{R}^{2n} : x_j \leq r\}$$

have a dimension equal to m. Set $J_0 = J_+ \cap J_-$.

By the Mayer-Vietoris theorem we have the following exact sequence

$$\begin{tikzcd}
H^{k-1}(J, \mathcal{O}_{\mathbb{C}^n}) \arrow[r, "r_{k-1}"] & H^{k-1}(J_+, \mathcal{O}_{\mathbb{C}^n}) \oplus H^{k-1}(J_-, \mathcal{O}_{\mathbb{C}^n}) \arrow[r, "\delta_{k-1}"] & H^{k-1}(J_0, \mathcal{O}_{\mathbb{C}^n}) \arrow[dll, "s"] \\
H^{k}(J, \mathcal{O}_{\mathbb{C}^n}) \arrow[r, "r_{k}"] & H^{k}(J_+, \mathcal{O}_{\mathbb{C}^n}) \oplus H^{k}(J_-, \mathcal{O}_{\mathbb{C}^n}) \arrow[r, "\delta_{k}"] & H^{k}(J_0, \mathcal{O}_{\mathbb{C}^n})
\end{tikzcd}$$

Since J_+ and J_- are proper subsets of J, we know that $c_{|J_+} = 0$ and that $c_{|J_-} = 0$.

If $k > 1$, the recurrence hypothesis shows that

$$H^{k-1}(J_0, \mathcal{O}_{\mathbb{C}^n}) = 0.$$

As a consequence we first get that r_k is injective and then that $c_{|J} = 0$.

If $k = 1$, Cousin's lemma asserts that the arrow

$$\begin{array}{ccccc}
\mathcal{O}_{\mathbb{C}^n}(J_+) & \oplus & \mathcal{O}_{\mathbb{C}^n}(J_-) & \overset{\delta_0}{\to} & \mathcal{O}_{\mathbb{C}^n}(J_0) \\
f_+ & & f_- & \mapsto & f_{+|J_0} - f_{-|J_0}
\end{array}$$

is surjective. From this fact, it follows that r_1 is injective and that $c_{|J} = 0$.

But, since $J \in E$, $c_{|J} \neq 0$ and we get a contradiction. □

The preceding result allows us to prove that

Proposition 3.6 *On a block P of $\mathbb{C}^n$, Cartan's theorem B is a consequence of theorem A.*

Proof. Let us assume that theorem A is true for P. We shall prove by decreasing recurrence on k that

$$H^k(P, \mathcal{F}) = 0$$

for every $k \in \mathbb{N}_0$ and every coherent analytic sheaf $\mathcal{F}$ on P. First, let us remark that the block P has a finite flabby dimension l since it is a compact subset of $\mathbb{R}^{2n}$. Thus

$$H^{l+1}(P, \mathcal{F}) = 0$$

for every sheaf $\mathcal{F}$ on P. Next, let us note that if $k > 1$ and if

$$H^k(P, \mathcal{F}) = 0$$

for every coherent analytic sheaf $\mathcal{F}$ on P, then there is by hypothesis an exact sequence of the form

$$0 \to \mathcal{G} \to \mathcal{O}^q \to \mathcal{F} \to 0$$

where $\mathcal{G}$ is a coherent analytic sheaf on P. Hence, the long exact sequence of cohomology together with the preceding proposition give that

$$H^{k-1}(P, \mathcal{F}) = H^k(P, \mathcal{G}) = 0.$$

The conclusion follows easily. $\square$

We are now in position to prove the main result of this section.

Proposition 3.7 *Theorems A and B hold on a block of $\mathbb{C}^n$.*

Proof. Let us prove theorem A, this will give also theorem B. We shall proceed by recurrence on the dimension of the block P. The case $\dim P = 0$ is clear, so we may assume that the result is true for $\dim P \leq m$ and prove it for $\dim P = m + 1$ by contradiction. Thus, assume that we have an analytic coherent sheaf $\mathcal{F}$ on P which does not have any global system of generators on P.

Consider the set

$$E = \{J \subset P : J \text{ block of } \mathbb{C}^n, \mathcal{F}_{|J} \text{ without system of generators}\}$$

ordered by inclusion.

If $\mathcal{J}$ is a totally ordered subset of E, then $\cap \mathcal{J} \in E$. As a matter of fact, if it is not the case, the sheaf $\mathcal{F}_{|\cap \mathcal{J}}$ has a system of generators $(g_j)_{j=1}^{l}$ where $g_j \in \mathcal{F}(\cap \mathcal{J})\ \forall j \in \{1, \ldots, l\}$. Since $\cap \mathcal{J}$ is compact, there is a neighbourhood V of $\cap \mathcal{J}$ in P and sections $g'_j \in \mathcal{F}(V)$ such that $g'_{j|\cap \mathcal{J}} = g_j$. But $\mathcal{F}$ is a coherent analytic sheaf on P, thus there is a neighbourhood W de $\cap \mathcal{J}$ such that $(g'_{j|W})_{j=1}^{l}$ generate $\mathcal{F}_{|W}$. Since P is compact, there is $J \in \mathcal{J}$ such that

$J \subset W$. It is then clear that $(g'_{j|_J})_{j=1}^{l}$ generate $\mathcal{F}$ on J. But this is not possible since $J \in E$.

From the preceding fact and Zorn's lemma, one gets a minimal block $J \in E$. The recurrence hypothesis asserts that $\dim J = m+1$. Let us choose a real number r and a natural number j in such a way that the blocks

$$\begin{array}{rcl} J_- & = & J \cap \{x \in \mathbb{R}^{2n} : x_j \leq r\} \\ J_+ & = & J \cap \{x \in \mathbb{R}^{2n} : x_j \geq r\} \end{array}$$

have a dimension equal to $m+1$ and let us set

$$J_0 = J_+ \cap J_-.$$

Since J is minimal, one can find systems of generators g_+, g_- of $\mathcal{F}$ on J_+, J_-. We know that $\dim J_0 = m$, thus Cartan's theorems are true on J_0. By consequence, the epimorphisms

$$\mathcal{O}^p_{|_{J_+}} \twoheadrightarrow \mathcal{F}_{|_{J_+}} \qquad \mathcal{O}^q_{|_{P_-}} \twoheadrightarrow \mathcal{F}_{|_{P_-}}$$

associated to the system of generators g_+ and g_- induce epimorphisms at the level of global sections on J_0. Thus $g_{+|_{J_0}}, g_{-|_{J_0}}$ generate $\mathcal{F}(J_0)$.

We can now apply lemma **??** and get a system of generator of $\mathcal{F}$ on J. But this condradicts the fact that $J \in E$, so the proof is complete. $\square$

4 The case of a brick

Definition 4.1 A *brick* of a Stein manifold X is a compact subset B of X such that one can find $N \in \mathbb{N}$, a holomorphic map $f : X \to \mathbb{C}^N$, a block Q of $\mathbb{C}^n$ and a relatively compact open neighbourhood U of B in X in such a way that

a) $f|_{\bar{U}} : \bar{U} \to \mathbb{C}^N$ is injective,

b) f is an immersion near every point of $\bar{U}$,

c) $B = U \cap f^{-1}(Q)$,

d) $\dot{U} \cap f^{-1}(Q) = \emptyset$.

Under these assumptions one sees that $\Omega = \mathbb{C}^N \setminus f(\dot{U})$ is an open neighbourhood of Q in $\mathbb{C}^N$, that $f|_U : U \to \Omega$ is a closed injective immersion and that $f_{|U}^{-1}(Q) = B$.

Proposition 4.2 *Cartan's theorems A and B are true for the bricks of a Stein manifold.*

Proof. Let us prove theorem B, theorem A will follow. Using the same notations as in the definition of the bricks of a Stein manifold, we know that

$$f_{|_U} : U \to \Omega$$

is a closed imbedding, that $f_{|_U}^{-1}(Q) = B$ and that $\Omega \supset Q$.

Let $\mathcal{F}$ be a coherent analytic sheaf on B. By definition, there is a open neighbourhood V of B and a coherent $\mathcal{O}_V$-module $\mathcal{G}$ on V such that $\mathcal{G}_{|_B} = \mathcal{F}$. Moreover, without any restriction, one may assume that $V \subset U$. It is then clear that $\omega = \Omega \setminus f(U \setminus V)$ is an open subset of $\mathbb{C}^N$ containing Q such that $f_{|_U}^{-1}(\omega) = V$ and where the map

$$f_{|_V} : V \to \omega$$

is a closed imbedding. Hence,

$$\mathbb{R}f_{|_V *}(\mathcal{G}) = f_{|_V *}(\mathcal{G})$$

is a coherent analytic sheaf on ω. Using Cartan's theorems for the blocks of $\mathbb{C}^N$ one gets that

$$H^k(Q, f_{|_V *}(\mathcal{G})) = 0$$

if $k > 0$. But, sheaf theory shows that

$$H^k(V \cap f^{-1}(Q), \mathcal{G}) = H^k(Q, \mathbb{R}f_{|_V *}(\mathcal{G}))$$

thus we get the requested result since $V \cap f^{-1}(Q) = B$ and $\mathcal{G}_{|_B} = \mathcal{F}$. □

5 The case of a Stein manifold

Lemma 5.1 (Separation) *For every holomorphically convex compact subset K of X and every compact subset K' not intersecting K, there are a natural number N, a block Q of $\mathbb{C}^N$ and a holomorphic application*

$$f : X \to \mathbb{C}^N$$

such that $K \subset f^{-1}(\overset{\circ}{Q})$ and that $K' \cap f^{-1}(Q) = \emptyset$.

Proof. Since $K' \subset X \setminus \hat{K}$, for every $z \in K'$, there is $f_z \in \mathcal{O}_X(X)$ such that $|f_z(z)| > 1 > \|f_z\|_K$. Replacing eventually f_z by one of its power, one may assume that

$$\sup\{\|\Re f_z\|_K, \|\Im f_z\|_K\} < 1$$

and that

$$\sup\{|\Re f_z(z)|, |\Im f_z(z)|\} > 1.$$

Since K' is compact, there is a finite number of sections $f_1, \dots, f_J$ of $\mathcal{O}_X(X)$ such that

$$K' \subset \bigcup_{j=1}^{J} \{z \in X : \sup\{|\Re f_j(z)|, |\Im f_j(z)|\} > 1\}$$

and that

$$\sup_{1 \le j \le J} \sup\{\|\Re f_j\|_K, \|\Im f_j\|_K\} < 1.$$

In order to conclude, we just have to set $f = (f_1, \dots, f_J)$ and

$$Q = ([-1,1] + i\,[-1,1])^J$$

and to note that $f^{-1}(Q) \cap K' = \emptyset$ and that $K \subset f^{-1}(\overset{\circ}{Q})$. □

Lemma 5.2 (Imbedding) *If K is a compact subset of a Stein manifold X, then there is $N \in \mathbb{N}_0$ and a holomorphic map $f : X \to \mathbb{C}^N$ such that $f|_K$ is injective and that f is an immersion near every point of K.*

Proof. For every holomorphic map $\varphi : X \to \mathbb{C}^n$, one sets

$$I_\varphi = \{x \in X : T_x\varphi \text{ injective}\}.$$

The family of open subsets I_φ covers X since X is a Stein manifold. Thus, there are holomorphic maps $\varphi_1, \dots, \varphi_J$ such that

$$K \subset \bigcup_{j=1}^{J} I_{\varphi_j}.$$

If we set $f = (\varphi_1, \dots, \varphi_J)$, it is clear that the map $f : X \to \mathbb{C}^{nJ}$ is an immersion near every point of K. Moreover, since K is compact, there are finitely many open subsets $V_1, \ldots, V_L$ covering K, f being injective on each of them. Let us set

$$V = \bigcup_{l=1}^{L} V_l \times V_l.$$

It is clear that V is an open neighbourhood of

$$\Delta_K = \{(x, x) : x \in K\}$$

and that

$$\Delta_K = V \cap (K \times K) \cap \{(x, y) : f(x) = f(y)\}.$$

Now, for every holomorphic function g on X, let us set

$$J_g = \{(x, y) : g(x) \neq g(y)\}.$$

Since X is Stein, the open subsets J_g cover the compact subset

$$K' = K \times K \setminus V \cap (K \times K).$$

Then, we find holomorphic functions $g_1, \ldots, g_M$ on X such that

$$K' \subset \bigcup_{m=1}^{M} J_{g_m}.$$

Consider the application

$$(f, g_1, \ldots, g_M) : X \to \mathbb{C}^{nJ+M}.$$

By construction, this application is injective on K. Moreover it is also an immersion near every point of K since this is true for f. □

Proposition 5.3 *For every open neighbourhood V of a holomorphically convex compact subset K of a Stein manifold X, there is a brick B of X such that*

$$K \subset \overset{\circ}{B} \subset B \subset V.$$

Proof. One can without any restriction assume that $\bar{V}$ is compact. Since

$$K \cap \dot{V} = \emptyset$$

the separation lemma gives us a holomorphic map

$$f_1 : X \to \mathbb{C}^{N_1}$$

and a block Q_1 of $\mathbb{C}^{N_1}$ such that

$$f_1^{-1}(\overset{\circ}{Q}_1) \supset K \qquad f_1^{-1}(Q_1) \cap \dot{V} = \emptyset.$$

Moreover, the imbedding lemma gives us an injective holomorphic map

$$f_2 : X \to \mathbb{C}^{N_2}$$

which is an immersion near every point of $\bar{V}$. Since $f_2(\bar{V})$ is a compact subset of $\mathbb{C}^{N_2}$, there is a block Q_2 of $\mathbb{C}^{N_2}$ such that

$$f_2(\bar{V}) \subset \overset{\circ}{Q}_2.$$

The map

$$(f_1, f_2) : X \to \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$$

is then an immersion near every point of $\bar{V}$ which is injective when restricted to this set. One checks easily that

$$\dot{V} \cap (f_1, f_2)^{-1}(Q_1 \times Q_2) = \dot{V} \cap f_1^{-1}(Q_1) = \emptyset$$

$$K \subset V \cap f_1^{-1}(\overset{\circ}{Q}_1) = V \cap (f_1, f_2)^{-1}(\overset{\circ}{Q}_1 \times \overset{\circ}{Q}_2).$$

From these relations one deduces that

$$B = V \cap (f_1, f_2)^{-1}(Q_1 \times Q_2)$$

is a brick and that

$$K \subset \overset{\circ}{B} \subset B \subset V$$

as requested. □

Proposition 5.4 *If X is a Stein manifold, then there is an increasing sequence $(B_m)_{m \in \mathbb{N}}$ of bricks of X such that,*

$$B_m \subset \overset{\circ}{B}_{m+1}$$

for every $m \in \mathbb{N}$ and that

$$X = \bigcup_{m \in \mathbb{N}} B_m.$$

Proof. Since X is countable at infinity, there is a fundamental sequence of compact subsets $(K_m)_{m\in\mathbb{N}}$ such that

$$K_m \subset \overset{\circ}{K}_{m+1}$$

for every $m \in \mathbb{N}$ and that

$$X = \bigcup_{m\in\mathbb{N}} K_m.$$

Since X is Stein, $\hat{K}_m$ is compact and one can build by recurrence a subsequence $(K_{m_k})_{k\in\mathbb{N}}$ such that

$$\hat{K}_{m_k} \subset \overset{\circ}{K}_{m_{k+1}}.$$

Then, by the preceding proposition one gets bricks $(B_k)_{k\in\mathbb{N}}$ such that

$$\hat{K}_{m_k} \subset \overset{\circ}{B}_k \subset B_k \subset \overset{\circ}{K}_{m_{k+1}}.$$

From this relation it follows that

$$B_k \subset \overset{\circ}{B}_{k+1}$$

for every $k \in \mathbb{N}$ and that

$$X \subset \bigcup_{k\in\mathbb{N}} K_{m_k} \subset \bigcup_{k\in\mathbb{N}} B_k$$

so the proof is complete. □

Definition 5.5 Let $\mathcal{F}$ be a coherent analytic sheaf on a brick B of a Stein manifold X. To every system of generators $g = (g_1, \dots, g_J)$ of $\mathcal{F}$ on B, let us associate the semi-norm $\|\cdot\|_{B,g}$ defined by

$$\|\sigma\|_{B,g} = \inf\{\sup_{1\le j\le J} \|f_j\|_B : f_1, \dots, f_J \in \mathcal{O}_X(B); \sigma = \sum_{j=1}^{J} f_j g_j\}.$$

One checks easily that if g_1, g_2 are two systems of generators of $\mathcal{F}$ on B, then the semi-norms $\|\cdot\|_{B,g_1}$ et $\|\cdot\|_{B,g_2}$ are equivalent. The *B-topology* on $\mathcal{F}(B)$ is then the canonical topology defined by any one of these semi-norms.

Lemma 5.6 *Let B, B' be two bricks of a Stein manifold X and let $\mathcal{F}$ be a coherent analytic sheaf on X. If $B \subset \overset{\circ}{B}{}'$ and if ρ denotes the restriction morphism between the space $\mathcal{F}(B')$ endowed with the B'-topology and the space $\mathcal{F}(B)$ endowed with the B-topology, then* *a)* ρ *is a continuous map,*

b) $\operatorname{im}\rho$ *is dense,*

c) $\rho(\overline{\{0\}}) = \{0\}$*,*

d) if σ_m *is a Cauchy sequence of* $\mathcal{F}(B')$*, then the sequence* $\rho(\sigma_m)$ *converges in* $\mathcal{F}(B)$*.*

Proof. a) is direct.

b) Assume first that $\mathcal{F} = \mathcal{O}_X$. Since B is a brick, there is an open neighbourhood U of B, an holomorphic map

$$f : X \to \mathbb{C}^N$$

and a block Q contained in an open subset Ω of $\mathbb{C}^N$ such that

$$f_{|_U} : U \to \Omega$$

is a closed imbedding and that

$$B = U \cap f^{-1}(Q).$$

Since the natural arrow $\mathcal{O}_\Omega \to f_{|_U *}\mathcal{O}_U$ is surjective, theorem B for the blocks of $\mathbb{C}^N$ shows that the arrow

$$\begin{array}{ccc} \mathcal{O}_\Omega(Q) & \to & \mathcal{O}_X(B) \\ h & \mapsto & h \circ f \end{array}$$

is also surjective. Let us fix $\sigma \in \mathcal{O}_X(B)$ and $\epsilon > 0$. The preceding result shows that there is a section $h \in \mathcal{O}_\Omega(Q)$ such that $\sigma = h \circ f$. Thus, Runge lemma for the blocks of $\mathbb{C}^N$ gives us a section $h' \in \mathcal{O}_{\mathbb{C}^N}(\mathbb{C}^N)$ such that

$$\|h - h'\|_Q \leq \epsilon.$$

From this relation it follows that $h' \circ f \in \mathcal{O}_X(X)$ and that

$$\|h' \circ f - \sigma\|_B \leq \epsilon$$

and this allows us to conclude since $h' \circ f|_{B'} \in \mathcal{O}_X(B')$.

Now let us deal with the general case. Since Cartan's theorems are true for the bricks B' and B, one gets the epimorphism

$$\mathcal{O}^l_{B'} \xrightarrow{g} \mathcal{F}_{|_{B'}}$$

where $g = (g_1, \ldots, g_l)$ generates $\mathcal{F}(B')$ and where $(g_{1|_B}, \ldots, g_{l|_B})$ generates $\mathcal{F}(B)$. Let us fix $\sigma \in \mathcal{F}(B)$ and $\epsilon > 0$. There are holomorphic functions $f_1, \ldots, f_l$ on B such that

$$\sigma = \sum_{j=1}^{l} f_j g_{j|_B}.$$

The preceding results concerning the sheaf $\mathcal{O}_X$ give us holomorphic functions $f'_1, \ldots, f'_l$ on B' such that

$$\left\| f'_j - f_j \right\|_B \leq \epsilon$$

for every $j \in \{1, \ldots, l\}$. To conclude, we just have to define $\sigma' \in \mathcal{F}(B')$ by setting

$$\sigma' = \sum_{j=1}^{l} f'_j g_j$$

and to note that

$$\left\| \sigma'_{|B} - \sigma \right\|_{B, g|_B} \leq \epsilon.$$

c) and d). Since Cartan's theorems are true on B', one can find an exact sequence of the following kind

$$\mathcal{O}^q_{|_{B'}} \to \mathcal{O}^p_{|_{B'}} \xrightarrow{g} \mathcal{F}_{|_{B'}} \to 0.$$

From this sequence, we also deduce the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathcal{O}^q_X(B') \arrow[r] \arrow[d] & \mathcal{O}^p_X(B') \arrow[r] \arrow[d] & \mathcal{F}(B') \arrow[r] \arrow[d, "u"] & 0 \\
\mathcal{O}^q_X(\mathring{B}') \arrow[r, "w"] \arrow[d] & \mathcal{O}^p_X(\mathring{B}') \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "v"] & 0 \\
\mathcal{O}^q_X(B) \arrow[r] & \mathcal{O}^p_X(B) \arrow[r] & \mathcal{F}(B) \arrow[r] & 0
\end{tikzcd}$$

where F is a Fréchet space since the map w has a closed range (cf. for example [**?**] p. 156). In this diagram, all the arrows are continuous if one endows the spaces in the first line with the B'-topology, those in the third line with the B-topology and those in the second with the usual Fréchet topology.

If $\sigma \in \mathcal{F}(B')$ is in the closure of $\{0\}$, it is clear that $u(\sigma) \in \overline{\{0\}}$. But F is a Fréchet space, thus $u(\sigma) = 0$ and $\sigma|_B = v(u(\sigma)) = 0$.

If $\sigma_m \in \mathcal{F}(B')$ is a Cauchy sequence, then $u(\sigma_m)$ is a Cauchy sequence in the Fréchet space F and thus converges in this space. From this it follows that the sequence $\sigma_{m|B} = v(u(\sigma_m))$ converges in $\mathcal{F}(B)$. □

Proposition 5.7 *If X is a Stein manifold and if $\mathcal{F}$ is a coherent analytic sheaf on X, then*

$$H^k(X, \mathcal{F}) = 0$$

if $k > 0$.

Proof. Proposition **??** gives us a sequence $(B_m)_{m\in\mathbb{N}}$ of bricks of X such that

$$B_m \subset \overset{\circ}{B}_{m+1}$$

for every $m \in \mathbb{N}$.

By the preceding lemma, the projective system consisting of the spaces $\mathcal{F}(B_m)$ endowed with their B_m-topology and the canonical restriction maps is of the generalised Mittag-Leffler type (cf. definition **??**). Moreover, one knows that $H^k(B_m, \mathcal{F}) = 0$ for every $k > 0$ since B_m is a brick. One may then use porposition **??** to conclude that

$$H^k(X, \mathcal{F}) = \underset{\substack{\leftarrow \\ m\in\mathbb{N}}}{\lim} H^k(B_m, \mathcal{F}) = 0$$

as requested. □

Proposition 5.8 *Cartan's theorems A and B hold on a Stein manifold and on each of its holomorphically convex compact subsets.*

Proof. The preceding proposition gives theorem B for X and we know that theorem A is one of its consequences. So, we just need to deal with the holomorphically convex compact subsets of X. Let K be such a subset. Proposition **??** shows that the bricks B of X such that $\overset{\circ}{B} \supset K$ make a fundamental system of neighbourhoods of K in X. It is then a classical result in sheaf theory that

$$H^k(K, \mathcal{F}) = \underset{\substack{\rightarrow \\ \overset{\circ}{B}\supset K}}{\lim} H^k(B, \mathcal{F}).$$

This allows us to conclude since $H^k(B, \mathcal{F}) = 0$ if $k > 0$. □

6 Appendix

Definition 6.1 A countable projective system $(E_m, u_m^{m+1})_{m\in\mathbb{N}}$ is of *generalized Mittag-Leffler* type if one can endow each space E_m with a pseudometric d_m such that

a) $\forall m \in \mathbb{N}$, the map $u_m^{m+1} : E_{m+1} \to E_m$ is continuous and has a dense range,

b) $\forall m \in \mathbb{N}$, $\exists M \geq m$ such that if $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in E_M, then the sequence $u_m^M(x_k)$ converges in E_m,

c) $\forall m \in \mathbb{N}$, $\exists M \geq m$ sucht that if $d_M(x,y) = 0$, then $u_m^M(x) = u_m^M(y)$.

Proposition 6.2 *If* (E_m, u_m^{m+1}) *is a countable projective system of generalized Mittag-Leffler type, then*

$$u_m(\lim_{\substack{\leftarrow \\ m\in\mathbb{N}}} E_m)$$

is a dense subset of E_m *for every* $m \in \mathbb{N}$*.*

Proof. It is clearly sufficient to prove that

$$u_0(\lim_{\substack{\leftarrow \\ m\in\mathbb{N}}} E_m)$$

is a dense subset of E_0. In order to achieve this, let us fix $e \in E_0$ and $\epsilon > 0$. We know that the maps u_k^j are continuous for $k \leq j$, hence it is clear that if e_j is is an arbitrary element of E_j then

$$\forall \epsilon > 0\ \exists \eta > 0 \qquad d_j(e', e_j) \leq \eta \quad \Rightarrow \quad \forall k \leq j\ \ d_k(u_k^j(e'), u_k^j(e_j)) \leq \epsilon 2^{-j-1}.$$

This fact allows us to use the density of $u_j^{j+1}(E_{j+1})$ in E_j to construct by recurrence a sequence $(e_j)_{j\in\mathbb{N}}$ such that $e_0 = e$ in such a way that

$$\forall k \leq j \quad d_k(u_k^{j+1}(e_{j+1}), u_k^j(e_j)) \leq \epsilon 2^{-j-1}.$$

It is then clear that $(u_k^j(e_j))_{j\geq k}$ is a Cauchy sequence of E_k for every $k \in \mathbb{N}$. It follows from part (b) of the definition of generalized Mittag-Leffler systems that the sequence $(u_k^j(e_j))_{j\geq k}$ has a limit a_k in E_k and that

$$d(a_k, u_k^j(e_j)) \leq \epsilon \sum_{l=j}^{+\infty} 2^{-l-1} \leq \epsilon.$$

Moreover, the continuity of u_k^j shows that

$$d(a_k, u_k^j a_j) = 0$$

if $k \leq j$.

Part (c) of the definition of generalised Mittag-Leffler systems gives us an increasing sequence of natural numbers M_k such that

$$d_{M_k}(x, y) = 0 \quad \Rightarrow \quad u_k^{M_k}(x) = u_k^{M_k}(y).$$

Let us set $b_k = u_k^{M_k}(a_{M_k})$. It follows from the construction of a_k that

$$u_k^{k+1}(b_{k+1}) = u_k^{M_{k+1}}(a_{M_{k+1}}) = u_k^{M_k}(u_{M_k}^{M_{k+1}} a_{M_{k+1}})$$

and since $d(a_{M_k}, u_{M_k}^{M_{k+1}} a_{M_{k+1}}) = 0$, the definition of M_k allows us to assert that

$$u_k^{k+1}(b_{k+1}) = b_k.$$

Moreover, one has

$$d(b_0, e) = d(u_0^{M_0}(a_{M_0}), e) = d(a_0, e) \leq \epsilon.$$

These two relations show that the sequence b_m defines a member

$$b \in \varprojlim_{m \in \mathbb{N}} E_m$$

and that $d(u_0(b), e) \leq \epsilon$. Since ϵ is an arbitrary positive real number, the proof is complete. $\square$

Proposition 6.3 *If*

$$0 \to E. \xrightarrow{u.} F. \xrightarrow{v.} G. \to 0$$

is an exact sequence of projective systems over $\mathbb{N}$ *and if* $E.$ *is of generalized Mittag-Leffler type, then the sequence*

$$0 \to \varprojlim E. \to \varprojlim F. \to \varprojlim G. \to 0$$

is exact.

Proof. Only the surjectivity of $\underset{\leftarrow}{\lim} v.$ needs a proof. Let us fix $z \in \underset{\leftarrow}{\lim} G.$ and let us denote by z_m the canonical image of z in G_m. Next, let us choose

$$y_m \in v_m^{-1}(z_m).$$

Since $\ker v_m = \operatorname{im} u_m$, the map

$$x \mapsto y_m + u_m(x)$$

gives an isomorphism between E_m and $v_m^{-1}(z_m)$. From this fact, it follows that the projective system $(v_m^{-1}(y_m))_{m\in\mathbb{N}}$ is of generalized Mittag-Leffler type and the preceding proposition gives that

$$L = \underset{\substack{\leftarrow \\ m\in\mathbb{N}}}{\lim} v_m^{-1}(z_m) \neq \emptyset$$

since its canonical image in $v_m^{-1}(z_m)$ is dense.

To conclude, we just have to note that, by construction, the canonical image y of $x \in L$ in $\underset{\leftarrow}{\lim} F$ satisfies $v(y) = z$. □

Corollary 6.4 *(cf. [?]) If $E.$ is a projective system over $\mathbb{N}$ of generalized Mittag-Leffler type, then*

$$\underset{\leftarrow}{\lim}^{(1)} E. = 0$$

where $\underset{\leftarrow}{\lim}^{(1)}$ denotes the first right derived functor of $\underset{\leftarrow}{\lim}$.

Proof. The result is a direct consequence of the preceding proposition if one imbeds $E.$ in a projective system F which is acyclic for the functor $\underset{\leftarrow}{\lim}$. □

Proposition 6.5 *(cf. [?]) If $\mathcal{F}$ is a sheaf on the topological space X, if $(F_m)_{m\in\mathbb{N}}$ is an increasing sequence of closed subsets of X such that*

$$X = \bigcup_{m\in\mathbb{N}} F_m$$

and if the projective systems

$$(H^k(F_m, \mathcal{F}))_{m\in\mathbb{N}}$$

are of generalized Mittag-Leffler type for $0 \leq k \leq l-1$, then one has

$$H^l(X, \mathcal{F}) = \underset{\substack{\leftarrow \\ m\in\mathbb{N}}}{\lim} H^l(F_m, \mathcal{F}).$$

Proof. Consider the complex of projective systems over $\mathbb{N}$

$$(\mathcal{C}^{\cdot}(F_m, \mathcal{F}))_{m\in\mathbb{N}}$$

built with the Godement's canonical flabby resolutions of $\mathcal{F}$ on the closed subsets F_m. By flabbiness, it is clear that the projective systems

$$(\mathcal{C}^k(F_m, \mathcal{F}))_{m\in\mathbb{N}}$$

are acyclic for $\varprojlim$. From this fact it follows that

$$\mathcal{C}^{\cdot}(X, \mathcal{F}) = \varprojlim_{m\in\mathbb{N}} \mathcal{C}^{\cdot}(F_m, \mathcal{F}) = \mathbb{R}\varprojlim_{m\in\mathbb{N}} \mathcal{C}^{\cdot}(F_m, \mathcal{F}).$$

But, by a classical result on derived functors, our hypothesis together with the preceding corollary gives that

$$H^l(\mathbb{R}\varprojlim_{m\in\mathbb{N}} \mathcal{C}^{\cdot}(F_m, \mathcal{F})) = \varprojlim_{m\in\mathbb{N}} H^l(F_m, \mathcal{F})$$

and the proof is complete. □

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