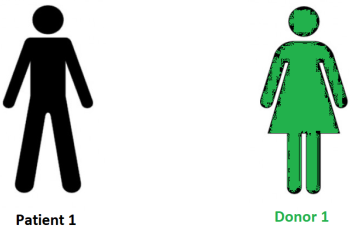
Locally stable exchanges

Marie BARATTO, Yves CRAMA, Joao Pedro PEDROSO, Ana VIANA

September 13th 2022

Kidney exchange problem



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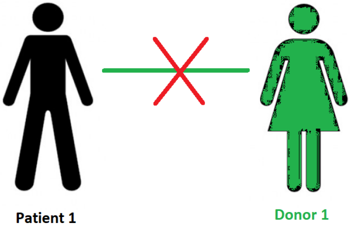

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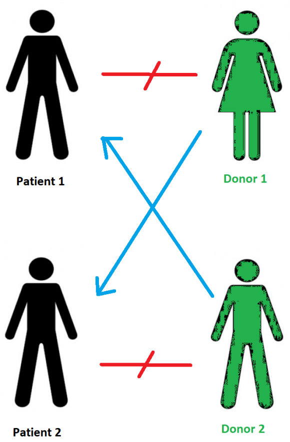

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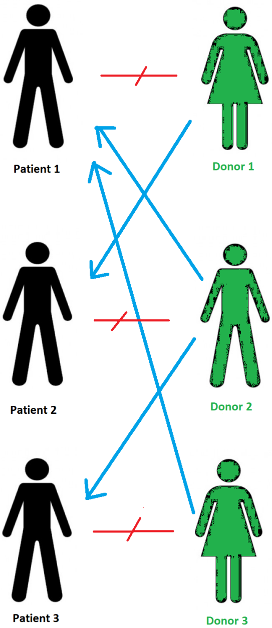
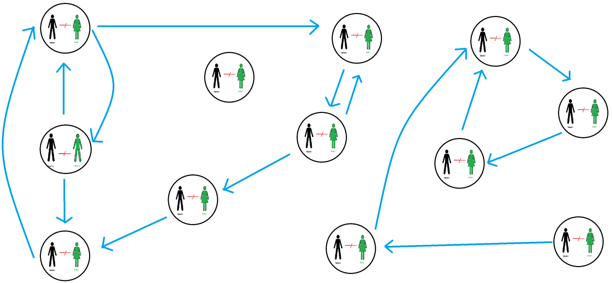
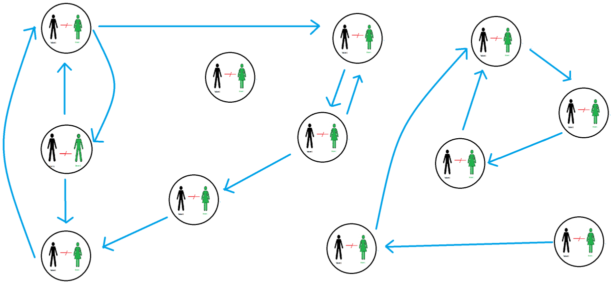


Pool of incompatible pairs



Compatibility graph



G=(V,A,w) where:

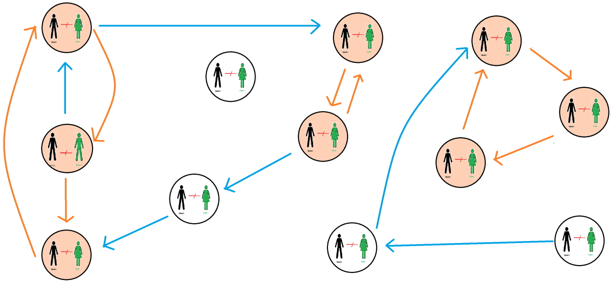
- $V = \{1, ..., n\}$ set of vertices, consisting of all patient-donor pairs.
- A, the set of arcs, designating compatibilities between the vertices. Two vertices i and j are connected by arc (i, j) if the donor in pair i is compatible with the patient in pair j.

We denote by $\mathcal{C}(G)$ be the set of feasible cycles of length at most K for $G = (V, A)$.

Possible exchanges

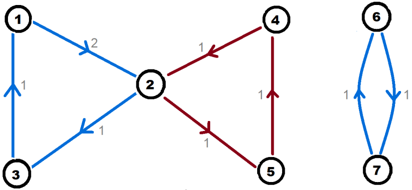
Definition

An **exchange** is a set of disjoint cycles in the directed graph. It is feasible if every cycle length does not exceed a given limit K.



Stable exchange

For each vertex $i \in V$, a preference order is given on the set of incoming arcs $\{(j, i) \in A\}$.
A preference p associated to arc (j, i) means that the recipient of the pair i ranks the donor of pair j at position p in its preferences list of acceptable donors.



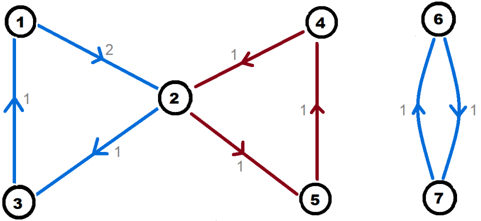
Stable exchange

Klimentova, X., Biró, P., Viana, A., Costa, V., and Pedroso, P. (2022). Integer programming models for the stable kidney exchange problem

Definition

A **blocking cycle** c for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either i is unmatched in $\mathcal{M}$,
- or i prefers c to $c' \in \mathcal{M}$ where $i \in V(c')$.
  We say that a vertex i prefers cycle c to c' if for $(k, i) \in A(c)$ and $(k', i) \in A(c')$, i strictly prefers k to k'.



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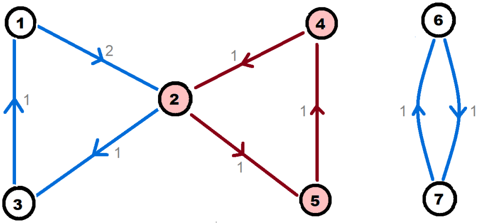

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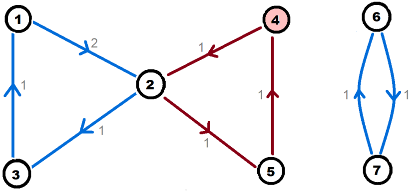

$\rightarrow$ Vertex 4 is unmatched in exchange $\mathcal{M}$ in blue

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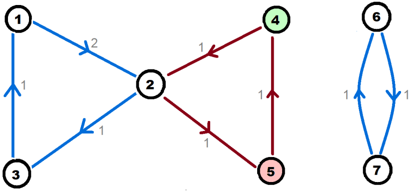

$\rightarrow$ Vertex 5 is unmatched in exchange $\mathcal{M}$ in blue

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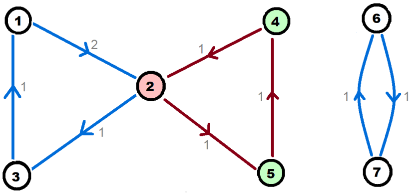

$\rightarrow$ Vertex 2 prefers cycle red because the donor of pair 4 is number one on its preference list and donor of pair 1 is at at the second position of its preference list.

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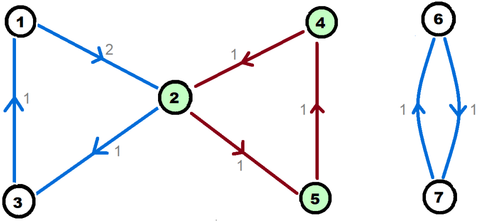

Definitions - stability

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Definition

Given a directed graph $G = (V, A)$, an exchange is called **stable** if no blocking cycle c exists for $\mathcal{M}$.

Stable exchange - Drawback

Definition

A **blocking cycle** c for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

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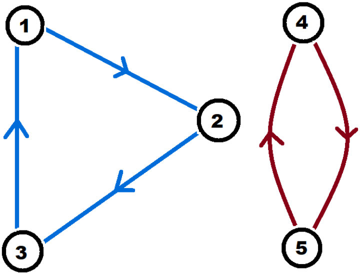


Locally Stable Exchange

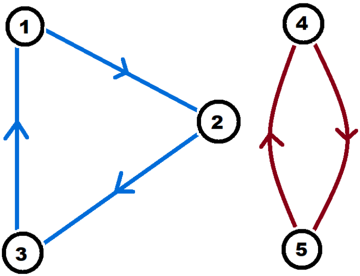
Definition

A **locally blocking cycle (L-blocking cycle)** c for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ but **has a vertex in common with** $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either i is unmatched in $\mathcal{M}$,
- or i prefers c to $c' \in \mathcal{M}$ where $i \in V(c')$.

Definition

Given a directed graph $G = (V, A)$, an exchange is called **locally stable (L-stable)** if no L-blocking cycle c exists for $\mathcal{M}$.



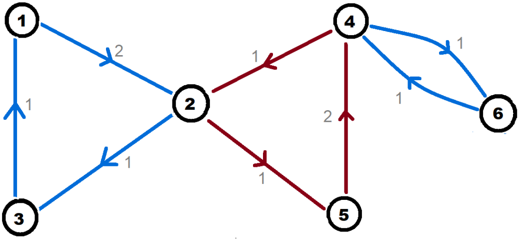
Definitions

Given a cycle c we define two sets:

- $PBC(c)$

Definition

For a given cycle c, a **potentially blocking cycle (PBC)** is another cycle c' non vertex disjoint with c and such that each vertex of c' is either not in c or prefers c' to c.



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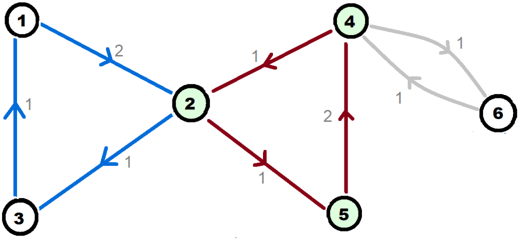

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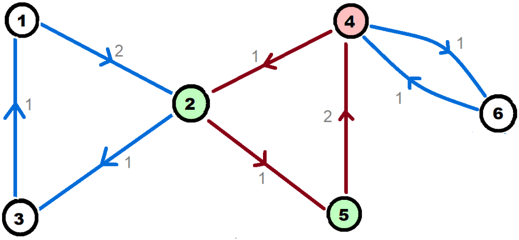


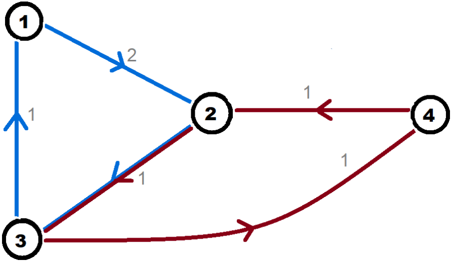
Definitions

Given a cycle c we define two sets:

- $\mathcal{A}(c)$

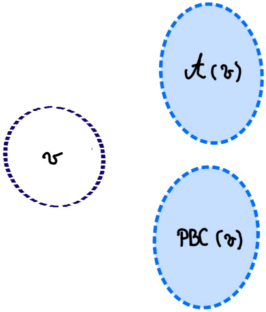
Definition

For a given cycle c, $\mathcal{A}(c)$ be the set of cycles that are not disjoint with c but they are not potentially blocking c and c does not potentially block them neither.



If two cycles c_u and c_v are not vertex disjoint then three situations can occur:

- $c_u \in PBC(c_v)$
- $c_v \in PBC(c_u)$
- $c_u \in \mathcal{A}(c_v)$ and $c_v \in \mathcal{A}(c_u)$

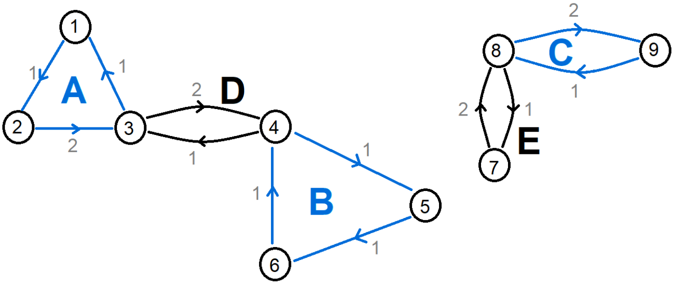


Result

For an exchange $\mathcal{M}$, the following conditions are equivalent:

(a) $\mathcal{M}$ is L-stable;

(b) for each cycle $v \notin \mathcal{M}$, if v is not disjoint from $\mathcal{M}$, then there exists $w \in \mathcal{M}$ such that $w \in PBC(v) \cup \mathcal{A}(v)$.



- $PBC(A) = \{D\}$
- $PBC(D) = \{B\}$
- $PBC(B) = \emptyset$
- $PBC(C) = \emptyset$
- $PBC(E) = \{C\}$

- $\mathcal{A}(c) = \emptyset$ for all cycles

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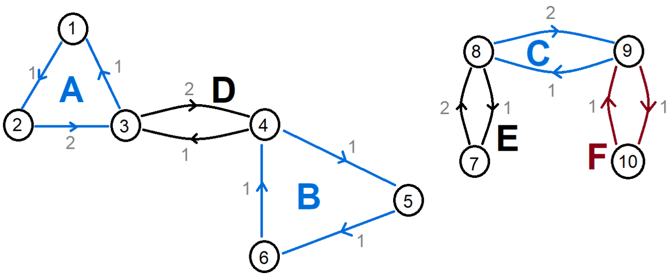

- $PBC(A) = \{D\}$
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- $PBC(B) = \emptyset$
- $PBC(C) = \{F\}$
- $PBC(E) = \{C\}$
- $PBC(F) = \emptyset$

- $\mathcal{A}(c) = \emptyset$ for all cycles

IP formulation

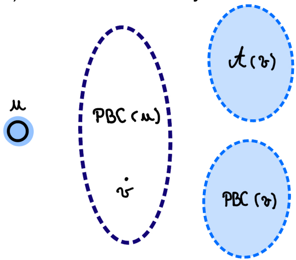
we introduce the natural variables y_v for all $v \in \mathcal{C}(G)$ and the following set of constraints:

$$y_u + y_v \leq 1 \qquad \forall u \in \mathcal{C}(G), \forall v \in PBC(u) \cup \mathcal{A}(u) \tag{1}$$

$$y_u \leq \sum_{w \in PBC(v) \cup \mathcal{A}(v)} y_w \qquad \forall u \in \mathcal{C}(G), \forall v \in PBC(u) \tag{2}$$

$$y_v \in \{0, 1\} \qquad \forall v \in \mathcal{C}(G) \tag{3}$$

1. Constraints (1) ensure the independence of the cycles selected
2. Constraints (2) ensure the stability of the exchange

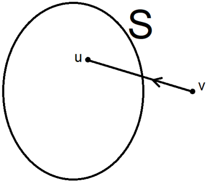


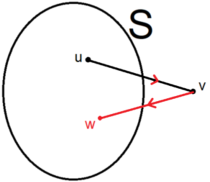
(Local) Kernel - Definitions

Definition

Given a directed graph $G = (V, A)$, subset $S \subseteq V$ is a **kernel** of G if it is independent and absorbing. That is:

- for all $(u, v) \in A$ either $u \notin S$ or $v \notin S$
- for every $v \notin S$ there exists a vertex $u \in S$ such that $(v, u) \in A$





Definition

A **local kernel** of G is an independent subset S of vertices such that every neighbor (or out-neighbor) of S is absorbed by S. In other words:

- for all $(u, v) \in A$ either $u \notin S$ or $v \notin S$
- if there exist $u \in S$ and $v \notin S$ such that $(u, v) \in A$, then there must exist $w \in S$ such that $(v, w) \in A$.

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- if there exist $u \in S$ and $v \notin S$ such that $(u, v) \in A$, then there must exist $w \in S$ such that $(v, w) \in A$.

Result

The empty set $S = \emptyset$ is an L-kernel. So, every directed graph has an L-kernel (but not necessarily a not empty one).

Result

Given a directed graph $G = (V, A)$, deciding whether G has a nonempty local kernel is NP-complete.

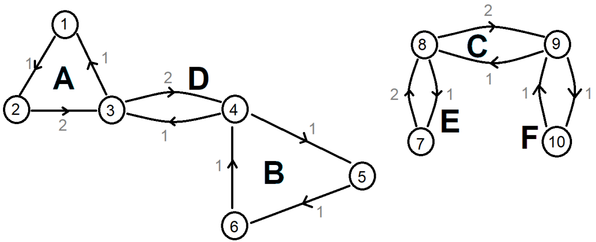
Reduction from SAT

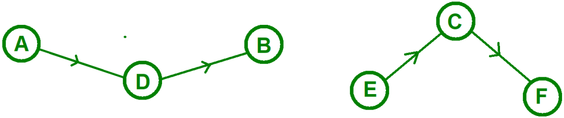
Blocking digraph

Starting from the initial directed graph $G = (V, A)$, lets construct a directed graph $G' = (V', A')$ such that:

- For each $c_v \in \mathcal{C}(G)$ there is a vertex c_v in V' representing that cycle.
- An arc $(c_u, c_v) \in A'$ if $c_v \in PBC(c_u)$ or if $c_v \in \mathcal{A}(c_u)$.

G' is the **blocking directed graph** associated to G.





Result

A L-kernel in G' defines a L-stable exchange in G.

and a kernel in G' defines a stable exchange in G

2. Formulation 2:

$$y_u + y_v \leq 1 \qquad \forall (u,v) \in A' \qquad (4)$$

$$y_u \leq \sum_{w \in N^+(v)} y_w \qquad \forall (u,v) \in A' \qquad (5)$$

$$y_v \in \{0,1\} \qquad \forall v \in V' \qquad (6)$$

- **Independence** constraint (4) can be replaced by

$$\sum_{c \in \mathcal{C}: i \in V(c)} x_c \leq 1 \qquad \forall i \in V \qquad (7)$$

- **Stability/Absorbing** constraint (5) can be replaced by fixing v and adding each constraint above for all $(u,v) \in A'$:

$$\sum_{w \in N^-(v)} y_w \leq |N^-(v)| \sum_{w \in N^+(v)} y_w \qquad \forall v \in V' \qquad (8)$$

where $N^-(v) = |\{w : (w,v) \in A'\}|$.

Comparison IP formulations

Comparison between 3 formulations:

1. Initial formulation for L-stable exchange (Form-LS)
2. Form-LS with independence constraints modified
3. Form-LS with independence constraints and stability constraints modified

Best IP formulation for L-stable exchanges in terms of computation time:

$$\sum_{v \in \mathcal{C}(G): i \in V(c)} y_v \leq 1 \qquad \forall i \in V$$

$$\sum_{w \in N^-(v)} y_w \leq |N^-(v)| \sum_{w \in N^+(v)} y_w \qquad \forall v \in V'$$

$$y_v \in \{0, 1\} \qquad \forall v \in V'$$

Aim

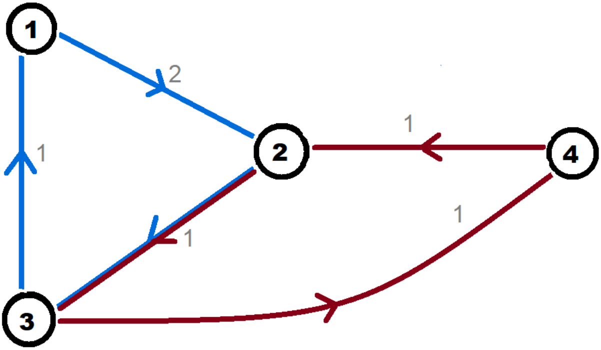
Stability vs *Local* stability

Stability vs *Local* stability

- Problem of maximum stable exchange(SE) and problem of maximum L-stable exchange (LSE) are not the same problems (not the same set of feasible solutions) **BUT** when the objective function of both problems is to maximize the total length of the cycles selected, the two formulations have the same objective.

- **SE** problem: Some instances do not have a solution
- **LSE** problem: All instances tested have a solution of cardinality greater than zero
  - for N=300,
    1. 5 out of 50 instances do not have a stable exchange
    2. 5: average optimal value is 74
    3. 45: average optimal value is 150, 08
  - for N=400,
    1. 11 out of 50 instances do not have a stable exchange
    2. 11: average optimal value is 141, 45
    3. 39: average optimal value is 205, 94

Local Strong Stability



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