Locally stable exchanges

Marie Baratto

Kidney exchange problem
Stable exchanges
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References
Kidney exchange problem

Patient 1

Donor 1
Kidney exchange problem

Patient 1

Donor 1
Kidney exchange problem

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Pool of incompatible pairs
Compatibility graph

\( G = (V, A, w) \) where:

- \( V = \{1, \ldots, n\} \) set of vertices, consisting of all patient-donor pairs.
- \( A \), the set of arcs, designating compatibilities between the vertices. Two vertices \( i \) and \( j \) are connected by arc \( (i, j) \) if the donor in pair \( i \) is compatible with the patient in pair \( j \).

We denote by \( C(G) \) be the set of feasible cycles of length at most \( K \) for \( G = (V, A) \).
Possible exchanges

Definition

An exchange is a set of disjoint cycles in the directed graph. It is feasible if every cycle length does not exceed a given limit $K$. 
For each vertex \( i \in V \), a preference order is given on the set of incoming arcs \( \{(j, i) \in A\} \).
A preference \( p \) associated to arc \((j, i)\) means that the recipient of the pair \( i \) ranks the donor of pair \( j \) at position \( p \) in its preferences list of acceptable donors.
Stable exchange


Definition

A **blocking cycle** $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either $i$ is unmatched in $\mathcal{M}$,
- or $i$ prefers $c$ to $c' \in \mathcal{M}$ where $i \in V(c')$.

We say that a vertex $i$ prefers cycle $c$ to $c'$ if for $(k, i) \in A(c)$ and $(k', i) \in A(c')$, $i$ strictly prefers $k$ to $k'$. 

![Diagram of blocking cycles](image-url)
A **blocking cycle** $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

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```
1  2  4
\downarrow\quad\downarrow\quad\downarrow
3  2  5
\uparrow\quad\uparrow\quad\uparrow
7  6
```
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We say that a vertex \( i \) prefers cycle \( c \) to \( c' \) if for \((k, i) \in A(c)\) and \((k', i) \in A(c')\), \( i \) strictly prefers \( k \) to \( k' \).

→ Vertex 4 is unmatched in exchange \( \mathcal{M} \) in blue
A **blocking cycle** $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

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We say that a vertex $i$ prefers cycle $c$ to $c'$ if for $(k, i) \in A(c)$ and $(k', i) \in A(c')$, $i$ strictly prefers $k$ to $k'$.

→ Vertex 5 is unmatched in exchange $\mathcal{M}$ in blue
A blocking cycle $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either $i$ is unmatched in $\mathcal{M}$,
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$\rightarrow$ Vertex 2 prefers cycle red because the donor of pair 4 is number one on its preference list and donor of pair 1 is at at the second position of its preference list.
Definition

A **blocking cycle** \( c \) for an exchange \( \mathcal{M} \) is a cycle that is not included in \( \mathcal{M} \) and such that, for every vertex \( i \in V(c) \),

- either \( i \) is unmatched in \( \mathcal{M} \),
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We say that a vertex \( i \) prefers cycle \( c \) to \( c' \) if for \( (k, i) \in A(c) \) and \( (k', i) \in A(c') \), \( i \) strictly prefers \( k \) to \( k' \).
### Definitions - stability

**Definition**

A **blocking cycle** $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either $i$ is unmatched in $\mathcal{M}$,
- or $i$ prefers $c$ to $c' \in \mathcal{M}$ where $i \in V(c')$.

**Definition**

Given a directed graph $G = (V, A)$, an exchange is called **stable** if no blocking cycle $c$ exists for $\mathcal{M}$. 

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**Definitions - stability**

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**Stable exchange - Drawback**

**Definition**

A **blocking cycle** $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either $i$ is unmatched in $\mathcal{M}$,
- or $i$ prefers $c$ to $c' \in \mathcal{M}$ where $i \in V(c')$. 

![Diagram of blocking cycle](image)
Definition

A **locally blocking cycle** (L-blocking cycle) $c$ for an exchange $\mathcal{M}$ is a cycle that is not included in $\mathcal{M}$ but **has a vertex in common with** $\mathcal{M}$ and such that, for every vertex $i \in V(c)$,

- either $i$ is unmatched in $\mathcal{M}$,
- or $i$ prefers $c$ to $c' \in \mathcal{M}$ where $i \in V(c')$.

Definition

Given a directed graph $G = (V, A)$, an exchange is called **locally stable** (L-stable) if no L-blocking cycle $c$ exists for $\mathcal{M}$.
Definitions

Given a cycle $c$ we define two sets:

- $PBC(c)$

Definition

For a given cycle $c$, a potentially blocking cycle (PBC) is another cycle $c'$ non vertex disjoint with $c$ and such that each vertex of $c'$ is either not in $c$ or prefers $c'$ to $c$. 
Definitions

Given a cycle \( c \) we define two sets:

- \( PBC(c) \)

**Definition**

For a given cycle \( c \), a **potentially blocking cycle (PBC)** is another cycle \( c' \) non vertex disjoint with \( c \) and such that each vertex of \( c' \) is either not in \( c \) or prefers \( c' \) to \( c \).
Definitions

Given a cycle $c$ we define two sets:

- $PBC(c)$

Definition

For a given cycle $c$, a potentially blocking cycle (PBC) is another cycle $c'$ non vertex disjoint with $c$ and such that each vertex of $c'$ is either not in $c$ or prefers $c'$ to $c$. 
Definitions

Given a cycle $c$, we define two sets:

- $\mathcal{A}(c)$

**Definition**

For a given cycle $c$, $\mathcal{A}(c)$ be the set of cycles that are not disjoint with $c$ but they are not potentially blocking $c$ and $c$ does not potentially block them neither.
If two cycles $c_u$ and $c_v$ are not vertex disjoint then three situations can occur:

- $c_u \in PBC(c_v)$
- $c_v \in PBC(c_u)$
- $c_u \in A(c_v)$ and $c_v \in A(c_u)$
Result

For an exchange $\mathcal{M}$, the following conditions are equivalent:

(a) $\mathcal{M}$ is L-stable;

(b) for each cycle $v \notin \mathcal{M}$, if $v$ is not disjoint from $\mathcal{M}$, then there exists $w \in \mathcal{M}$ such that $w \in PBC(v) \cup A(v)$.

\[ PBC(A) = \{D\} \]
\[ PBC(D) = \{B\} \]
\[ PBC(B) = \emptyset \]
\[ PBC(C) = \emptyset \]
\[ PBC(E) = \{C\} \]
\[ A(c) = \emptyset \text{ for all cycles} \]
Result

For an exchange \( \mathcal{M} \), the following conditions are equivalent:

(a) \( \mathcal{M} \) is L-stable;

(b) for each cycle \( v \notin \mathcal{M} \), if \( v \) is not disjoint from \( \mathcal{M} \), then there exists \( w \in \mathcal{M} \) such that \( w \in PBC(v) \cup \mathcal{A}(v) \).

\[\text{• } PBC(A) = \{D\}\]
\[\text{• } PBC(D) = \{B\}\]
\[\text{• } PBC(B) = \emptyset\]
\[\text{• } PBC(C) = \{F\}\]
\[\text{• } PBC(E) = \{C\}\]
\[\text{• } PBC(F) = \emptyset\]
\[\text{• } \mathcal{A}(c) = \emptyset \text{ for all cycles}\]
we introduce the natural variables $y_v$ for all $v \in C(G)$ and the following set of constraints:

$$y_u + y_v \leq 1 \quad \forall u \in C(G), \forall v \in PBC(u) \cup A(u) \quad (1)$$

$$y_u \leq \sum_{w \in PBC(v) \cup A(v)} y_w \quad \forall u \in C(G), \forall v \in PBC(u) \quad (2)$$

$$y_v \in \{0, 1\} \quad \forall v \in C(G) \quad (3)$$

1. Constraints (1) ensure the independence of the cycles selected
2. Constraints (2) ensure the stability of the exchange
(Local) Kernel - Definitions

**Definition**

Given a directed graph $G = (V, A)$, subset $S \subseteq V$ is a *kernel* of $G$ if it is independent and absorbing. That is:

- for all $(u, v) \in A$ either $u \notin S$ or $v \notin S$
- for every $v \notin S$ there exists a vertex $u \in S$ such that $(v, u) \in A$

**Definition**

A *local kernel* of $G$ is an independent subset $S$ of vertices such that every neighbor (or out-neighbor) of $S$ is absorbed by $S$. In other words:

- for all $(u, v) \in A$ either $u \notin S$ or $v \notin S$
- if there exist $u \in S$ and $v \notin S$ such that $(u, v) \in A$, then there must exist $w \in S$ such that $(v, w) \in A$. 

Definition

A **local kernel** of $G$ is an independent subset $S$ of vertices such that every neighbor (or out-neighbor) of $S$ is absorbed by $S$. In other words:

- for all $(u, v) \in A$ either $u \notin S$ or $v \notin S$
- if there exist $u \in S$ and $v \notin S$ such that $(u, v) \in A$, then there must exist $w \in S$ such that $(v, w) \in A$.

Result

The empty set $S = \emptyset$ is an L-kernel. So, every directed graph has an L-kernel (but not necessarily a not empty one).

Result

Given a directed graph $G = (V, A)$, deciding whether $G$ has a nonempty local kernel is NP-complete.

*Reduction from SAT*
Starting from the initial directed graph \( G = (V, A) \), lets construct a directed graph \( G' = (V', A') \) such that:

- For each \( c_v \in C(G) \) there is a vertex \( c_v \) in \( V' \) representing that cycle.
- An arc \((c_u, c_v) \in A' \) if \( c_v \in PBC(c_u) \) or if \( c_v \in A(c_u) \).

\( G' \) is the **blocking directed graph** associated to \( G \).

Result

A L-kernel in \( G' \) defines a L-stable exchange in \( G \).

and a kernel in \( G' \) defines a stable exchange in \( G \)
Formulation 2:

\[ y_u + y_v \leq 1 \quad \forall (u, v) \in A' \]  
\[ y_u \leq \sum_{w \in N^+(v)} y_w \quad \forall (u, v) \in A' \]  
\[ y_v \in \{0, 1\} \quad \forall v \in V' \]

- **Independence** constraint (4) can be replaced by
  \[ \sum_{c \in C : i \in V(c)} x_c \leq 1 \quad \forall i \in V \] (7)

- **Stability/Absorbing** constraint (5) can be replaced by fixing \( v \) and adding each constraint above for all \((u, v) \in A'\):
  \[ \sum_{w \in N^-(v)} y_w \leq |N^-(v)| \sum_{w \in N^+(v)} y_w \quad \forall v \in V' \] (8)

where \( N^-(v) = |\{w : (w, v) \in A'\}|. \)
Comparison IP formulations

Comparison between 3 formulations:
1. Initial formulation for L-stable exchange (Form-LS)
2. Form-LS with independence constraints modified
3. Form-LS with independence constraints and stability constraints modified

Best IP formulation for L-stable exchanges in terms of computation time:

$$
\sum_{v \in C(G): i \in V(c)} y_v \leq 1 \quad \forall i \in V
$$

$$
\sum_{w \in N^-(v)} y_w \leq |N^-(v)| \sum_{w \in N^+(v)} y_w \quad \forall v \in V'
$$

$$
y_v \in \{0, 1\} \quad \forall v \in V'
$$
Stability vs Local stability
Stability vs *Local* stability

- Problem of maximum stable exchange (SE) and problem of maximum L-stable exchange (LSE) are not the same problems (not the same set of feasible solutions) **BUT** when the objective function of both problems is to maximize the total length of the cycles selected, the two formulations have the same objective.

- **SE** problem: Some instances do not have a solution
- **LSE** problem: All instances tested have a solution of cardinality greater than zero
  - for N=300,
    1. 5 out of 50 instances do not have a stable exchange
    2. 5: average optimal value is 74
    3. 45: average optimal value is 150, 08
  - for N=400,
    1. 11 out of 50 instances do not have a stable exchange
    2. 11: average optimal value is 141, 45
    3. 39: average optimal value is 205, 94
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