Temporal Display of Gestures in Diagrammatic Proof

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Abstract According to the deductivist view of mathematics which became the rule during the nineteenth century, formal proofs working with symbolic formulas replaced the intuitive knowledge that used to be gained by the step-by-step construction of geometric figures and diagrams. Twentieth century epistemological reflection on symbolic formulas and formal proofs, however, took them to be diagrams respectively exhibiting formal relations and transformations. The claim was also made that, for such diagrams to be proofs, temporal displays of transformations—and of other speech acts—were required. By returning to the significant elements of contemporary theories of mathematical proofs, we will here show how these proofs came to be a matter for semiotics and pragmatics as much as for formal logic. Once this is done, we will provide a few arguments against the standard objections to the ability of diagrams to present temporal order or perform speech acts.

Keywords. Gesture; Diagram; Proof; Geometric Figure; Semiotics.

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1. Mathematical proofs and temporal constructions

Immanuel Kant notoriously claimed that arithmetic is grounded on the temporal form of sense intuition because any numerical presentation or arithmetical operation requires a progressive display of units, which can only happen in time. Displaying units one after the other is the «schema» for numbers, i.e. the general method for providing a numerical concept with its intuitive image:

> The schema is to be distinguished from an image. Thus if I place five points in a row, •••••, this is an image of the number five. On the contrary, if I only think a number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (e.g., a thousand) in accordance with a certain concept than the image itself, which in this case I could survey and compare with the concept only with difficulty. Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept.[1]

Numbers are not reducible to sets of dots, but drawing dots in a row is a general procedure for assigning an image to any natural number.[2] According to Kant, the whole set of fundamental laws of natural numbers' arithmetic is involved in this «general procedure»: if I add five dots to a row of seven dots, I will get a row of twelve dots. Arithmetical judgments are not analytic (i.e., merely based on the analysis of the concepts involved); they must instead be based on the intuitive—and therefore temporal—display of their conceptual content:

> The concept of sum of 7 and 5 contains nothing more than

1 Kant 1998 [1781-1787] (hereafter *CPR*), A140/B179. A schema, Kant says, is «a general procedure of the imagination for providing a concept with its image».

2 *CPR*, A140-142/B179-181.

> number in which 7 and 5 are united—that is all. When I have the thought of the sum of 5 and 7, I do not thereby have the thought of 12; no matter how long I spend analysing my conception of such a possible sum, I won't find 12 in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise.[3]

Bernard Bolzano or Gottlob Frege's answers to such claims are equally notorious. According to Bolzano, an arithmetical judgment such as «7+2=9» is analytic and is not based on the temporal form of sense intuition:

> The propositions of arithmetic do not require the intuition of time in any way. We shall only analyse a single example. Kant gave the proposition 7 + 5 = 12, instead of which, to make it easier, we shall take the shorter 7 + 2 = 9. The proof of this proposition is not difficult as soon as we assume the general proposition a + (b + c) = (a + b) + c, i.e. that with an arithmetic sum one only looks at the number of terms not their order (certainly a wider concept than sequence in time). This proposition excludes the concept of time rather than presupposing it. But having accepted it, the proof of the above proposition can be carried out in the following way: the statements 1 + 1 = 2, 7 + 1 = 8, 8 + 1 = 9 are mere definitions and conventions. Therefore, 7 + 2 = 7 + (1 + 1) (*per def.*) = (7 +

3 *CPR*, B15.

1) + 1 (*per propos. praeced.*) = 8 + 1(*per def.*) = 9 (*per def.*).[4]

Frege likewise challenges the idea that arithmetical judgments require the intuitive and progressive display of dots or fingers.[5] For Frege, time «has nothing to do with the concept of number».[6] According to his logicist views, arithmetical judgments are analytic and do not rely on sense intuition, whether it be pure or empirical. More generally, Frege claims just like many 19th century mathematicians that if mathematical reasoning is to be rigorous and reliable, it should not rest on intuition but merely on deduction, i.e. on the strict application of previously stated rules of inference:[7]

> Considerably higher demands must be placed on the conduct of proof than is customary in arithmetic. A few methods of inference must be marked out in advance, and no step may be taken that is not in accordance with one of these. Thus in passing on to a new judgment one must not be satisfied, as the mathematicians have nearly always been hitherto, with the transition's being evidently correct; rather one must split it into the logically simple steps of which it is composed —and of which there are frequently not a few. In this way no presupposition can pass unnoticed; every axiom required must be uncovered. It is indeed precisely the presuppositions made tacitly and without clear awareness that obstruct our insight into the epistemological nature of a law.[8]

Frege's own formal language or «conceptual notation» (*Begriffsschrift*) aims to make explicit the exact form of the judgments involved in mathematical reasoning in such a way that the strict conformity of inferences to rules of inference can be easily checked.[9] Just as in

4 BOLZANO 2004 [1837], 83-137, § 8, 135.
5 FREGE 1963 [1884], § 5, 6.
6 FREGE 1963 [1884], § 40, 52-3.
7 «One may not appeal to intuition as a means of proof» (FREGE 1979 [1880-1881], 32).
8 FREGE 1964 [1893], § 0, 29.
9 FREGE 1972 [1879], 104.

Leibniz' project, providing mathematics (and science more generally) with a «*lingua characteristica*» makes it possible to reduce mathematical reasoning to some «*calculus ratiocinator*», i.e. mere computation of the formulae:

> That it made it possible to perform a type of computation, it was precisely this fact that Leibniz saw as a principal advantage of a script which compounded a concept out of its constituents.[10]

However, the question arises whether computation itself may require display in time.

Contemporary to Frege, Charles Sanders Peirce claims that formal proofs rest on the construction of diagrams. In these diagrams, formulae are icons of the formal relations between the contents of thought; the whole proof is an icon of the step by step transformation of the first formula into the last one:[11]

> All deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in imagination, and of observing the result so as to discover unnoticed and hidden relations among its parts.[12]

At each step of the proof, the transformation of one formula into the next can be seen to conform to some rule of transformation:

> Our purpose is to study the workings of necessary inference. What we want, in order to do this, is a method of representing

10 FREGE 1979 [1880-1881], 9.

11 PEIRCE 1958 (hereafter *CP*), vol. 2, § 279. In algebra, Peirce says, signs rather than their contents are the object of attention and of investigation (PEIRCE 1982, vol. I, 173).

12 *CP*, vol. 3, § 363.

> diagrammatically any possible set of premises, this diagram to be such that we can observe the transformations of these premises into the conclusion by a series of steps each of the utmost possible simplicity.[13]

David Hilbert similarly likens formal proofs to figures, which present formal relations and formal transformations «in front of the eyes»: a formal proof «is a figure, which we must be able to view as such».[14] In this case, mathematical reasoning is not so much about the mathematical items themselves (arithmetical entities, geometrical figures, etc.) as about the formulae and proofs that reflect their formal relations and the regulated transformations of these relations:

> The axioms, formulae and proofs that make up this formal edifice are precisely what the number-signs were in the construction of elementary number-theory […] and with them alone, as with the number-signs in number theory, contentual thought takes place—i.e., only with them is actual thought practiced.[15]

Ludwig Wittgenstein writes that formal proofs are computations of formal relations between signs:

> Whether a proposition belongs to logic can be calculated by

13 *CP*, vol. 4, § 429. This of course requires that formal relations can be made visually salient, as Gestalt theory claims, and that regulated transformations can be made visually salient, as experimental cognitive psychology testifies (see for instance KIRSHNER & AWTRY 2004, 224-57). The choice of notation can however influence the visual salience of forms and transformations, and therefore the obviousness of evidence.

14 HILBERT 1996 [1922], 1115-33, 1126. Such a claim notoriously contrasts with Hilbert's earlier claim according to which «A theorem is only proved when the proof is completely independent of the diagram» (HILBERT 2004, 75).

15 HILBERT 1996 [1922], 1123. See also p. 1127: «For concrete-intuitive number-theory, which we treated first, the numbers were the objectual and the displayable, and the proofs of theorems about the numbers fell into the domain of the thinkable. In our present investigation, proof itself is something concrete and displayable; the contentual reflections follow the proofs themselves».

> calculating the logical properties of the symbol. And this we do when we prove a logical proposition. For without troubling ourselves about a sense and a meaning, we form the logical propositions outs of others by mere symbolic rules.[16]

Looking at both the formal relations between signs in the formulae and the formal transformations between formulae in the proof provides the required «intuition» for the proof to be convincing:

> To the question whether we need intuition for the solution of mathematical problems it must be answered that language itself here supplies the necessary intuition. The process of calculation brings about just this intuition.[17]

According to the later Wittgenstein, this means that formal proofs are just as diagrammatic as geometrical proofs made from the (regulated) construction of spatial figures:

> Cogency of logical proof stands and falls with its geometrical cogency […] That is to say: logical proof, e.g., of the Russellian kind, is cogent only so long as it also possesses geometrical cogency.[18]

Kant himself had already observed that algebraic formulae are a way of presenting arithmetic judgments and reasoning similar to how intuitive geometrical constructions present geometric judgments and reasoning.[19] He even claims that, as a way of providing arithmetic concepts, judgments and reasonings with their intuitive image, algebraic formula are schemas likely to supersede the progressive display of units.[20]

16 WITTGENSTEIN 1921, § 6.126.
17 WITTGENSTEIN 1921, § 6.233-6.2331.
18 WITTGENSTEIN 1978, 174-5.
19 *CPR*, B745.
20 *CPR*, B762.

However, Kant's point does not in the least entail that algebraic computations could, unlike the intuitive computation of dots or fingers, go without temporality.

As far as geometrical constructions are concerned, Kant commented on proofs of synthetic judgements such as the theorem that the sum of the angles in a triangle equals two right angles:[21]

> Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.[22]

21 On Kant's intuitive constructions (regulated by concepts) and the way they ground universally valid synthetic—or «ampliative»—judgements, see LECLERCQ 2016.

22 *CPR*, A716/B744.

Kant is here describing the geometrical construction that leads to the following figure:

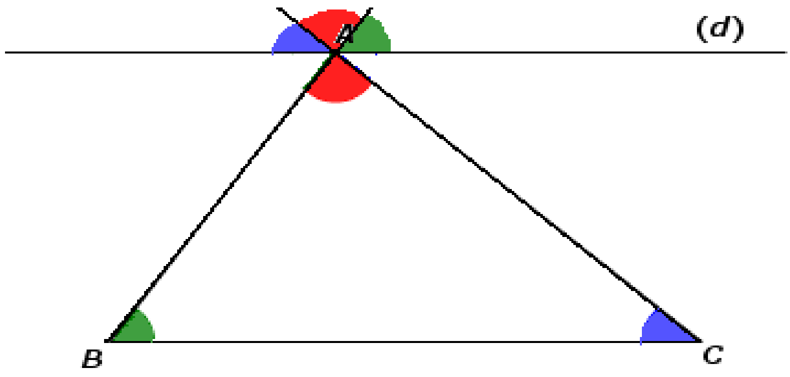

Figure 1. Diagrammatic proof that the sum of the angles of a triangle equals 180°.

Each step of such a construction involves some regulated transformation of the initial figure: the extension of [BA] along the straight line BA; the extension of [CA] along the straight line CA; the drawing of straight line d, which is parallel to BC; … .The whole sequence of transformations proves that angle A + angle B + angle C = 180°.

Bolzano's algebraic proof that 7+2=9 is similarly made from successive regulated transformations of an initial formula:

$7 + 2$	
$= 7 + (1 + 1)$	(def. of 2)
$= (7 + 1) + 1$	(assoc. of +)
$= 8 + 1$	(def. of 8)
$= 9$	(def. of 9)

Figure 2. Algebraic proof that 7+2 = 9 (the brackets at the right of the proof tell us by which rule a transformation is allowed)

If proofs, geometric or algebraic, rest on the construction of diagrams, the question still arises whether they rest on the *dynamic* display of such diagrams (in which case they are temporal) or merely on the resulting *static* diagrams (in which case they are not).

According to Kant, it is in the nature of any extensive quantity that a whole can only be presented through the presentation of its parts.[23] The whole can therefore be presented only through a progressive construction.[24] Just like the presentation of an arithmetic sum by the progressive addition of dots, the presentation of a geometric figure or an algebraic formula and its transformation must also require the «successive synthesis» of productive imagination.[25] The presentation therefore rests on the temporality of such a synthesis.

Is this really the case, however? Is time constitutive—a «condition of possibility»—of a proof? Or is time, as Frege claims, just a psychological requisite for subjectively working out, expressing or understanding the proof? In such a case, it would not be a requisite of the objective proof itself, which lies in what is worked out, expressed and understood. Although proving is a (temporal) process made of a sequence of steps, it is not clear whether the proof itself consists in this process or just in its «content» or «result».

Of course, the proof does not lie in the last step of the process, the *Quod Erat Demonstrandum*; it does not lie in the algebraic formula that had to be proved and now appears on the last line of the proof. Instead, the proof lies in the sequence of transformations that leads from the first to the last step. Although each step «follows» from the previous step according to some permitted transformation, however, does «follow» here have a temporal or a logical meaning? Frege is surely right to claim that time has nothing to do with the validity of inferential links. The question remains whether a proof is merely a set of valid inferential links or whether it is an ordered set of gestures.

23 *CPR*, A162/B203.
24 *CPR*, A714/B742.
25 *CPR*, A163/B204.

2. Mathematical proofs as sequences of gestures

According to Frege, the radical anti-psychologist, mental acts are not part of a proof; they are only part of the way a cognitive subject can «grasp» the proof. However, after having championed anti-psychologism himself, Ludwig Wittgenstein turns away from such a view. He claims that proofs need not only be valid but also convincing:

> The idea that there are two kinds of proofs: the real proof —the proof which gives a firm ground to the proposition, so that it is unshakeable and won't fall —and the proof that is to convince you (…), this idea comes from a false view of what a proof actually does.[26]

Inferential validity is a necessary but not a sufficient condition of proofs; proofs must also be laid out in such a way that they convince cognitive subjects:[27]

> What convinces us —that is the proof. A configuration that does not convince us is not the proof.[28]

This means that the whole «reasoning» (i.e., the whole process of going from one step of the proof to the next while checking the validity of each transformation) is constitutive of the proof. This reasoning requires mental acts—and more generally «gestures»—

26 WITTGENSTEIN 1939, 238.

27 Such a view (according to which both inferential validity and convincing force are necessary conditions for a proof, but neither of them are on their own sufficient conditions) seems to be a reasonable middle term between the opposing radical stances distinguished by Doyle et al. According to a «Baroque» perspective, a proof's status is underwritten by the formal correctness of its inferences, while according to a «Romantic» perspective, it is underwritten by apparent evidentiary force (DOYLE ET AL. 2014).

28 WITTGENSTEIN 1939, § III-39, p. 171.

anchored in time.

As has been pointed out by several philosophers of logic and mathematics, proofs are not just made of deductive relations between statements (that should only be «grasped» by knowing subjects). They are made of deductive *inferences*, where a deductive inference is:

> an epistemic *action* that can bring an agent from one epistemic step to another, for instance, from the state of knowing or believing the premises to one of knowing or believing the conclusion.[29]

Taken as a sequence of (epistemic) actions that carry out some specific plan, a proof surely conforms to some temporal organisation. When trying to give an account of planning in mathematical proofs (i.e., of the rational, rather than arbitrary, sequence of their deductive steps) Yacin Hamami and Rebeca Morris compare proofs to the activity of traveling:

> To understand the notion of proof activity, it might be useful to consider it in analogy with another familiar activity such as traveling. First, proof activities and traveling activities are both intended to bring the agent from one state to another. In the case of proof activities, the objective is to get from one epistemic state to another, while in the case of traveling activities, the objective is to get from one position state to another. Second, these two types of activities proceed through a sequence of moves —which are the actions constituting the building blocks of these activities —bringing the agent closer and closer to the desired state.[30]

By focusing on proof «activity» (i.e., on inferential «moves» and how they are «planned»), contemporary philosophy of mathematics

29 HAMAMI & MORRIS (forth); Hamami & Morris name the following papers that accord with this view: PRAWITZ 2012; SUNDHOLM 2012; BOGHOSSIAN 2014; BROOME 2014.

30 HAMAMI & MORRIS 2020.

requires a dynamic view of proofs:

> Mathematical proofs have traditionally been conceived as static, agent-free objects. But if one approaches them through their primary epistemic function —namely to bring knowledge of the associated theorem —one is forced to see them in a dynamic way, and to bring the mathematical agent back into the picture.[31]

This position clearly stresses the temporal structure of proofs, which should be reflected in the diagrams that express them.[32] Furthermore, as contemporary philosophers of mathematics using speech act theory rightly point out[33], mathematical proofs are not made solely of assertions and inferences—both of which, as Frege knew, are already speech acts.[34] They also involve many other «gestures». These gestures include definitions, which are declarative, and instructions for construction, which are directive.

If diagrams are to be proofs and proofs are sequences of speech acts, this means, on the semiotic side, that diagrams should be able to display a temporal order as well as to display various speech acts.

As far as temporal order is concerned, it does not initially seem easy for a single diagram to exhibit a path of successive steps. As Groupe µ observes, spatial figures do not lend themselves naturally to a linear reading, unlike linguistic texts:

> In the visual field, the syntagmatic relation is not linear (as is

31 HAMAMI & MORRIS 2020, 4.

32 «Proof activities are (…) temporally extended in the sense that each deductive inference stands in a particular relation to the inferences that are prior and posterior to it in the activity, that is, each deductive inference is inscribed within the overall temporal structure of the activity» (HAMAMI & MORRIS 2020, 6).

33 RUFFINO, SAN MAURO & VENTURI 2020a and forth. See also TANSWELL (forth.).

34 That Frege was aware of this is shown by the fact that he uses specific signs to express assertions and inferences, such as a vertical stroke for assertion or a long horizontal split bar for inferences. J.L. Austin, as the English translator of Frege's work, would have remembered Frege's signs when theorizing about speech acts, including assertion.

the case in the linguistic field): it is spatial. As a consequence, chronological factor is in principle absent: it cannot be asserted that the receiver looks first at some place, then at some other one, then again at some other one.[35]

This seems to be a major problem if proofs are supposed to be diagrams and at the same time require a temporal ordering of gestures. In «Proofs Without Words and Beyond — PWWs and Mathematical Proof», Doyle et al. make this exact comment concerning an allegedly purely visual proof of Viviani's theorem provided by Samuel Wolf:

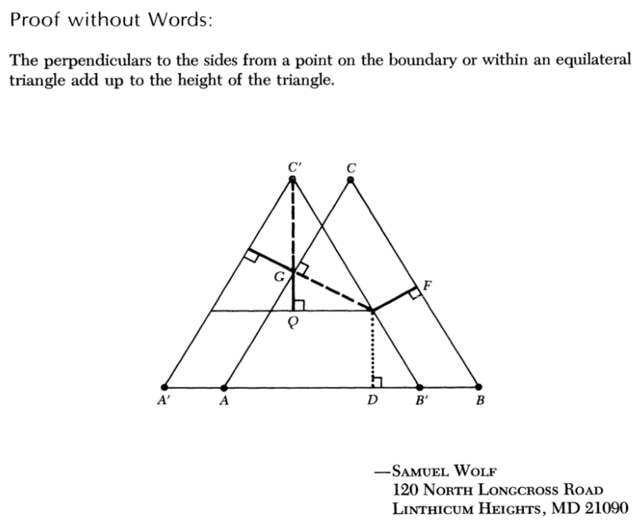

Figure 3. Wolf's Proof Without Words of Viviani's Theorem

However, Groupe μ's *Traité du signe visuel* also shows how spatial figures can be organized in such a way that they end up favouring some linear reading and therefore allowing some temporally ordered path connecting several steps. An alignment of figures that show

35 Groupe μ 1992, 316.

similarities in many respects (sides, global shape, colour, etc.) and only a few variations tends to be read as a sequence of successive transformations of one and the same figure:[36]

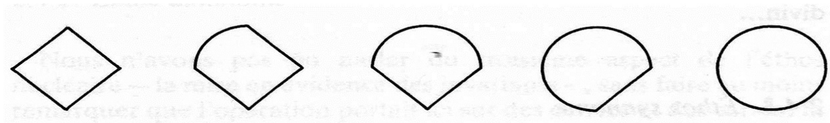

Figure 4. Mediation of square and circle

Due to the influence of linguistic semiotics on visual semiotics, the alignment tends to be read as a progressive transformation from left to right. Bolzano's proof (Fig. 2) similarly tends to be read as a progressive transformation from top to bottom.

Another way of providing a figure with a temporal order, and therefore of forcing a dynamic reading of a static image, lies in the use of conventional signs such as arrows. This feature occurs in Peirce's diagrammatic proofs of the validity of propositional logical reasonings:

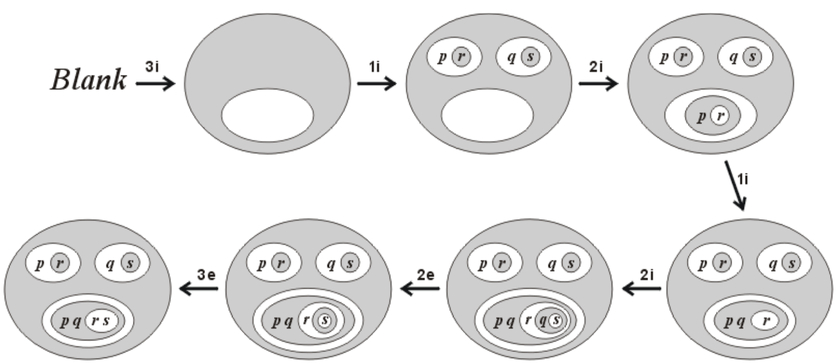

Figure 5. Diagrammatic proof that If *p* implies *r* and *q* implies *s*, then *p* and *q* implies *r* and *s*

36 GROUPE µ 1992, 328.

Diagrammatic proofs are not necessarily static. With some ordering of its parts, a diagram can exhibit dynamic processes of transformations. As shown in Peirce's proof (Fig. 5), and in Bolzano's (Fig. 2), division and the ordering of a diagram into sub-diagrams allow the exhibition of the ideal (logical or mathematical) relations the proof is about (in each sub-diagram) and also their successive transformations (in the juxtaposition of slightly different forms).

As Groupe μ point out, a «homeo-semiotic sequence» (i.e., the alignment of figures which are close in size and shape and are seen from the same viewpoint) suggests a transformative process. As Fontanille and Dondero note, however, the focus can also be put on specific transformations by varying the perspective on the figures:

> The homo-semiotic sequences express, when they apply to the same object: either a narrative transformation, if the point of view is the same, or a rhetorical-persuasive function, if the point of view changes.[37]

This narrative and rhetorical dimension of the diagram is what seems to be missing from Wolf's visual proof of Viviani's theorem. The diagram shows the last form but does not clearly show the transformations through which it has been constructed. Of course, the previous steps of the construction are displayed within the last diagram. They are, however, «hidden» by the lack of a clear presentation of this last diagram's genesis, i.e. of the successive transformations that led to it. In a teaching context, the last diagram is usually constructed progressively so that the students can see each step as well as each transformation that leads to the next step. Here the exhibition of the final diagram is supposed to summarise the whole process, but it takes a skilful receiver to «extract» the whole genesis from the final (static) picture and find the proof in it. The same is true of the visual proof of the sum of the angles in a triangle (Fig. 1), even if it is easier in that case to see how the diagram has been constructed

37 DONDERO AND FONTANILLE 2014, 118.

and to be convinced by its reconstruction.

3. Animated diagrams versus juxtaposed sub-diagrams

Several solutions have been worked out that try to compensate for the above problem. The first solution is to juxtapose the steps of the proof next to (or below) each other, just as in the case of Peirce's or Bolzano's proofs. The proof then consists of a supra-diagram made up of sub-diagrams exhibiting different phases of the figure's transformation. The chronological order is thus spatialised.

The second solution is to deliver animated proofs, as is now common in tutorials[38]. The dynamic nature of the proof is then expressed by the actual chronological sequence of pictures: transformations are literally displayed by being performed in time. The proof is explicitly temporal; rather than a single diagram, it consists in a temporal sequence of diagrams.

The evidentiary force of two solutions can be compared. This evidentiary force is what Daniel Archambault and Helen Purchase measure concerning subjects' cognitive grasp of a graph's evolution.[39] According to their experimental results, it appears that animated graphs are more «cognitively efficient» than «small multiples» (the same graph at various states in separate windows) only when «pictural stability» is weak, i.e. when lots of «nodes» are moving along the transformation(s). When pictural stability is strong (i.e., when the sequence is homeo-semiotic), conversely, the alignment of figures in one and the same diagram is often more cognitively efficient than an animated graph in which those figures appear consecutively.

It is easy to see the reason for these results. Although it can be useful for complex transformations with viewpoint changes to be performed before the eyes, this kind of dynamic change has the disadvantage that the initial state has disappeared by the time the final state appears.

38 e.g., for Viviani's theorem: https://www.geogebra.org/m/nhn6u7mv
39 ARCHAMBAULT & PURCHASE 2016.

However, for the transformation's regularity to be checked, the initial and final states must be carefully compared. Such a comparison requires that they both be visually present at the same time, so that the eye can go from one to the other one in order to spot similarities and modifications. This is the case in Bolzano's proof (Fig. 2) and Peirce's proof (Fig. 5).

In the case of geometrical proofs through the construction of figures, such as the proof of the sum of the angles in a triangle (Fig. 1) or the proof of Viviani's theorem (Fig. 3), animated graphs could be efficient in so far as each step of the construction stays «present» in the next step(s) and the animation precisely reveals how these steps follow from each other. For other geometrical proofs, however, the alignment of steps in one and the same picture is likely to be more cognitively efficient than an animated graph. For instance, the following proof of Pythagoras' theorem is very efficient:

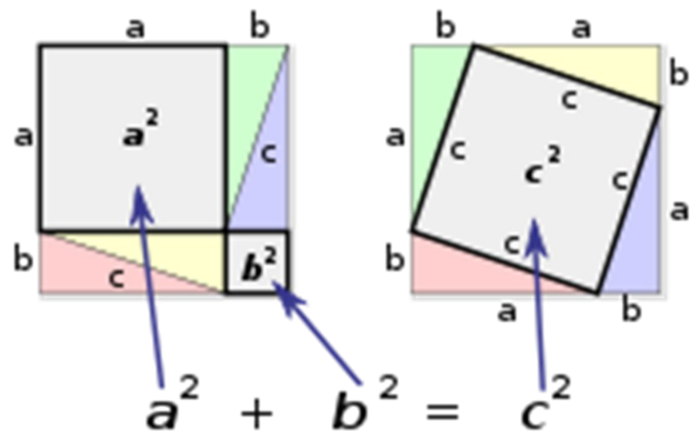

Figure 6. Diagrammatic proof of Pythagoras theorem

Animated versions of this proof exist, but it is not obvious that they make it more convincing.[40] The reason is once again that the convincing force partly rests on the ability to move visually from one figure to the other to check the transformations' regularity.

Similarly, the cogency of an animated proof of Viviani's theorem is not obviously greater than the following diagram, in which arrows highlight transformations between sub-diagrams:

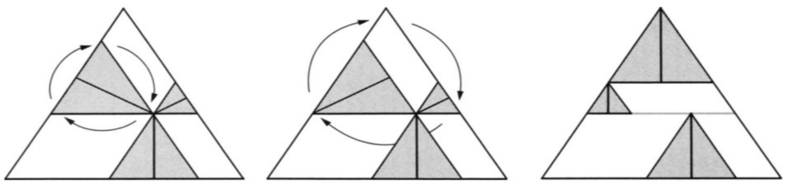

Figure 7. Kawasaki's Proof Without Words of Viviani's Theorem

Static pictures do not prevent their being read dynamically. Proofs can therefore be both diagrammatic and anchored in time.

4. Diagrams and proving gestures

If temporality is not the problem, can diagrams really express all the speech acts—assertions, inferences, declarations, directives, etc.—of which proofs are made? This is a difficult and highly debated question

40 See for instance: http://www.davis-inc.com/pythagor/
https://en.wikipedia.org/wiki/File:Pythagoras-2a.gif
https://giphy.com/gifs/mathematics-proof-pythagorean-theorem-RbOUwWPBinvFe
https://giphy.com/gifs/wolfram-research-pythagorean-theorem-l41JGHqVSThnCbSOA

in visual semiotics. Many would like to claim that diagrams cannot perform such speech acts «on their own». By themselves, diagrams can exhibit formal relations and transformations, but they cannot assert the truth of these relations or validate these transformations, let alone lay down their own underlying principles or command their own active interpretation.

However, this raises the question of the meaning of «by themselves» and «on their own» in this context. Does this mean that diagrams need to be accompanied by linguistic phrases that let them perform these speech acts? Not necessarily. Just as with temporality, what makes diagrams able to perform such «gestures» is mostly a question of use. Some habits of reading pictures, gained through education and practice, make us read them as temporal sequences. Likewise, some habits make us—or trained mathematicians—read them as asserting claims or validating transformations.

As Peirce stresses, decoding visual proofs surely requires some «familiarization»:

> Geometrical schemata are linear figures with letters attached; the perfect imaginability, on the one hand, and the extreme familiarity, on the other hand, of spatial relations are taken advantage of, to enable us to see what will necessarily be true under supposed conditions. The algebraical schemata are arrays of characters, sometimes in series, sometimes in blocks, with which are associated certain rules of permissible transformation. With these rules the algebraist has perfectly to familiarize himself. By virtue of these rules, become habits of association, when one array has been written or assumed to be permissibly scriptible, the mathematician just as directly perceives that another array is permissibly scriptible, as he perceives that a person talking in a certain tone is angry, or [is] using certain words in such and such a sense.[41]

If familiarization is all that is required, however, then diagrams can do

41 *CP*, vol. 4, § 246. See also *CP*, vol. 4, § 368.

the job «on their own» for people who have been trained to use them for the purpose of proving.

When J.L. Austin stressed that linguistic sentences can be used to perform various speech acts, he was aware that they did not do it 'by themselves' but only through some social training.[42] Why, then, would the situation be different for diagrams?

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42 Austin 1962.

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