Expansions in Cantor real bases and alternate bases:
Combinatorial, algebraic and ergodic properties

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Outline

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1. Definitions

2. Combinatorial properties
   - Combinatorial properties of Cantor real base expansions
   - More combinatorial properties of alternate base expansions

3. Algebraic properties of alternate base expansions

4. Ergodic properties of alternate base expansions

Integer base representations:

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{b^{n+1}}$$

with $b \in \mathbb{N}_{\geq 2}$

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with $\beta \in \mathbb{R}_{>1}$

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Cantor real base representations:

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with $\beta_0, \beta_1, \ldots \in \mathbb{R}_{>1}$

Cantor real bases

A ***Cantor (real) base*** is a sequence $\beta = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$.

A ***β-representation*** of a non-negative real number x is an infinite word $a_0 a_1 \cdots \in \mathbb{N}^{\mathbb{N}}$ such that

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \frac{a_2}{\beta_0\beta_1\beta_2} + \cdots$$
$$= \sum_{n\in\mathbb{N}} \frac{a_n}{\beta_0\cdots\beta_n}.$$

Example

In base $\beta = (2, \varphi, \varphi^2, \varphi^2, \varphi^2, \cdots)$ with $\varphi = \frac{1+\sqrt{5}}{2}$, the sequence 1110^ω has value

$$\frac{1}{2} + \frac{1}{2\varphi} + \frac{1}{2\varphi^3} \simeq 0.93.$$

Greedy expansions

For $x \in [0,1]$, a distinguished $\boldsymbol{\beta}$-representation $a_0 a_1 a_2 \cdots$ is computed thanks to the ***greedy algorithm***:

- $a_0 = \lfloor \beta_0 x \rfloor$ and $r_0 = \beta_0 x - a_0$
- $a_n = \lfloor \beta_n r_{n-1} \rfloor$ and $r_n = \beta_n r_{n-1} - a_n$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation of x is denoted by $d_{\boldsymbol{\beta}}(x)$ and is called the ***(greedy) $\boldsymbol{\beta}$-expansion*** of x.

Notation:
For all $n \in \mathbb{N}$,

$$\boldsymbol{\beta}^{(n)} = (\beta_n, \beta_{n+1}, \ldots).$$

Example

Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$. Consider $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ defined by

$$\beta_n = \begin{cases} \alpha & \text{if } |\mathrm{rep}_2(n)|_1 \equiv 0 \pmod 2 \\ \beta & \text{otherwise} \end{cases}, \quad \forall n \in \mathbb{N}$$

We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) =$

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$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$
- $a_0 = \lfloor x\beta_0 \rfloor$ and $r_0 = x\beta_0 - a_0$
- $a_n = \lfloor r_{n-1}\beta_n \rfloor$ and $r_n = r_{n-1}\beta_n - a_n$

Let's compute $d_{\boldsymbol{\beta}}(1)$.

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We get $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ and $d_{\boldsymbol{\beta}}(1) = 2$

$d_{\boldsymbol{\beta}}(x) = (a_n)_{n\in\mathbb{N}}$

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- $a_0 = \lfloor \alpha \rfloor = 2, \quad r_0 = \alpha - 2 \simeq 0.30$
- $a_1 = \lfloor (\alpha - 2)\,\beta \rfloor = 0, \quad r_1 = (\alpha - 2)\,\beta \simeq 0.43$

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$\vdots$

Alternate bases

We call an ***alternate base*** a periodic Cantor base

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots).$$

In this case we simply write

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$$

and the integer p is called the ***length*** of $\boldsymbol{\beta}$.

Example

Let $\beta = (3, \varphi, \varphi)$. We obtain

$$\beta = (3, \varphi, \varphi), \quad \beta^{(1)} = (\varphi, \varphi, 3), \quad \beta^{(2)} = (\varphi, 3, \varphi)$$

and

- $d_\beta(1) = 30^\omega$
- $d_{\beta^{(1)}}(1) = 110^\omega$
- $d_{\beta^{(2)}}(1) = ??$

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- $a_0 = \lfloor \varphi \rfloor = 1, \quad r_0 = \varphi - 1 = \frac{1}{\varphi}$
- $a_1 = \left\lfloor \frac{1}{\varphi} 3 \right\rfloor = 1, \quad r_1 = \frac{3}{\varphi} - 1$

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Let $\beta = (3, \varphi, \varphi)$. We obtain

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$\vdots$

Example

Let $\beta = (3, \varphi, \varphi)$. We obtain

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A note on lazy expansions

All this presentation is about greedy β-expansions, however in my thesis, I also study ***lazy*** $\boldsymbol{\beta}$***-expansions***.
I link them with the greedy expansions and I prove analogue results.

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Combinatorial properties of expansions in Cantor real bases

From now on, let β be a Cantor base.

Proposition (Charlier & C. 2021)

The β-expansion of a real number $x \in [0,1]$ is lexicographically maximal among all β-representations of x.

Proposition (Charlier & C. 2021)

The function d_β is increasing:

$$\forall x, y \in [0,1], x < y \Leftrightarrow d_\beta(x) <_{lex} d_\beta(y)$$

Definition of the quasi-greedy expansion of 1

The **quasi-greedy β-expansion of** 1 denoted $d^*_\beta(1)$ is defined recursively as follows:

$$d^*_\beta(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite} \\ a_0 \cdots a_{\ell-2}(a_{\ell-1}-1)d^*_{\beta^{(\ell)}}(1) & \text{if } d_\beta(1) = a_0 \cdots \underbrace{a_{\ell-1}}_{\neq 0} 0^\omega. \end{cases}$$

Proposition (Charlier & C. 2021)

$d^*_\beta(1)$ is the greatest of all the **infinite** β-representations of 1 with respect to the lexicographic order.

Example

Let $\beta = (3, \varphi, \varphi)$,

- $d_\beta(1) = 30^\omega$ and $d^*_\beta(1) =??$
- $d_{\beta^{(1)}}(1) = 110^\omega$ and $d^*_{\beta^{(1)}}(1) =??$
- $d_{\beta^{(2)}}(1) = 1(110)^\omega = d^*_{\beta^{(2)}}(1)$.

$d^*_\beta(1) =$

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$$d^*_\beta(1) = 2\underbrace{}$$

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$$d^*_\beta(1) = 2\underbrace{\qquad\qquad\qquad}$$

$$d^*_{\beta^{(1)}}(1) = 10\underbrace{\qquad\qquad\qquad}$$

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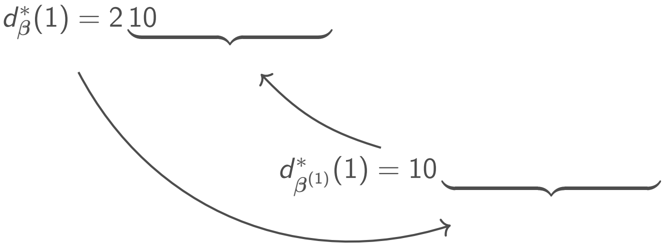

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$$d^*_\beta(1) = 2\,\underbrace{10\qquad\qquad}$$

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- $d_{\beta^{(2)}}(1) = 1(110)^\omega = d^*_{\beta^{(2)}}(1)$.

$$d^*_\beta(1) = 2\,\underbrace{10210\cdots}_{} = (210)^\omega$$

$$d^*_{\beta^{(1)}}(1) = 10\,\underbrace{210210\cdots}_{} = (102)^\omega$$

Generalization of Parry's theorem in Cantor bases

We define $D_\beta = \{d_\beta(x) : x \in [0,1)\}$.

Theorem (Charlier & C. 2021)

An infinite word a over $\mathbb{N}$ belongs to D_β if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) <_{lex} d^*_{\beta^{(n)}}(1)$.

Example

Let $\beta = (3, \varphi, \varphi)$.

- $d^*_\beta(1) = (210)^\omega$
- $d^*_{\beta^{(1)}}(1) = (102)^\omega$
- $d^*_{\beta^{(2)}}(1) = 1(110)^\omega$

The sequence $a = 210000100(110)^\omega$ belongs to D_β because

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Example

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- $d^*_\beta(1) = 210\underline{2}10210210\cdots$
- $d^*_{\beta^{(1)}}(1) = 102102102102\cdots$
- $d^*_{\beta^{(2)}}(1) = 110110110110\cdots$

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The sequence $a = 210000100(110)^\omega$ belongs to D_β because

$$a = 210000\underline{1}00110110110110\cdots$$

β-shift

Let S_β denote the topological closure of D_β:

$$S_\beta = \overline{D_\beta}.$$

Proposition (Charlier & C. 2021)

An infinite word a over $\mathbb{N}$ belongs to S_β if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) \leq_{\text{lex}} d^*_{\beta^{(n)}}(1)$.

However, S_β is not shift-invariant.

Corollary (Charlier & C. 2021)

For all $a \in S_\beta$, we have $\sigma(a) \in S_{\beta^{(1)}}$.

We set

$$\Delta_\beta = \bigcup_{n\in\mathbb{N}} D_{\beta^{(n)}} \quad \text{and} \quad \Sigma_\beta = \overline{\Delta_\beta}.$$

Proposition (Charlier & C. 2021)

The sets Δ_β and Σ_β are both shift-invariant and we have

$$\mathrm{Fac}(D_\beta) = \mathrm{Fac}(S_\beta) = \mathrm{Fac}(\Delta_\beta) = \mathrm{Fac}(\Sigma_\beta).$$

We call Σ_β the β-***shift***.

Outline

More combinatorial properties of alternate base expansions

From now on, we consider an alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$.

Proposition (Charlier & C. 2021)

The $\boldsymbol{\beta}$-expansion of 1 is never purely periodic.

The $\boldsymbol{\beta}$-expansion of 1 can be ultimately periodic with a period q such that p and q are coprime.

Example

Let $\boldsymbol{\beta} = (\sqrt{6}, 3, \frac{1}{3}(2+\sqrt{6}))$, we have $d_{\boldsymbol{\beta}}(1) = 2(10)^{\omega}$.

The quasi-greedy expansion of 1

At most p steps are needed in order to construct $d^*_\beta(1)$:

- either within the first p recursive calls we reach an infinite greedy expansion
- or these first p steps completely determine $d^*_\beta(1)$.

Proposition (Charlier & C. 2021)

The quasi-greedy expansion $d^*_\beta(1)$ is ultimately periodic if and only if, within the first p recursive calls to the definition of the quasi-greedies, either an infinite ultimately periodic greedy expansion is reached or only finite greedy expansions are involved.

A corollary of the generalization of Parry's theorem

Proposition (Charlier & C. 2021)

A β-representation a of 1 is the greedy β-expansion of 1 if and only if for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1, p-1]\!]$, $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\beta^{(i)}}(1)$.

Remark:

When $p \geq 2$, this does not provide us a purely combinatorial condition. But, a purely combinatorial condition cannot exist.

Example

Consider $a = 2(10)^\omega$. Then $\text{val}_{\boldsymbol{\alpha}}(a) = \text{val}_{\boldsymbol{\beta}}(a) = 1$ for both $\boldsymbol{\alpha} = (1+\varphi, 2)$ and $\boldsymbol{\beta} = (\frac{31}{10}, \frac{420}{341})$ but $d_{\boldsymbol{\alpha}}(1) = a$ and $d_{\boldsymbol{\beta}}(1) \neq a$.

A generalization of Bertrand-Mathis' theorem

An alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ is a ***Parry alternate base*** if $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic for all $i \in [\![0, p-1]\!]$.

Theorem (Charlier & C. 2021)

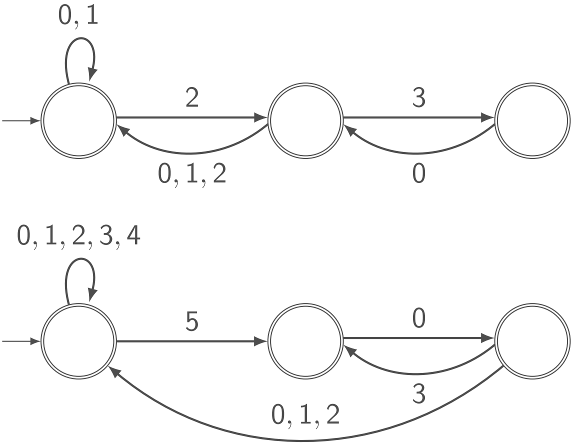
The $\boldsymbol{\beta}$-shift $\Sigma_{\boldsymbol{\beta}}$ is sofic if and only if $\boldsymbol{\beta}$ is a Parry alternate base.

Sketch of the proof:

- $\Leftarrow$ Use p automata (one for each expansion) with the same idea as in the β-shift where the states are duplicated p times to remember the index modulo p of the letter we are reading.
- $\Rightarrow$ More difficult.

Example

Let $\beta = (\varphi^2, 2\varphi^2)$. We have $d_\beta(1) = 2(30)^\omega$ and $d_{\beta^{(1)}}(1) = 5(03)^\omega$.



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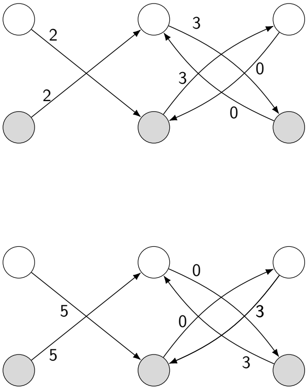

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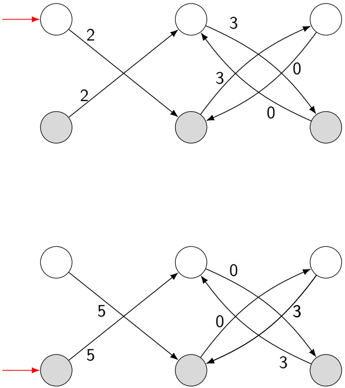

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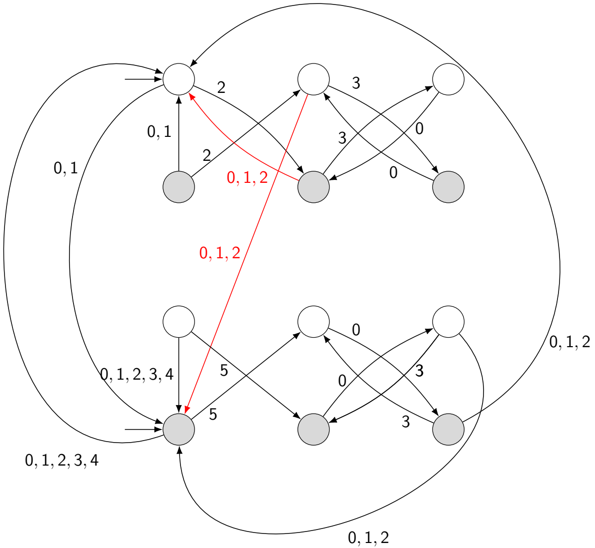


Remark:
If $d_{\beta^{(i)}}(1)$ is finite for all $i \in [\![0, p-1]\!]$, we do not necessarily have that Σ_β is of finite type.

Example

Let $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(1) = 2010^\omega$$
$$d_{\beta^{(1)}}(1) = 110^\omega$$

and

$$d^*_\beta(1) = 200(10)^\omega$$
$$d^*_{\beta^{(1)}}(1) = (10)^\omega.$$

So, all words in $2(00)^*2$ are avoided by Σ_β.

Outline

Alternate alphabet

An ***alternate alphabet of length*** p is a sequence

$$\boldsymbol{D} = (D_0, \ldots, D_{p-1}, D_0, \ldots, D_{p-1}, \ldots)$$

where $D_0, \ldots, D_{p-1}$ are finite alphabets of integers containing 0.

A word $a_0 a_1 a_2 \cdots$ is ***written over*** $\boldsymbol{D}$ if for all $n \in \mathbb{N}$, $a_n \in D_n$.

Normalization

The ***normalization function***

$$\nu_{\beta,\boldsymbol{D}} \colon (\cup_{i=0}^{p-1} D_i)^{\mathbb{N}} \to (\cup_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!])^{\mathbb{N}}$$

is the partial function mapping any $\boldsymbol{\beta}$-representation written over $\boldsymbol{D}$ of a real number $x \in [0, 1)$ to the greedy $\boldsymbol{\beta}$-expansion of x.

We say that $\nu_{\beta,\boldsymbol{D}}$ is ***computable by a finite Büchi automaton*** if there exists a finite Büchi automaton accepting the set

$$\{(u, v) : \ \mathrm{val}_{\boldsymbol{\beta}}(u) \in [0, 1) \text{ and } v = \nu_{\beta,\boldsymbol{D}}(u)\}.$$

Such a Büchi automaton is called a ***normalizer*** in base β over $\boldsymbol{D}$.

Algebraicity of Parry alternate bases

From now on, we consider an alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ and we let $\delta = \prod_{i=0}^{p-1} \beta_i$.

Theorem (Charlier, C., Masáková & Pelantová 2022)

If $\boldsymbol{\beta}$ is a Parry alternate base, then δ is an algebraic integer and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$.

Theorem* (Charlier, C., Masáková & Pelantová 2022)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\boldsymbol{\beta}$ is a Parry alternate base.

* Proven by using properties of an ***alternate base spectrum***.

Remarks:

- The product δ being a Pisot number is not *necessary* (already for $p = 1$ since there exist Parry numbers which are not Pisot).

- The product δ being a Pisot number is not *sufficient*.

Example

Consider $\beta = (\sqrt{\delta}, \sqrt{\delta})$ where $\delta \simeq 1.3247\cdots$ is the smallest Pisot number, that is the positive zero of the polynomial $x^3 - x - 1$. The β-expansion of 1 is equal to $d_{\sqrt{\delta}}(1)$, which is aperiodic.

- The bases $\beta_0, \ldots, \beta_{p-1}$ need not be algebraic integers.

Example

Let $\boldsymbol{\beta} = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have $d_{\boldsymbol{\beta}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$. However, $\frac{5+\sqrt{13}}{6}$ is not an algebraic integer.

- For the same non Pisot algebraic integer δ, there may exist a Parry alternate base $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{p-1})$ and a non-Parry alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$ such that

$$\alpha_0, \ldots, \alpha_{p-1}, \beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$$

and

$$\prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i = \delta.$$

A sufficient condition for the normalization

Theorem (Charlier, C., Masáková & Pelantová 2022)

If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$, then the normalization function $\nu_{\boldsymbol{\beta}, \boldsymbol{D}}$ is computable by a finite Büchi automaton.

Sketch of the constructive proof:

- If δ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\boldsymbol{\beta}$ is a Parry alternate base. Hence, we modify the automaton accepting $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ in order to obtain a finite Büchi automaton accepting $D_{\boldsymbol{\beta}}$.
- We define a ***zero automaton*** and a ***converter**** in base $\boldsymbol{\beta}$ over the alternate alphabet $\boldsymbol{D}$.
- Altogether, we have a normalizer.

* Both are proven to be finite thanks to the study of the ***alternate base spectrum***.

Outline

Greedy transformations

Let $\boldsymbol{\beta}$ be a Cantor base. For all $n \in \mathbb{N}$, consider

$$T_{\beta_n} \colon [0,1) \to [0,1), \; x \mapsto \beta_n x - \lfloor \beta_n x \rfloor .$$

For all $x \in [0,1)$, we have

$$a_n = \left\lfloor \beta_n \big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) \big) \right\rfloor .$$

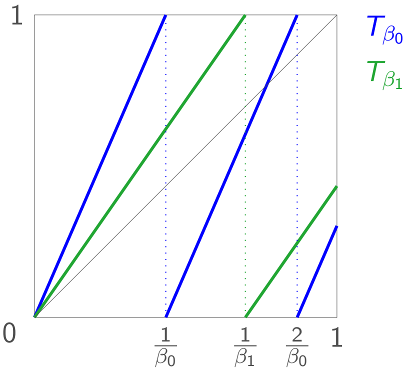
If $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})$, we alternate the p transformations

$$T_{\beta_0}, \ldots, T_{\beta_{p-1}}.$$

Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

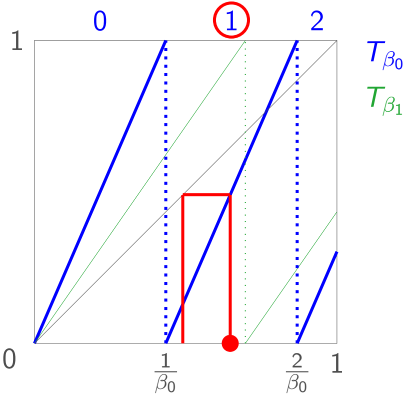
$$d_\beta(\frac{1+\sqrt{5}}{5}) =$$



Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

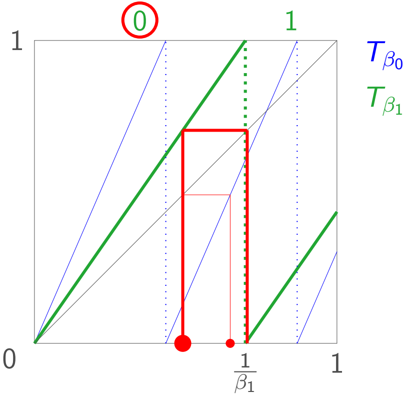
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 1$$



Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

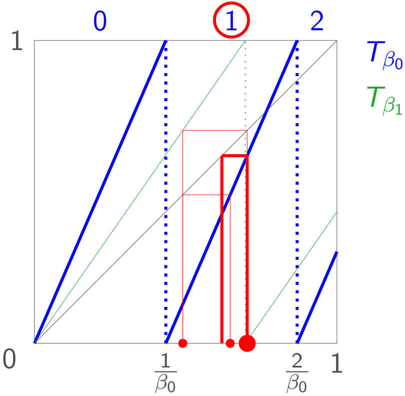
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 10$$



Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

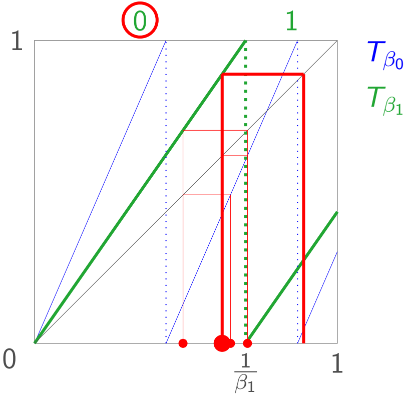
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 101$$



Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

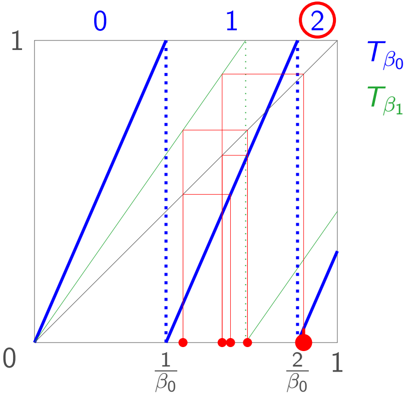
$$d_\beta(\frac{1+\sqrt{5}}{5}) = 1010$$



Example

Consider $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$. We have

$$d_\beta(\tfrac{1+\sqrt{5}}{5}) = 10102\cdots$$



Greedy β-transformation

Consider $\boldsymbol{\beta}=(\beta_0,\ldots,\beta_{p-1})$. We define the $\boldsymbol{\beta}$-***transformation***

$$T_{\boldsymbol{\beta}} : [\![0,p-1]\!] \times [0,1) \to [\![0,p-1]\!] \times [0,1)\,,$$
$$(i,x) \mapsto (i+1 \bmod p,\, T_{\beta_i}(x))\,.$$

Iterating the map $T_{\boldsymbol{\beta}}$, we get

$$T_{\boldsymbol{\beta}}^n(0,x) = (n \bmod p,\ \ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)).$$

So, $d_{\boldsymbol{\beta}}(x)$ is obtained by iterating the transformation $T_{\boldsymbol{\beta}}$ on $(0,x)$.

Dynamical properties of T_β

Consider the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}([0,1)) \right\}$$

over $[\![0, p-1]\!] \times [0,1)$.

We define the ***p*-Lebesgue measure** on $\mathcal{T}_p$ as follows:

$$\lambda_p\left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \lambda(B_i).$$

Theorem (Charlier, C. & Dajani 2021)

There exists a unique T_β-invariant probability measure on $\mathcal{T}_p$ that is absolutely continuous with respect to λ_p. Furthermore, this measure is equivalent to λ_p on $\mathcal{T}_p$ and the associated dynamical system is ergodic and has entropy $\frac{1}{p}\log(\delta)$.

Sketch of the proof:

- For all $i \in [\![0, p-1]\!]$, we consider

$$T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i}$$

and we find a $(T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i})$-invariant probability measure μ_i, equivalent to Lebesgue and such that $T_{\beta_{i+p-1}} \circ \cdots \circ T_{\beta_i}$ is ergodic.
- With the probability measures μ_i we construct a probability measure on $\mathcal{T}_p$ as follows:

$$\mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1} (\{i\} \times B_i) \right) = \frac{1}{p} \sum_{i=0}^{p-1} \mu_i(B_i)$$

- We show the properties of $\mu_{\boldsymbol{\beta}}$ thanks to the ones of μ_i for all $i \in [\![0, p-1]\!]$.

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