LIÈGE université
Sciences Appliquées

Groundwater modeling: flow simulations

A. Dassargues

References:

Dassargues A., 2018. Hydrogeology: groundwater science and engineering, 472p. Taylor & Francis CRC press (chapters 4, 12 & 13)

Dassargues A. 2020. Hydrogéologie appliquée: science et ingénierie des eaux souterraines, 512p. Dunod (chapitres 4, 2 et 13)

Groundwater modeling: flow simulation

- *Groundwater flow equations*
  - *REV concept*
  - *Potentiometric/piezometric head*
  - *Porosities, hydraulic conductivty, Darcy's law, transmissivity*
  - *Steady state equations*
  - *Transient equations*
  - *Extending to partially saturated flow*
- *Flow Boundary Conditions*
  - *Dirichlet BC's*
  - *Neumann BC's*
  - *Cauchy BC's*
- *Introduction to solving methods*
  - *Finite Difference method*
  - *Time integrations schemas*
  - *Finite Difference method (FD): practical recommendations*
  - *Finite Element method: summary*
  - *Finite Volume method: summary*
- *References*

Assumptions and REV concept

→ *REV is the volume of geological medium considered as representative for quantification of the value of the different properties (by averaged, equivalent values)*

→ - *large enough to be relevant with regard to the studied problem (avoid microscopic scale for a real case study)*
- *small enough for avoiding too smoothed values that will corrupt our description of the processes*

… this REV concept implicitly implies that the medium is considered as continuous (and porous)

- *the REV depends on the kind of problem being studied and the study objectives*
- *the REV is used for groundwater flow and solute transport … but also in all other fields where a quantification is needed for properties of the geological medium*

(de Marsily 1986, Dagan 1989, Bear & Verruijt 1987)

Assumptions, REV concept and scale

Scale

Micro ↔ **Fluid dynamic and pore studies**

Homogenisation

Macro ↔ **Lab tests (from dm to a few m)**

Homogenisation

Mega ↔ **Detailed numerical models – Physically consistent simulations of the reservoir**

Homogenisation

Giga ↔ **Balance models, black-box models, transfer functions, etc.**

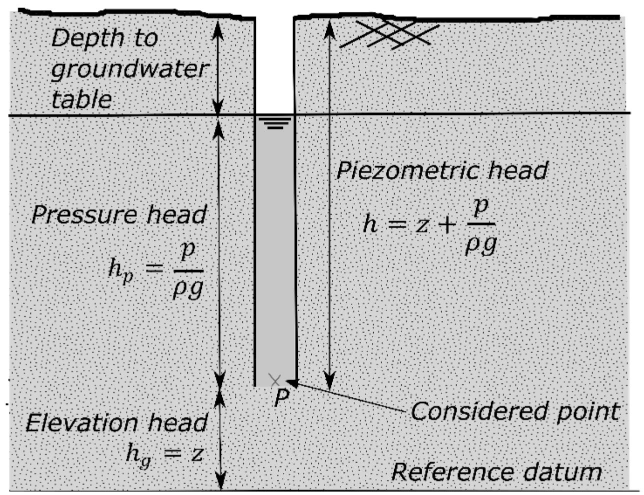
parameter values can be quite different in function of the considered scale

(Dassargues & Monjoie 1993, Dassargues 2018 and 2020, Hoffmann et al. 2021)

Hydrostatic

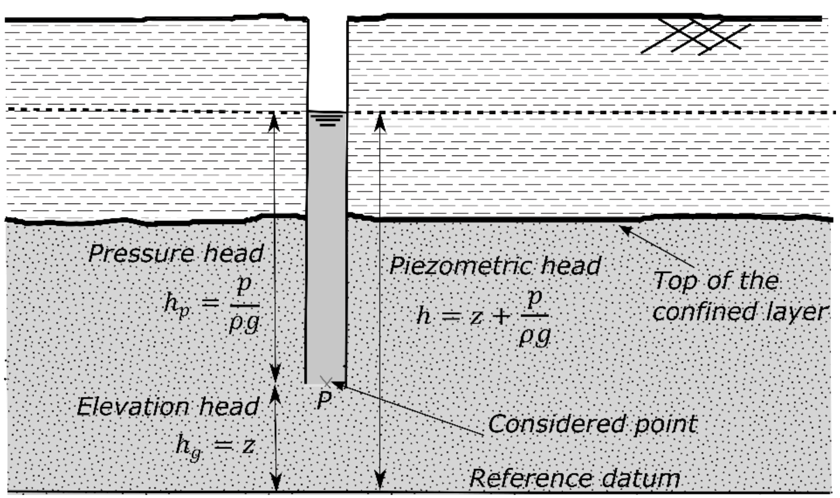
… the potential energy is expressed usually in 'water head' or hydraulic head or piezometric head:

Unconfined conditions



$$h = \frac{\Phi_{tot}}{g} = z + \frac{p}{\rho . g}$$

Confined conditions



Water pressure vs piezometric head

… a direct link between hydraulic/piezometric head h and water pressure p:

$$p = (h - z)\rho.g$$

$$\frac{\partial p}{\partial t} = \rho.g \frac{\partial h}{\partial t}$$

for groundwater flow problems, the main variable is the piezometric head or the water pressure

piezometric heads can be compared only if groundwater has everywhere the same temperature and the same salt content

if it is not the case, … density will vary … and to a same water pressure correspond different piezometric heads (of groundwater with different salt content)

Rmq: if density effect (salwater) is considered work with pressure or with 'equivalent freshwater piezometric heads'

corrections in function of the salinity must be done

(Carabin and Dassargues 1999)

Porosity

… two components in the total porosity:

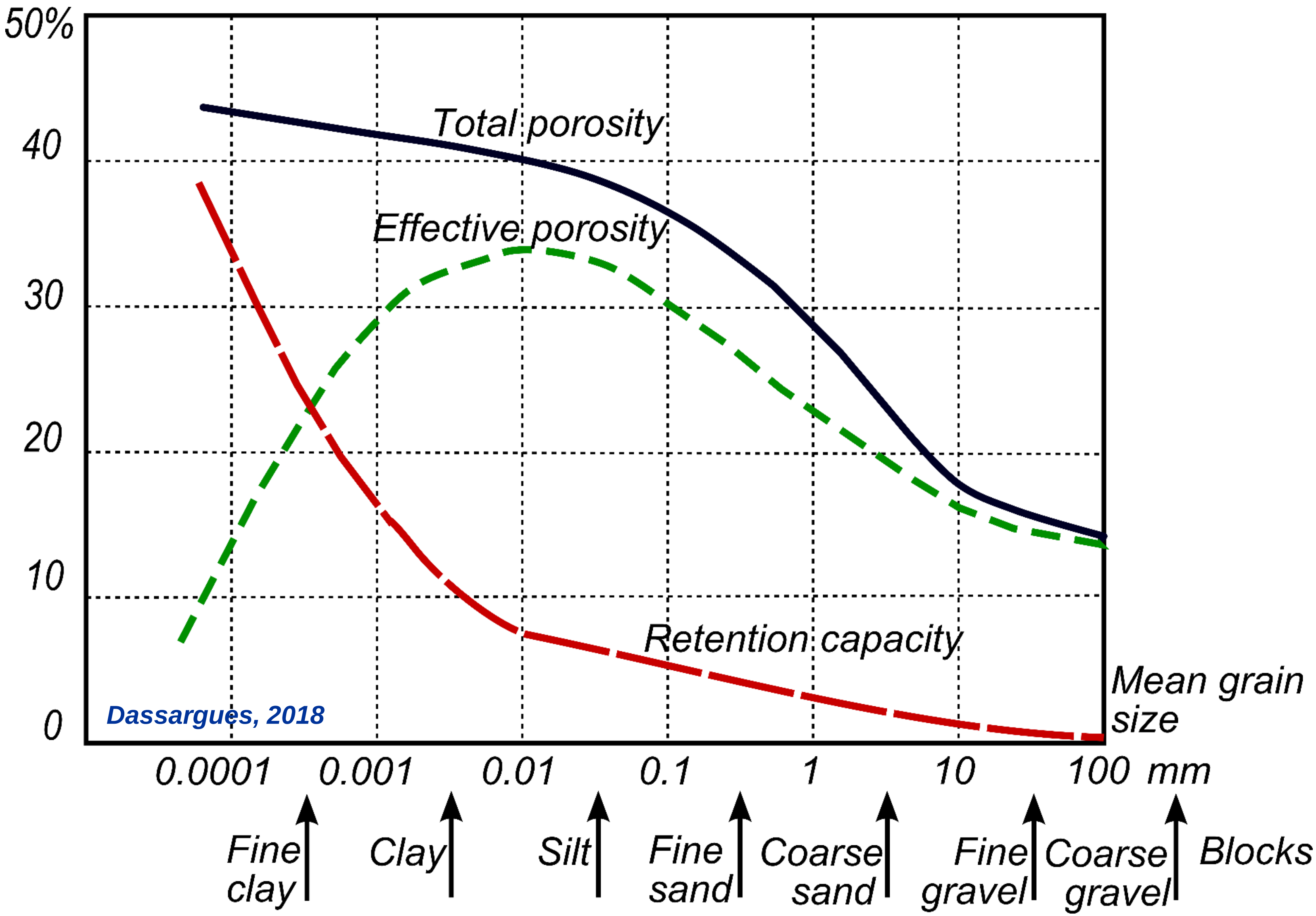
- *cinematic porosity* n_c
- *retention capacity' or 'specific retention')* S_r

$$n_c + S_r = n \qquad \textit{with} \qquad n_c = \frac{V_m}{V_t} \qquad S_r = \frac{V_{im}}{V_t}$$

- *the cinematic porosity is difficult to be measured in practice (… after which duration ? … at which pressure ? …)*

- *drainage porosity*
  = effective porosity $n_e \cong n_c$
  corresponding to the drainable water by gravity
  (also called mobile water or free water)
  …. also called   = "specific yield" S_y

Porosity and granular analysis



(Eckis 1934, Castany 1963)

Hydraulic conductivity and Darcy's law

… experimental law

- *quantity of water per time unit through a porous medium:*

$$Q = K.A.\frac{\Delta h}{L}$$

- K *permeability coefficient, hydraulic conductivity, water permeability (by abuse of language: permeability) of the porous medium (m/s)*

- *… specific flux or flow rate (specific discharge):*

$$q = \frac{Q}{A}$$

in $m^3/(m^2.s)$ … in m/s

Hydraulic conductivity and Darcy's law

This specific discharge is often called 'Darcy's velocity' … it is only a flow rate Q divided by a surface A

this surface is not the groundwater flow section

- *the actual groundwater flow section is :* $A.n_e$

- *… to obtain a mean (averaged/equivalent on the REV) groundwater velocity :*

$$v_e = \frac{q}{n_e} = \frac{K}{n_e} \cdot \frac{\Delta h}{L}$$

m/s

' advection velocity' (effective velocity)

…about mobile water porosities

the 'mobile water porosity' to be considered for groundwater flow is typically higher than the 'mobile water porosity' acting in solute transport processes *(Payne et al. 2008, Hadley and Newell, 2014)*

Box 8.1 About effective transport porosity

Effective transport porosity is in reality very dependent on what we define as mobile and immobile groundwater in the representative volume of geological medium. For many contamination problems and associated risk assessments, it is very important to predict the first arrival time of solute contaminant at a "target." This is the case, for example, for protection zone delineation around a pumping well, for security assessment of waste disposal, for assessment of risk created by industrial contamination hazards, and for groundwater vulnerability mapping. To be conservative (i.e., on the security side) with regards to most practical applications where solute transport is calculated, one should choose to define the effective porosity relevant to solute advection as a relatively small part of the total porosity. This is especially true and important if the geological medium is known as a "dual" porosity medium or is highly heterogeneous at the micro- and macroscales. If the Darcy flux (i.e., specific discharge, see Chapter 4, Section 4.4) would be divided by the total porosity n, this would produce a kind of averaged velocity between the mobile and immobile groundwater in the medium which can be highly misleading (i.e., leading to underestimated velocities). However, some can argue that this could be partially and artificially compensated or corrected by considering larger values of the longitudinal dispersivity.

(Dassargues 2018 and 2020)

→ *useful porosity for solute transport is 'effective transport porosity' < drainage porosity*

Hydraulic conductivity and intrinsic permeability

K *depends on:*

- *fluid properties:*
  - *viscosity*
  - *density*
- *porous medium properties:*
  - *granular proportions,*
  - *grains shapes,*
  - *pore distribution and shapes*
  - *intergranular porosity*

$$K = \frac{k.\rho.g}{\mu}$$

intrinsic permeability or permeability (m^2)

volume mass of the fluid (kg/m^3)

gravity acceleration (m/s^2)

dynamic viscosity ($kg/(m.s)$, $N.s/m^2$ or $Pa.s$

Generalisation of the Darcy's law in 3D

hydraulic conductivity and intrinsic permeability described by tensors:

$$\boldsymbol{K} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{zx} & K_{zy} & K_{zz} \end{bmatrix}$$

and

$$\boldsymbol{K} = \boldsymbol{k}\rho g/\mu$$

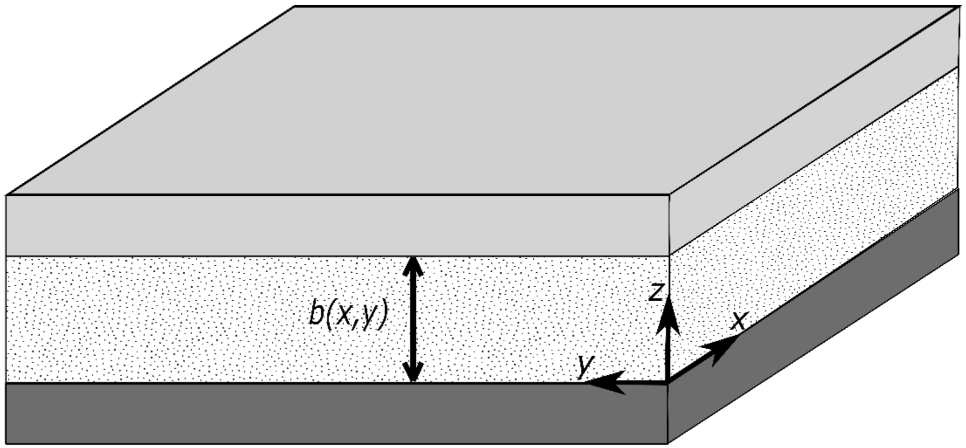
elevation of the considered point (with regards to the reference datum)

$$\boldsymbol{q} = -\boldsymbol{K} \cdot \nabla h = -\frac{\boldsymbol{k}\rho g}{\mu} \cdot \nabla h = -\frac{\boldsymbol{k}}{\mu} \cdot (\nabla p + \rho g \nabla z)$$

$$\boldsymbol{grad}\, h = \nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right)$$

Transmissivity

… for a confined aquifer



(Dassargues, 2018 & 2020)

transmissivity (m^2/s) in a point

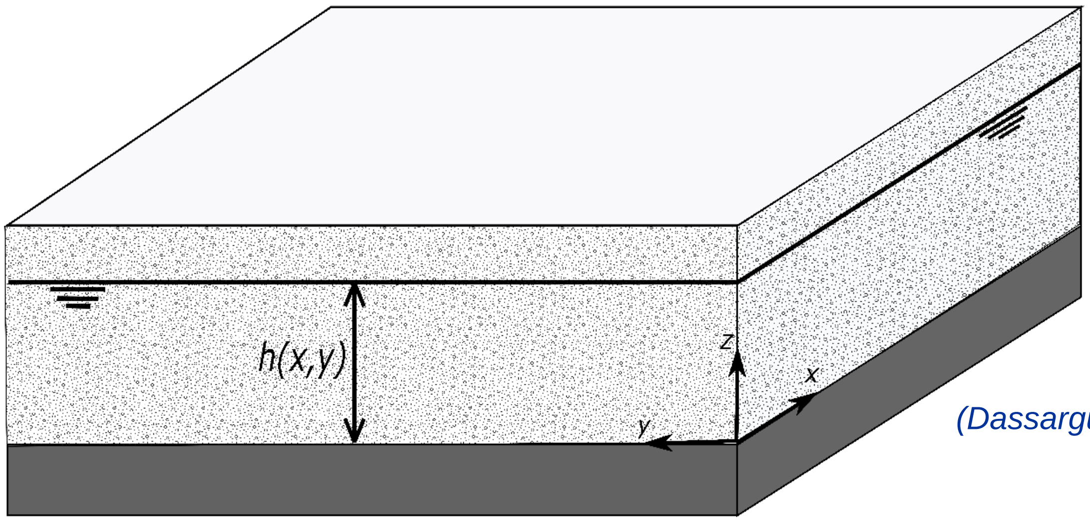
$$T(x,y) = \int_0^{b(x,y))} K(x,y,z)dz = K_{avg}(x,y)b(x,y)$$

thickness of the confined aquifer at the concerned point

mean value of the hydraulic conductivity on the vertical of the concerned point

Transmissivity

… for an unconfined aquifer



(Dassargues, 2018 & 2020)

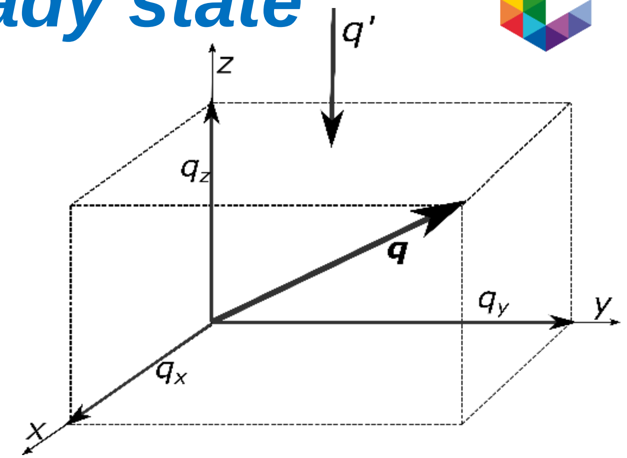
the saturated thickness of the unconfined aquifer at the point (of horizontal coordinates x and y)

depends on the piezometric head!

$$T(x,y) = \int_0^{h(x,y)} K(x,y).dz$$

(Dupuit 1863, Delleur 1999)

Groundwater flow equation in steady state



$$\nabla \cdot (\rho\, \boldsymbol{K} \cdot \nabla h) + \rho q' = 0$$

terms are kg/(m³s)

$$\frac{\partial}{\partial x_i}\left(\rho K_{ij} \frac{\partial h}{\partial x_j}\right) + \rho q'_i = 0$$

in indicial notation

$$\frac{\partial}{\partial x}\left(K_{xx} \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy} \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{zz} \frac{\partial h}{\partial z}\right) + q' = 0$$

if density is assumed constant and the principal anisotropy directions of the K tensor are known and aligned with the selected coordinate system – terms are in s^{-1}

$$\frac{\partial}{\partial x}\left(K_{xx} \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_{zz} \frac{\partial h}{\partial z}\right) + q' = 0$$

if 2D vertical flow, terms are in s^{-1}

$$\frac{\partial}{\partial x}\left(T_{xx} \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_{yy} \frac{\partial h}{\partial y}\right) + q'' = 0$$

if 2D horizontal flow, terms are in m/s

Equation in transient conditions

… in transient flow,
storage variation in function of the time:

$$\frac{\partial (n\,\rho)}{\partial t}$$

specific storage coefficient (m^{-1})

$$S_s = \rho\, g\, \alpha$$

$$\Rightarrow \quad \frac{\partial (n\,\rho)}{\partial t} = \rho^2 g\left(\alpha + \cancel{n\beta_s} + \cancel{n\beta_w}\right)\frac{\partial h}{\partial t} = \rho\, S_s\, \frac{\partial h}{\partial t}$$

Volume (REV) compressibility of the porous medium (Pa^{-1})

Solid grain compressibility (Pa^{-1})

Water compressibility (Pa^{-1})

Specific storage coefficient

… often, the influence of the water compressibility and the solid grain compressibility can be neglected with regards to the volume compressibility of the porous medium (as a whole)

$$S_s = \rho\, g\, \alpha$$

… this link between the volume compressibility and the specific storage coefficient is showing clearly the direct coupling between saturated transient groundwater flow and geomechanical behaviour in compressible porous media

the volume compressibility is dependent on

- *effective stress variation*
- *the effective preconsolidation stress of the porous medium*

Rmq: S_s is actually variable as it depends on compressibility that is dependent on the effective stress state that is dependent on the water pressure,… (Dassargues et al. 1991 and 1993, Dassargues 1995, 1997, 1998)

Equation in transient conditions

Storage coefficient = water volume (m^3) stored or drained per aquifer surface unit (m^2) for a unit variation of piezometric head (m)

… vertical integration

$$S(x,y)=\int_0^{e(x,y)} S_s(x,y).dz$$

confined aquifer

$$S = S_s.e$$

$$S = n_e + \int_{z_1}^{h} S_s.dz$$

unconfined aquifer

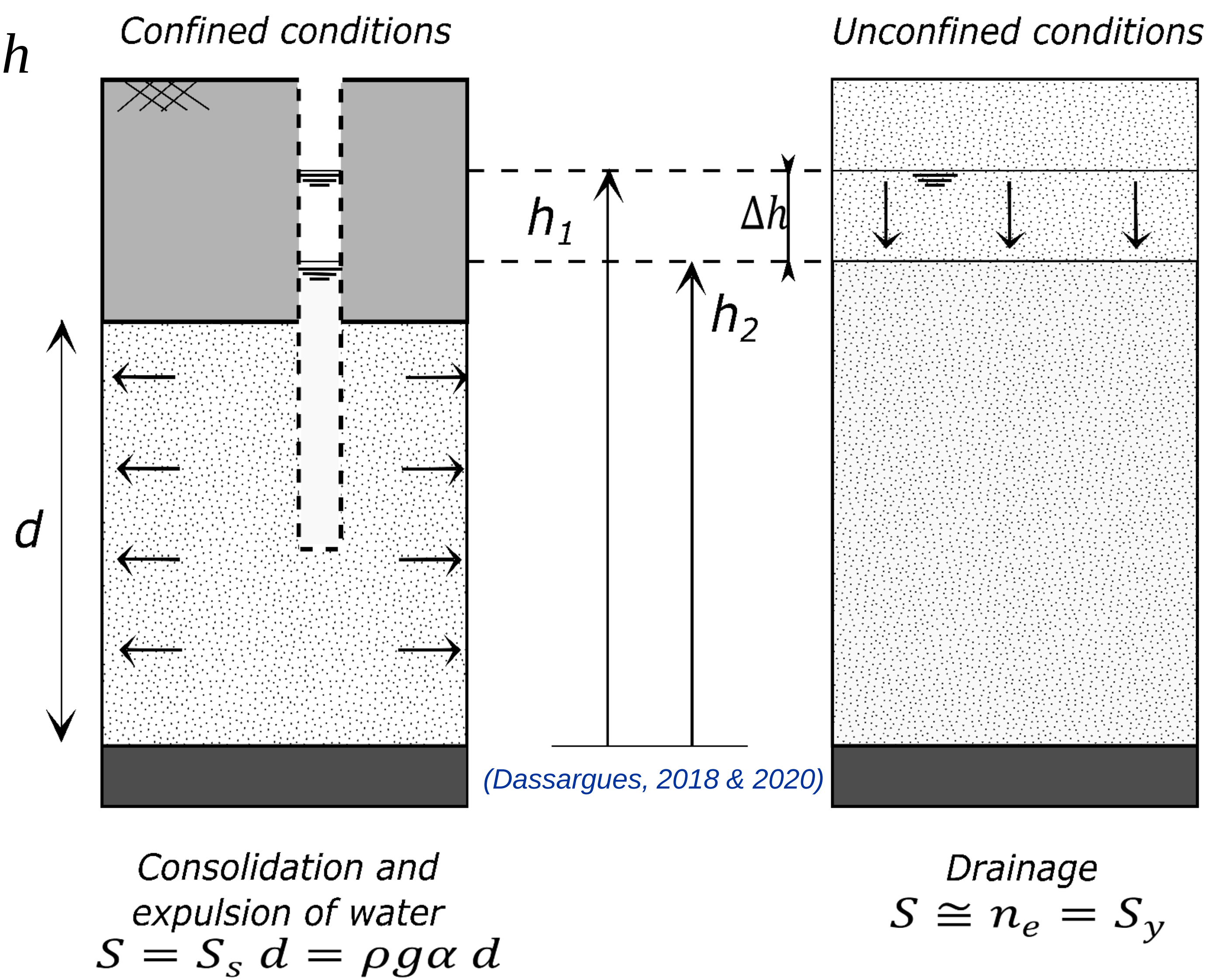
the most important part of the storage is due to saturation/drainage of the porous medium

(Dassargues 1995b, 2018, 2020)

Storage coefficient

$$S = n_e + S_s.h$$

$$S \cong n_e$$



(Dassargues, 2018 & 2020)

reference datum = bottom of the aquifer layer

2D groundwater flow equations in transient conditions (horizontal flow)

confined aquifer

$$\nabla \cdot (\boldsymbol{T} \cdot \nabla h) + q'' = S\frac{\partial h}{\partial t}$$

terms are in m/s

$$\frac{\partial}{\partial x_i}\left(T_{ij}\frac{\partial h}{\partial x_j}\right) + q''_i = S\frac{\partial h}{\partial t}$$

in indicial notation

$$\frac{\partial}{\partial x}\left(T_{xx}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_{yy}\frac{\partial h}{\partial y}\right) + q'' = S\frac{\partial h}{\partial t}$$

principal anisotropy directions aligned with the selected coordinate system

unconfined aquifer

$$\nabla \cdot (\boldsymbol{T}(h) \cdot \nabla h) + q'' = n_e\frac{\partial h}{\partial t} = S_y\frac{\partial h}{\partial t}$$

terms are in m/s

$$\frac{\partial}{\partial x_i}\left(T_{ij}\frac{\partial h}{\partial x_j}\right) + q''_i = n_e\frac{\partial h}{\partial t} = S_y\frac{\partial h}{\partial t}$$

in indicial notation

$$\frac{\partial}{\partial x}\left(T_{xx}\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T_{yy}\frac{\partial h}{\partial y}\right) + q'' = S\frac{\partial h}{\partial t}$$

principal anisotropy directions aligned with the selected coordinate system

Particular case: 2D horizontal confined flow

vertical integration of the 3D equation on the thickness of the confined aquifer

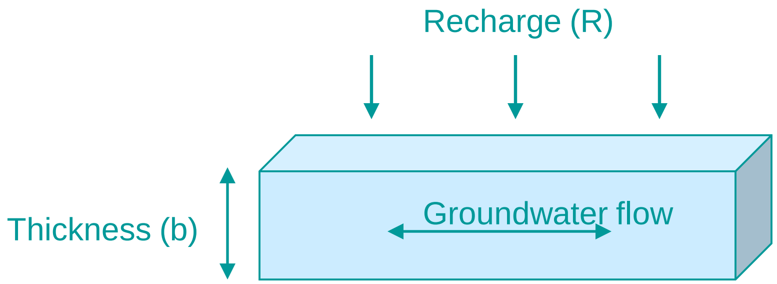
$$\frac{\partial}{\partial x}\left(T_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_{yy}\frac{\partial h}{\partial y}\right)=S\frac{\partial h}{\partial t}-R+L$$

where *T : transmissivity [L^2 T^{-1}], K $\times$ thickness*

S : storage coefficient [-], S_s $\times$ thickness

R : recharge rate [L^3 L^{-2} T^{-1}] = [L T^{-1}]

L : leakance [L T-1]



Particular case: 2D horizontal confined flow

… leakance calculation for a quasi 3D approach

Multi-aquifer system

Leakance (vertical flow) through the aquitard

Darcy's law

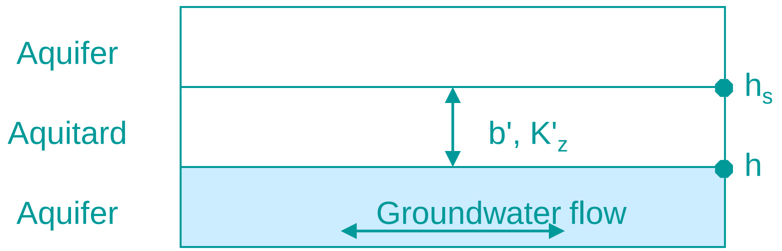
$$L = -\frac{K_z^{'}(h_s - h)}{b^{'}}$$

où *K'_z : vertical hydraulic conductivity of the aquitard*

b' : aquitard thickness

h_s : surperfical aquifer piezometric head

h : main aquifer piezometric head (main variable)



Groundwater flow equations including the partially saturated zone

$$\nabla \cdot \rho\left[\boldsymbol{K}\left(h_p\right) \cdot \nabla h_p + \boldsymbol{K}\left(h_p\right) \cdot \nabla z\right] + \rho q' = \rho C\left(h_p\right) \frac{\partial h_p}{\partial t}$$

(Celia et al. 1990, Dassargues 1997)

with water pressure head as main variable

terms are kg/(m^{3}s)

$$\nabla \cdot \rho\left[\boldsymbol{K}(\theta) \cdot \nabla h_p + \boldsymbol{K}(\theta) \cdot \nabla z\right] + \rho q' = \rho \frac{\partial \theta}{\partial t}$$

(Richards 1931)

in a mixed way as a function of the water content and the pressure head

needs relations between

- *θ and h_p*
- *θ and K*

…

… van Genuchten relations and others

$$p_c/\rho g = -h_p = -p/\rho g = z - h$$

Flow Boundary Conditions

- *Dirichlet conditions: prescribed piezometric head*
- *Neumann conditions: prescribed flux*
- *Cauchy or mixed conditions: flux depending on piezometric head*

Practical examples where groundwater flow BCs are discussed on various practical cases are available from the experience of the author and researchers from his team in the following references:

Dassargues et al. 1988, Dassargues 1991, Stefanescu & Dassargues 1996, Dassargues 1997, Carabin & Dassargues 2000, Brouyère et al. 2004, Peeters et al. 2004, Rojas & Dassargues 2007, Rocha et al. 2007, Rojas et al. 2008, Jusseret et al. 2009, Goderniaux et al. 2009, Brouyère et al. 2009, Rojas et al. 2010, Wildemeersch et al. 2010, Orban et al. 2010, Goderniaux et al. 2011, Wildemeersch et al. 2014, César et al. 2014, Pujades et al. 2016)

Flow BC's

Prescribed piezometric head

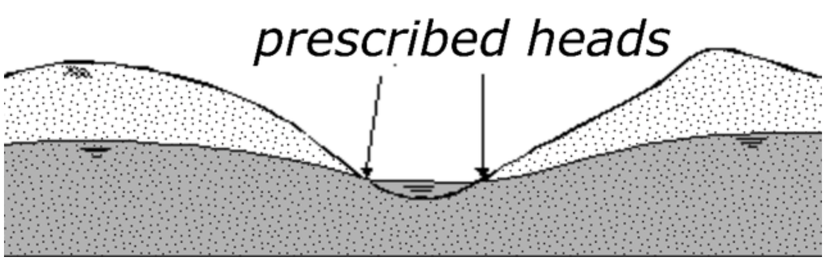
(Dirichlet condition)

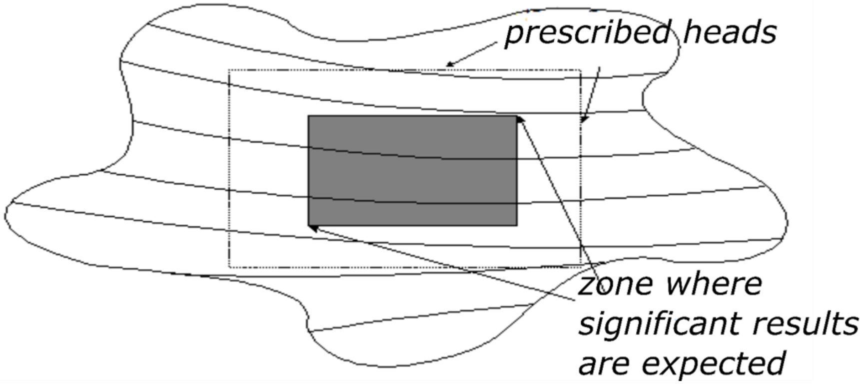
Prescribed piezometric head on the concerned boundary:

$$h(x, y, z, t) = f'(x, y, z, t)$$

f' *can vary in space and time*
(one value per node and per time step)

➡ *a flux will be computed per concerned node*





Flow BC's

Prescribed flux (Neumann condition)

The first derivative of the piezometric head is prescribed on the concerned boundary:

$$\nabla h \cdot \boldsymbol{n} = \frac{\partial h}{\partial n}(x, y, z, t) = f''(x, y, z, t)$$

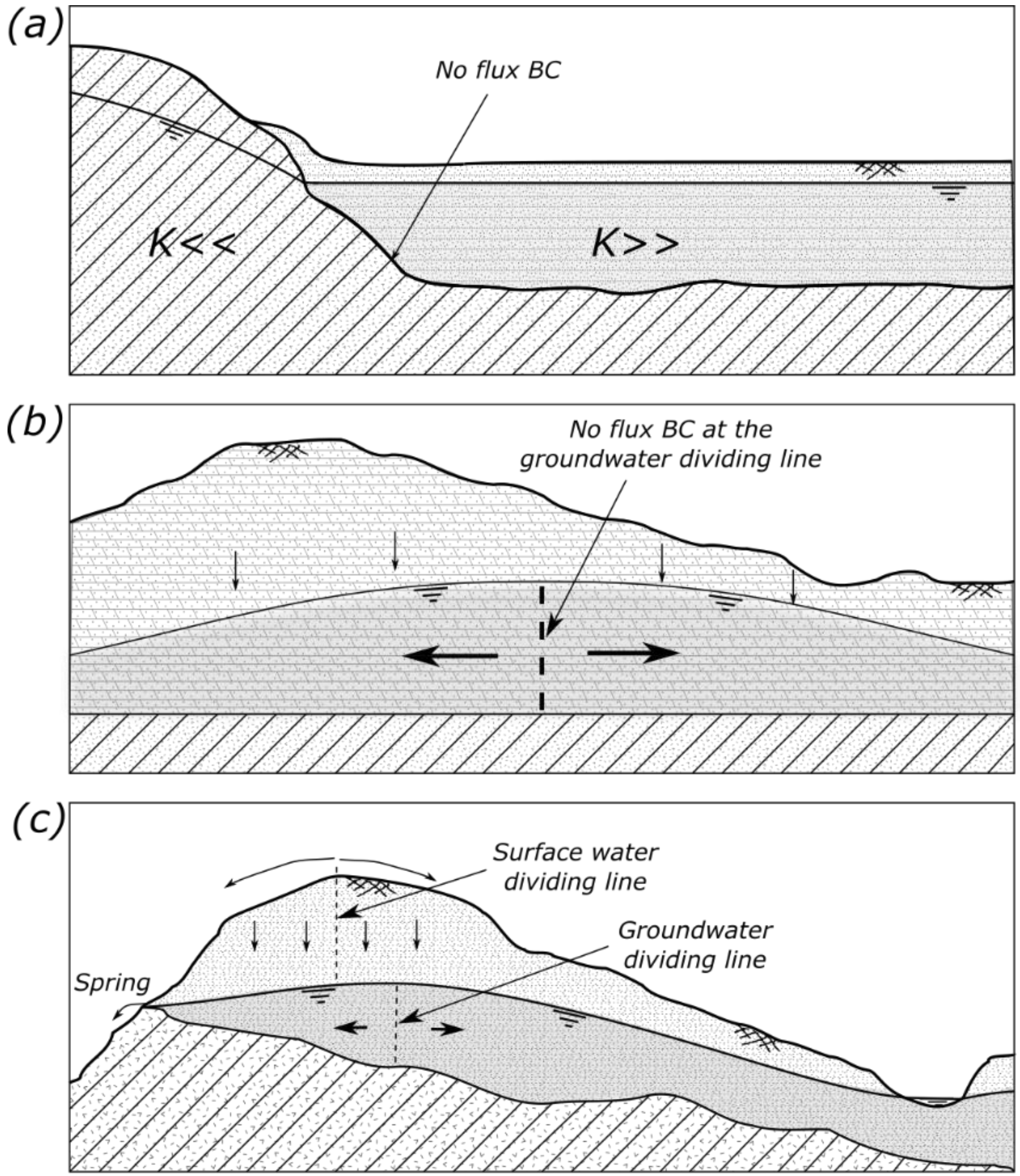
f'' *piezometric gradient normal to the concerned boundary, its value can vary in space and time (one value per concerned node and per time step)*
Applying the Darcy's law, it is a way of prescribing the water flux through the boundary:

$$K \frac{\partial h}{\partial n}(x, y, z, t) = q''(x, y, z, t)$$

q'' *: precribed flux through the boundary (m/s)*

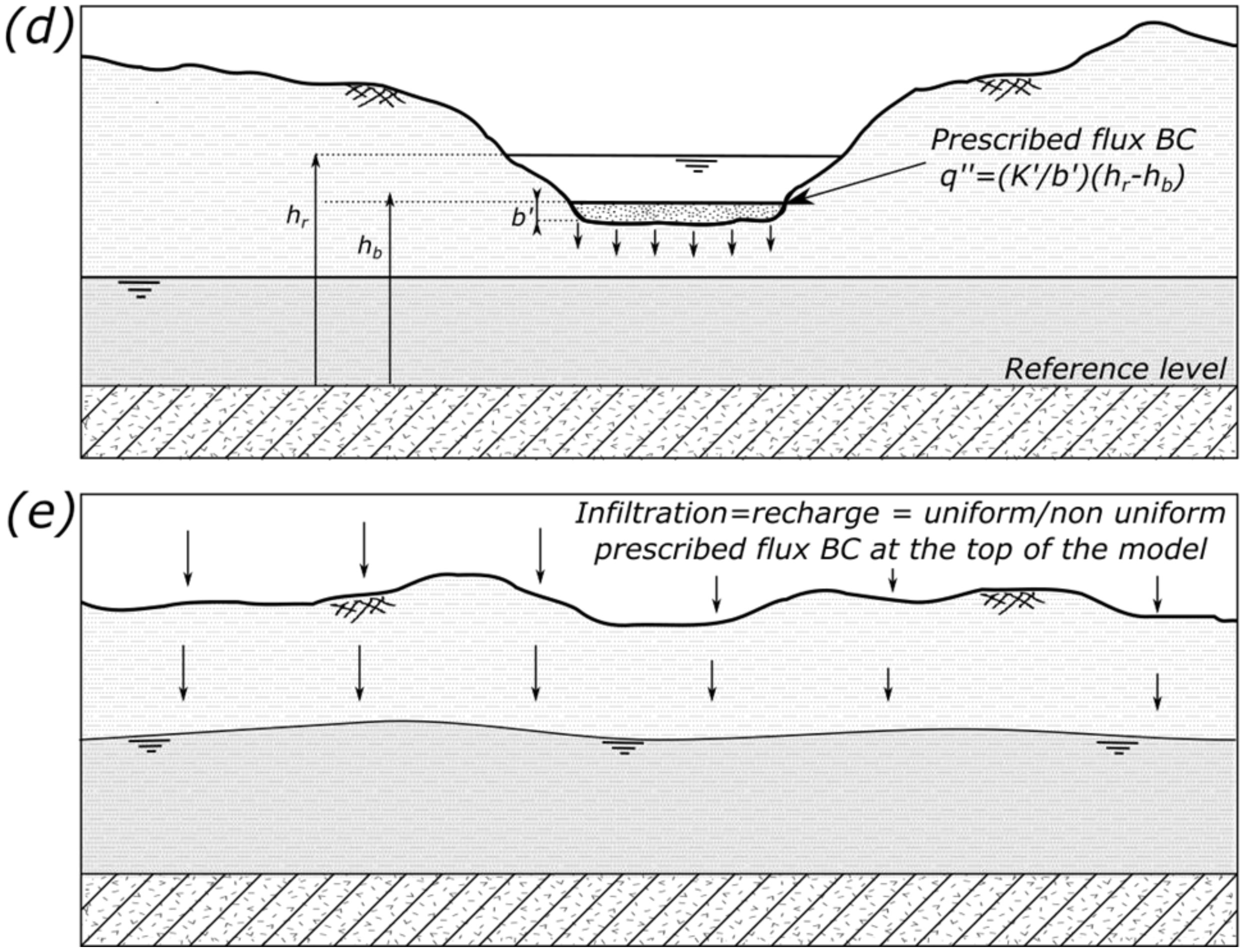
particular case: $f'' = 0$

Flow BC's *Prescribed flux (Neumann condition)*



(Dassargues, 2018)

Flow BC's *Prescribed flux (Neumann condition)*



(Dassargues, 2018)

Flow BC's

Flux depending on the piezometric head
(mixed condition or Cauchy condition)

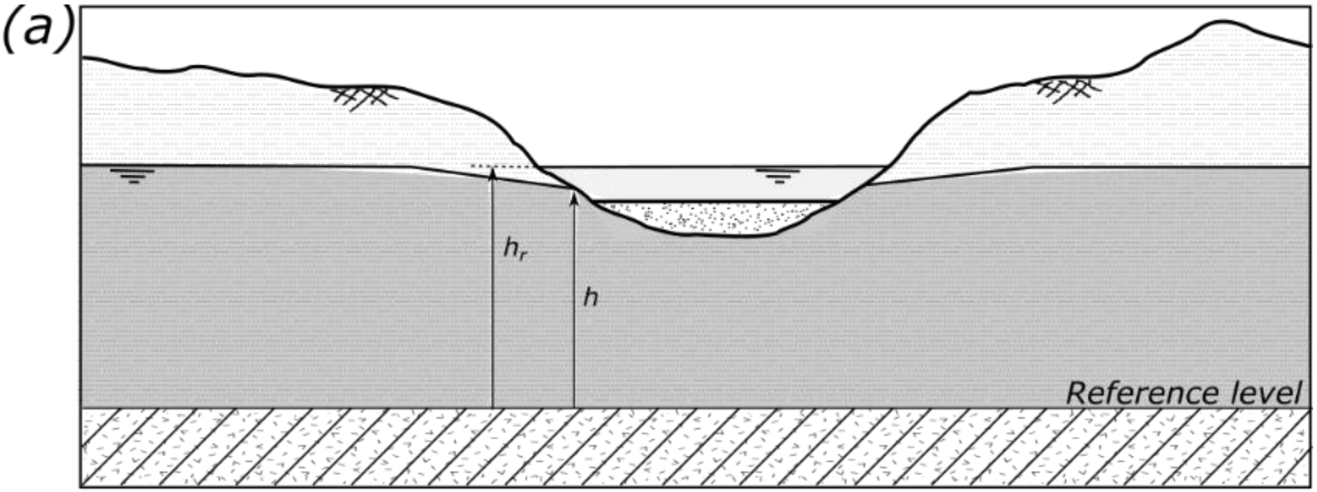
A combination (linear relation) of the piezometric head and its first derivative is prescribed on the boundary:

$$a.\frac{\partial h}{\partial n}(x,y,z,t)+b.h(x,y,z,t)=f'''(x,y,z,t)$$

f''' *can vary in space and in time*
(one value per concerned node and per time step)

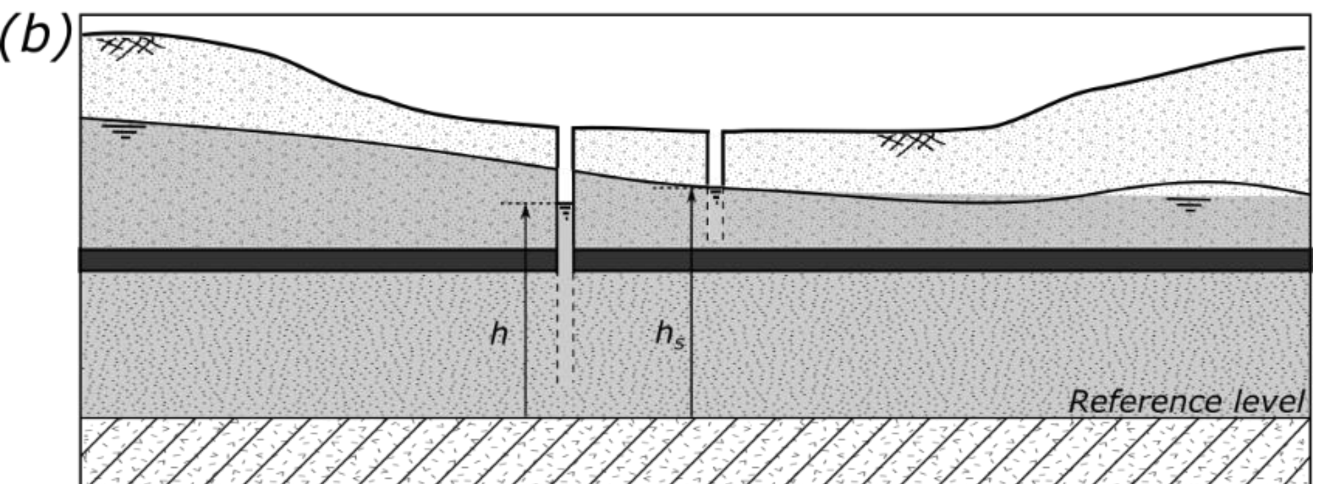
- *interactions between surface water bodies and groundwater*
- *interactions between different aquifers*

Flow BC's *Flux depending on the piezometric head (mixed condition or Cauchy condition)*



conductance

$$q'' = -K\frac{\partial h}{\partial n} = \frac{K'}{b'}(h_r - h)$$

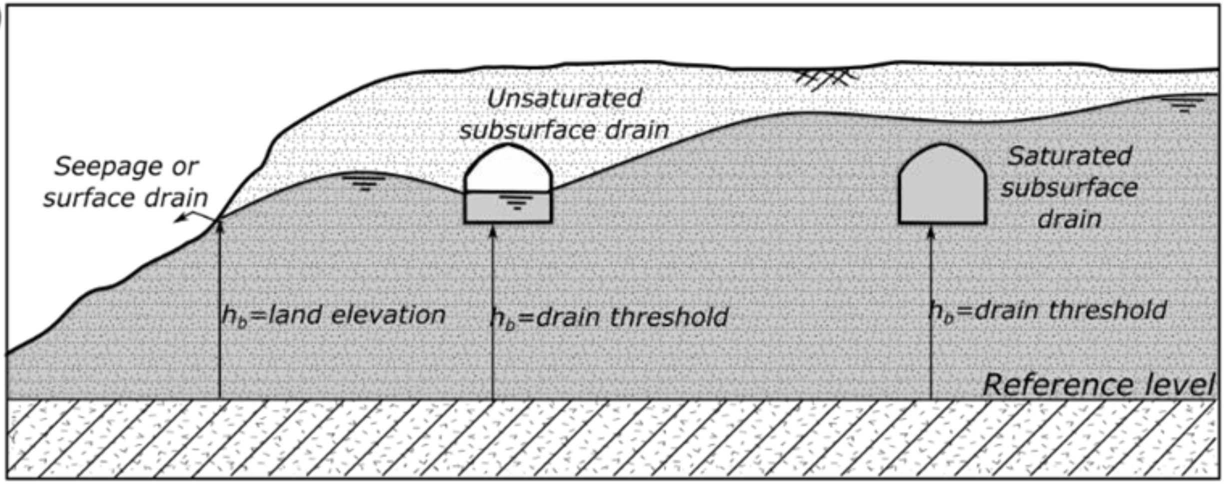


conductance

$$q'' = -K\frac{\partial h}{\partial n} = \frac{K'}{b'}(h_s - h)$$

Flow BC's *Flux depending on the piezometric head (mixed condition or Cauchy condition)*

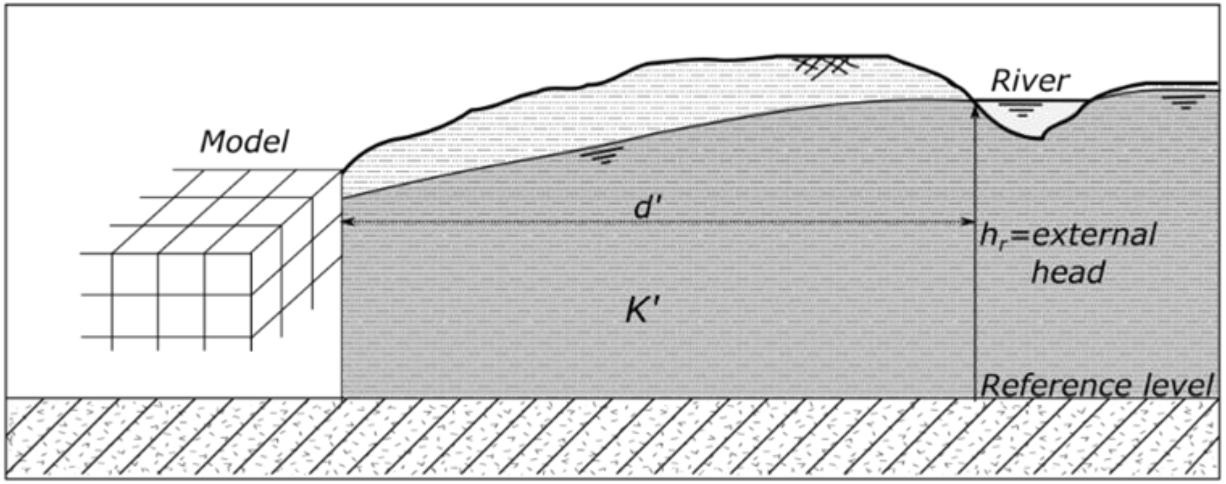
(c)



conductance

$$q'' = -K\frac{\partial h}{\partial n} = \frac{K'}{b'}(h_b - h)$$

(d)

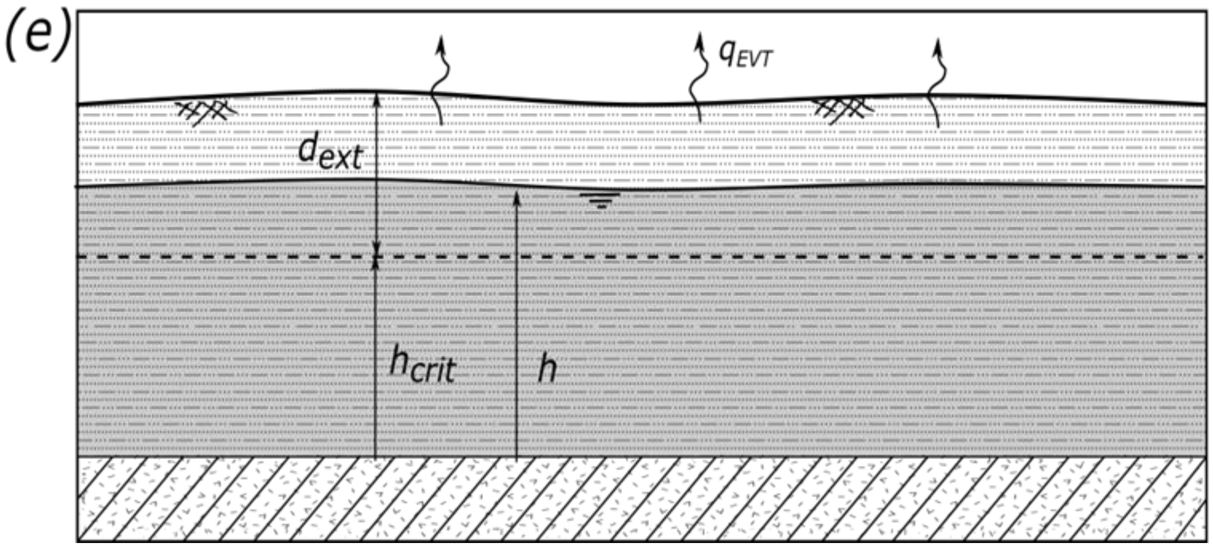


conductance

$$q'' = -K\frac{\partial h}{\partial n} = \frac{K'}{d'}(h_r - h)$$

prescribing an 'external head' (i.e. not on the true boundary but outside the modelled zone) so that a groundwater flux across the boundary is computed from the difference between this 'external head' and the piezometric head on the model boundary using a given conductance

Flow BC's *Flux depending on the piezometric head (mixed condition or Cauchy condition)*



in arid and semi-arid zones

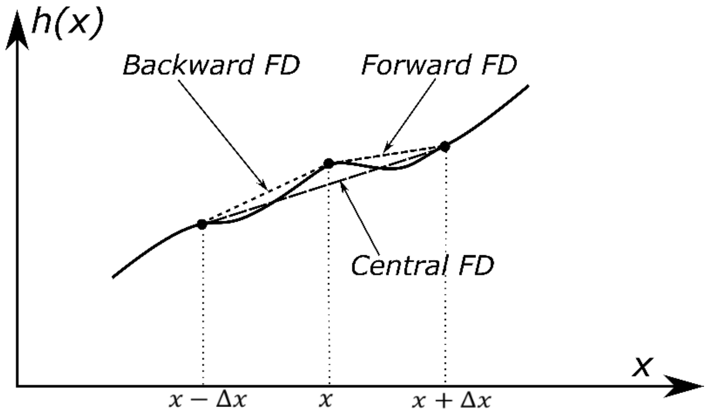
conductance

$$q_{EvT} = \frac{R_{EvT}}{d_{ext}}\left(h(x,y,z,t) - h_{crit}(x,y,z,t)\right)$$

(Anderson et al. 2015)

represent an evapotranspiration flux leaving the model but dependent on the 'depth to water' (i.e. the land surface elevation minus piezometric head). An extinction depth d_{ext} corresponding to a critical head h_{crit} can be defined so that EvT occurs only if the water table is higher

Solving a gw flow problem with a Finite Difference schema



Simple case…

Assumptions:

- *steady state* ➡ $\frac{\partial h}{\partial t} = 0$
- *no infiltration, no sink/source term*
- *homogeneity of the medium*
- *2D problem* ➡ $I = 0 \quad q = 0$

➡ $T = Cst$

GW flow equation in steady state ➡ $$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

Solving a gw flow problem with a FD schema

Simple case…

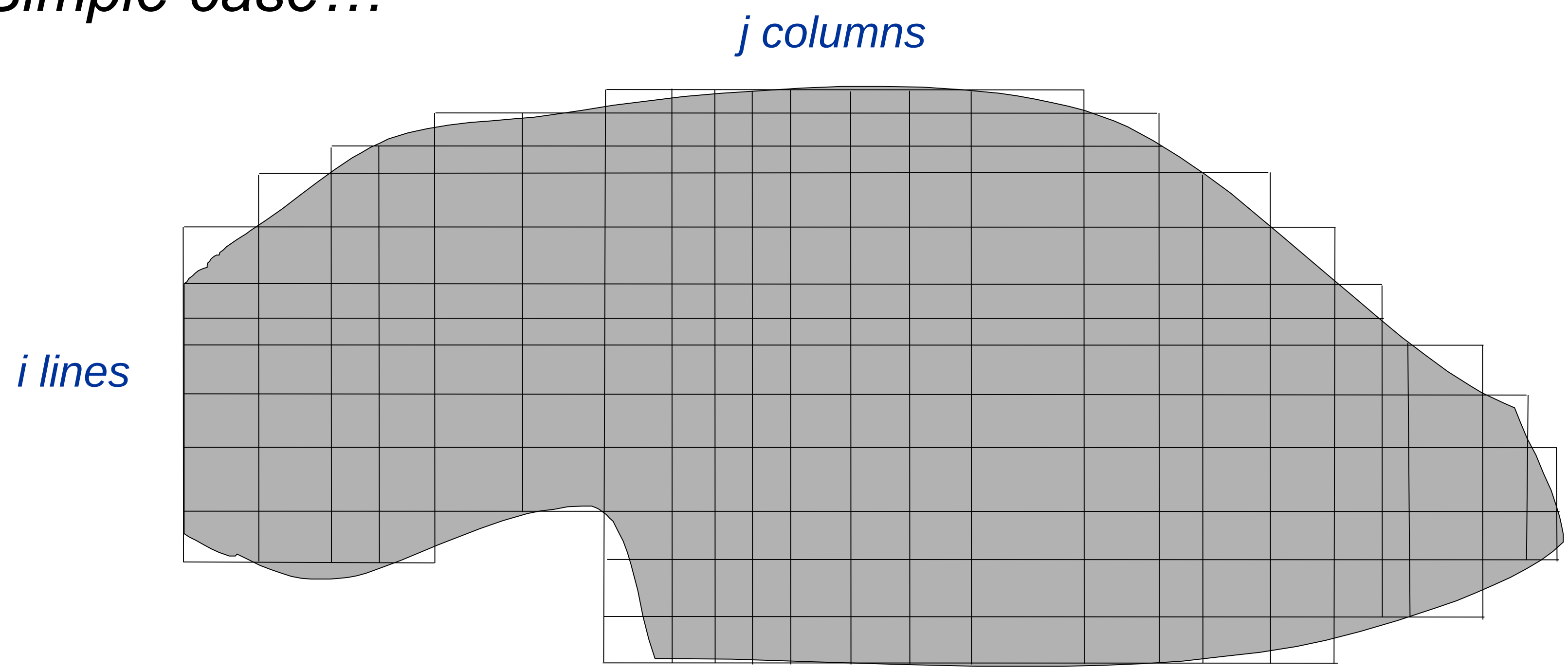
Definition of a partial derivative of a function $h(x)$ *of the variable* x

$$\frac{\partial h}{\partial x} = \lim_{\Delta x \to 0}\left(\frac{h(x+\Delta x)-h(x)}{\Delta x}\right)$$

- *spatial discretisation with a grid*
- *the nodes are the central points of rectangular cells ('Block Centered Finite Difference method' = BCFD)*
- *the cells are homogeneous … the continuous variation of the variable is replaced by a discrete variable defined at the central points of the cells*
- *the approximation of the differential equation is better as the cells are small*

Solving a gw flow problem with a FD schema

Simple case…



➡ *the nodes are numbered sequentially, index* i, j *and piezometric head values* h_{ij} *are attributed*

Solving a gw flow problem with a FD schema

Simple case…

Taylor series for a continuous function $h(x)$

$$h(x+\Delta x)=h(x)+\Delta x.\frac{\partial h(x)}{\partial x}+\frac{(\Delta x)^2}{2!}.\frac{\partial^2 h(x)}{\partial x^2}+\frac{(\Delta x)^3}{3!}.\frac{\partial^3 h(x)}{\partial x^3}+\ldots+\frac{(\Delta x)^n}{n!}.\frac{\partial^n h(x)}{\partial x^n}$$

… for h_{ij} and the x direction:

… terms of the 3rd order and more are neglected

$$h_{i+1\,j}=h_{ij}+(x_{i+1}-x_i).\frac{\partial h}{\partial x}+\frac{(x_{i+1}-x_i)^2}{2}.\frac{\partial^2 h}{\partial x^2}$$

… for $\Delta x > 0$

$$h_{i-1,j}=h_{ij}+(x_{i-1}-x_i).\frac{\partial h}{\partial x}+\frac{(x_{i-1}-x_i)^2}{2}.\frac{\partial^2 h}{\partial x^2}$$

… for $\Delta x < 0$

$$\Rightarrow \frac{h_{i+1\,j}-h_{ij}}{(x_{i+1}-x_i)}=\frac{\partial h}{\partial x}+\frac{(x_{i+1}-x_i)}{2}\frac{\partial^2 h}{\partial x^2}$$

and

$$\frac{h_{i-1\,j}-h_{ij}}{(x_{i-1}-x_i)}=\frac{\partial h}{\partial x}+\frac{(x_{i-1}-x_i)}{2}\frac{\partial^2 h}{\partial x^2}$$

$$\Rightarrow \frac{(h_{i+1\,j}-h_{ij})}{(x_{i+1}-x_i)}+\frac{(h_{i-1\,j}-h_{ij})}{(x_i-x_{i-1})}=\frac{1}{2}(x_{i+1}-x_{i-1}).\frac{\partial^2 h}{\partial x^2}$$

Solving a gw flow problem with a FD schema

Simple case…

$$\frac{\partial^2 h}{\partial x^2} = \frac{2}{(x_{i+1} - x_{i-1})} . \left\{ \frac{h_{i+1\,j}}{(x_{i+1} - x_i)} - \left(\frac{1}{(x_{i+1} - x_i)} + \frac{1}{(x_i - x_{i-1})} \right) . h_{ij} + \frac{h_{i-1\,j}}{(x_i - x_{i-1})} \right\}$$

if $\quad (x_{i-1} - x_i) = \Delta x = (x_{i+1} - x_i)$

$$\frac{\partial^2 h}{\partial x^2} = \frac{h_{i+1\,j} - 2.h_{ij} + h_{i-1\,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{h_{i\,j+1} - 2.h_{ij} + h_{i\,j-1}}{(\Delta y)^2}$$

if $\quad \Delta x = \Delta y = \Delta m = Cst$

$$T . \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = \frac{T}{(\Delta m)^2} . \left(h_{i+1\,j} + h_{i-1\,j} + h_{i\,j+1} + h_{i\,j-1} - 4h_{ij} \right) = 0$$

$$h_{ij} = \frac{1}{4} . (h_{i+1\,j} + h_{i-1\,j} + h_{i\,j+1} + h_{i\,j-1})$$

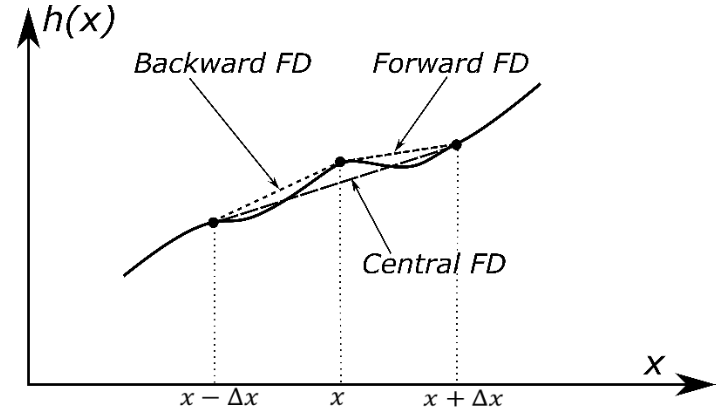
Introduction to solving methods: FD

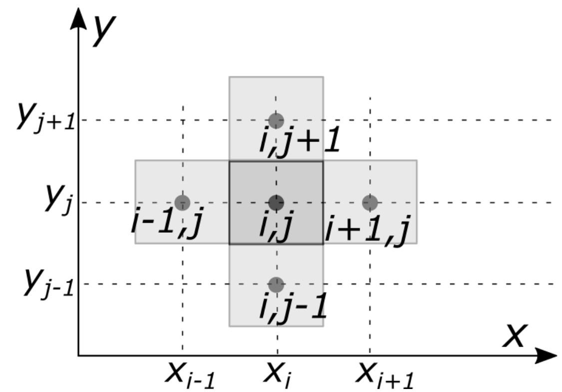
1D spatial approximation of the gradient by a finite difference:

Forward FD $$\frac{\partial h}{\partial x} \approx \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

Central FD $$\frac{\partial h}{\partial x} \approx \frac{h(x+\Delta x) - h(x-\Delta x)}{2\Delta x}$$

$$\frac{\partial^2 h}{\partial x^2} \approx \frac{h_{i+1j} - 2h_{ij} + h_{i-1j}}{(\Delta x)^2}$$



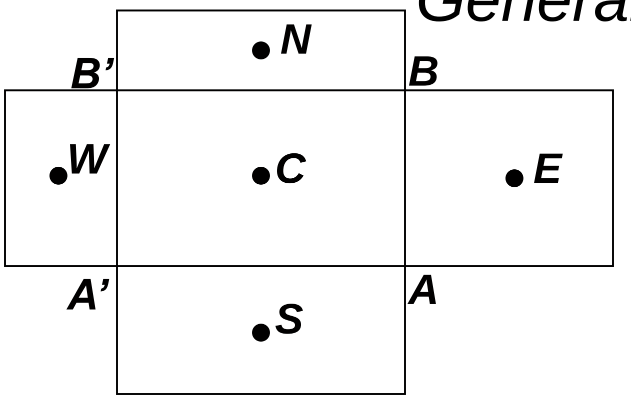


In 2D, with a 2nd order accurate FD:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \approx \frac{\left(h_{i+1j} + h_{i-1j} + h_{ij+1} + h_{ij-1} - 4h_{ij}\right)}{(\Delta m)^2} = 0$$

Solving a gw flow problem with a FD schema

Generalisation



$$\int_{AB} T_x \frac{\partial h}{\partial x} dy + \int_{BB'} T_y \frac{\partial h}{\partial y} dx +$$

$$\int_{B'A'} T_x \frac{\partial h}{\partial x} dy + \int_{A'A} T_y \frac{\partial h}{\partial y} dx = Q_C$$

$$T_{xEC}\left(\frac{h_E - h_C}{x_E - x_C}\right)(y_B - y_A) + T_{yNC}\left(\frac{h_N - h_C}{y_N - y_C}\right)(x_B - x_{B'})$$

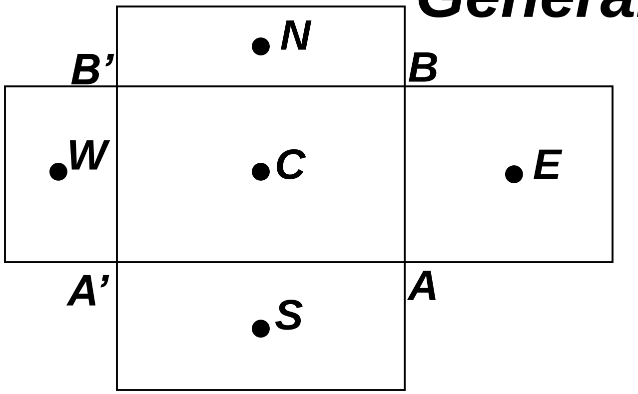
$$+ T_{xWC}\left(\frac{h_W - h_C}{x_W - x_C}\right)(y_{B'} - y_{A'}) + T_{ySC}\left(\frac{h_S - h_C}{y_S - y_C}\right)(x_A - x_{A'}) = Q_C$$

… if rectangular cells : $\Delta x = a$ and $\Delta y = b$

$$T_{xEC}\frac{b}{a}(h_E - h_C) + T_{yNC}\frac{a}{b}(h_N - h_C) + T_{xWC}\frac{b}{a}(h_W - h_C) + T_{ySC}\frac{a}{b}(h_S - h_C) = Q_C$$

Solving a gw flow problem with a FD schema

Generalisation



$$T_{xEC}\frac{b}{a}(h_E-h_C)+T_{yNC}\frac{a}{b}(h_N-h_C)+T_{xWC}\frac{b}{a}(h_W-h_C)+T_{ySC}\frac{a}{b}(h_S-h_C)=Q_C$$

➡ *a ratio of maximum 1/10 for dimensions of the rectangular cells (for good computation conditions)*

… if $a=b$

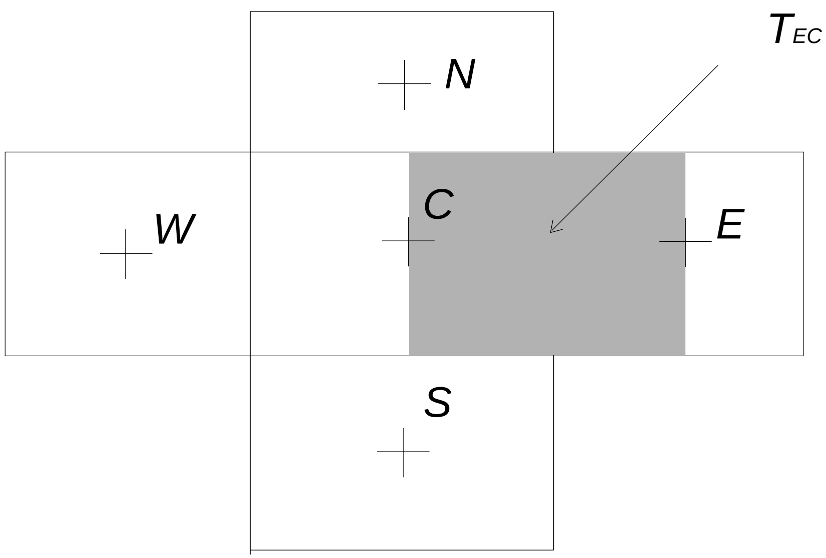
$$T_{xEC}(h_E-h_C)+T_{yNC}(h_N-h_C)+T_{xWC}(h_W-h_C)+T_{ySC}(h_S-h_C)=Q_C$$

Solving a gw flow problem with a FD schema

Equivalent values for parameters

… on the basis of the continuity principle

➡ *'averaged' or 'equivalent' values between cells*



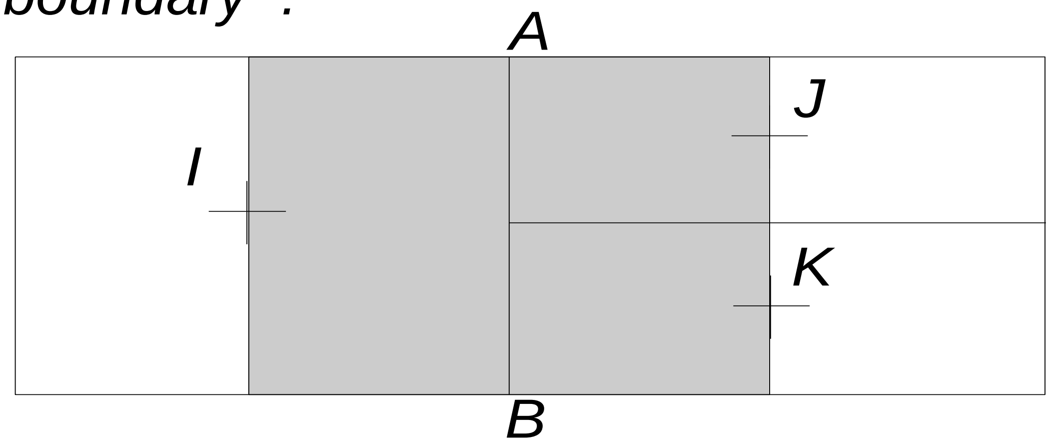
$$T_{xEC} = \frac{2.T_{xE}.T_{xC}}{T_{xE} + T_{xC}}$$

$$T_{EC} = \frac{2.T_{E}.T_{C}}{T_{E} + T_{C}}$$

Solving a gw flow problem with a FD schema

Equivalent values for parameters

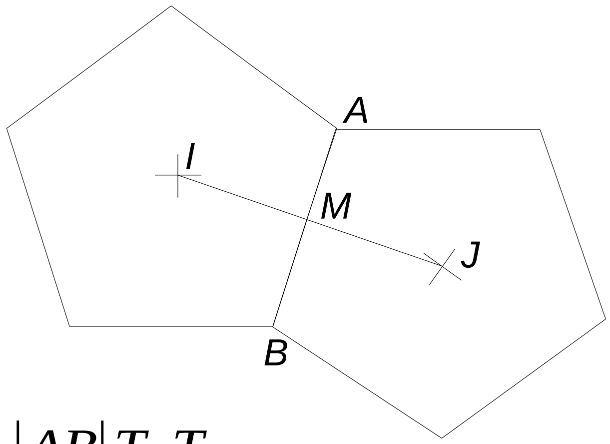
… for more complex mesh (here nested mesh), the water flux on the boundary :



$$\int_{AB} (T_x . \frac{\partial h}{\partial x}).dy = \frac{4.T_I .T_J .T_K}{(T_I .T_J + T_J .T_K + T_K .T_I)} .(h_J - h_I + h_K - h_I)$$

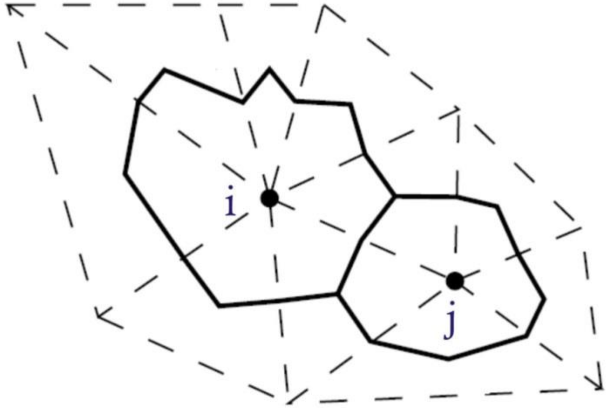
Solving a gw flow problem with a FD schema

Equivalent values for parameters

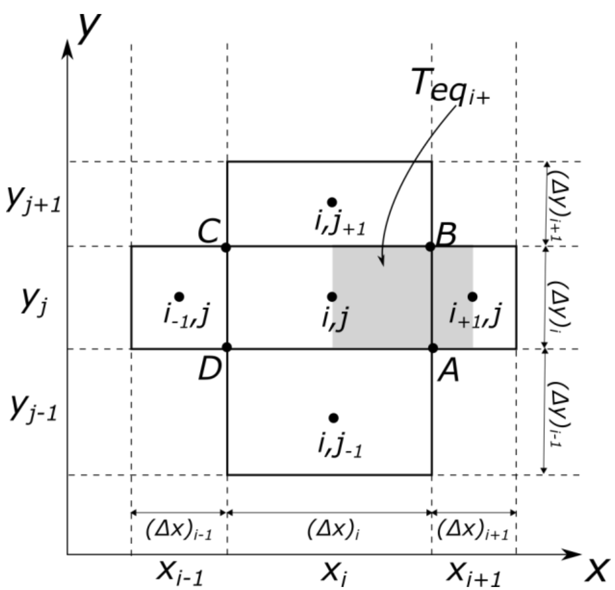


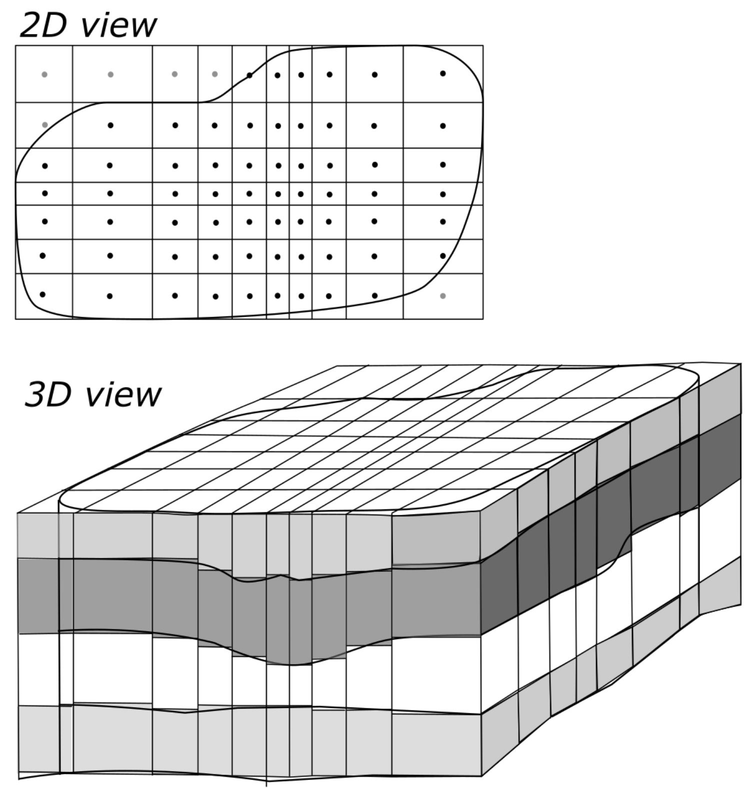
$$\int_{AB} T.\left(\frac{\partial h}{\partial x}.n_x + \frac{\partial h}{\partial y}.n_y\right) dS = \frac{|AB|.T_I.T_J}{(|IM|.T_J + |MJ|.T_I)}.(h_J - h_I)$$

… for Control Volume Finite Elements: same principle



Introduction to solving methods: BCFD



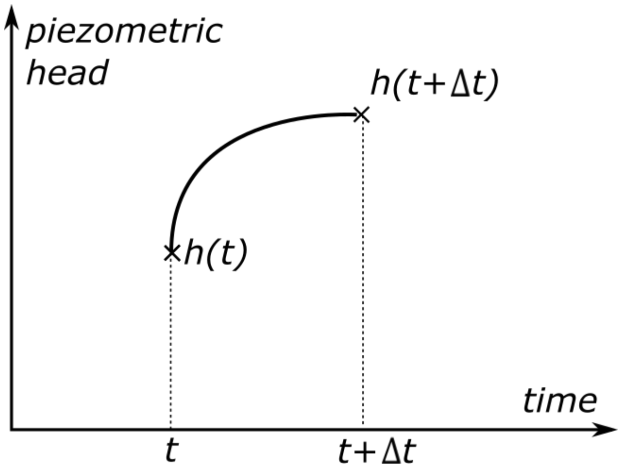


$$T_{eq_{i+}} = \frac{2T_{i+1j}T_{ij}}{T_{ij} + T_{i+1j}}$$

Introduction to solving methods: time integration scheme

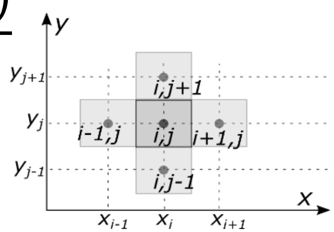
$$T\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\approx T\frac{\left(h_{i+1j}+h_{i-1j}+h_{ij+1}+h_{ij-1}-4h_{ij}\right)}{(\Delta m)^2}=S\frac{\partial h}{\partial t}$$

$$\frac{\partial h}{\partial t}=\frac{h(t+\Delta t)-h(t)}{\Delta t}$$



at what time do we consider the piezometric head values?

$$\frac{T}{(\Delta m)^2}\left(h_{i+1j}+h_{i-1j}+h_{ij+1}+h_{ij-1}-4h_{ij}\right)+Q_{ij}=S\frac{h_{ij}(t+\Delta t)-h_{ij}(t)}{\Delta t}$$



Explicit

$$h_{ij}(t+\Delta t)=h_{ij}(t)+\frac{Q_{ij}\Delta t}{S}+\frac{T\Delta t}{(\Delta m)^2 S}\left(h_{i+1j}(t)+h_{i-1j}(t)+h_{ij+1}(t)+h_{ij-1}(t)-4h_{ij}(t)\right)$$

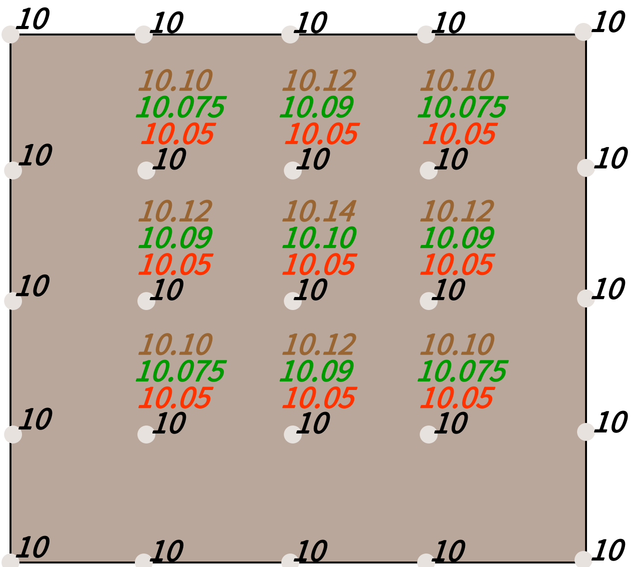
- *physically: not so accurate*
- *numerically: stability problem when the time step becomes larger*
- *respect a stability criterion*

$$T=Cst$$

$$\Delta x=\Delta y=\Delta m=Cst$$

Explicit method

$$h_{ij}(t+\Delta t) = h_{ij}(t) + \frac{I.\Delta t}{S} + \frac{T.\Delta t}{(\Delta m)^2.S}.\left(h_{i+1\,j}(t) + h_{i-1\,j}(t) + h_{i\,j+1}(t) + h_{i\,j-1}(t) - 4h_{ij}(t)\right)$$



Example:

- *squared island*
- *initial value h = 10 m*
- *BC's : h = 10 m*
- *infiltration: 0.002 m/day*
- *S = 0.4 ; T = 100 m²/day*
- *Δt =10 days Δm =50m*

$$\frac{I.\Delta t}{S} = 0.05 \qquad \frac{T.\Delta t}{(\Delta m)^2.S} = 0.25$$

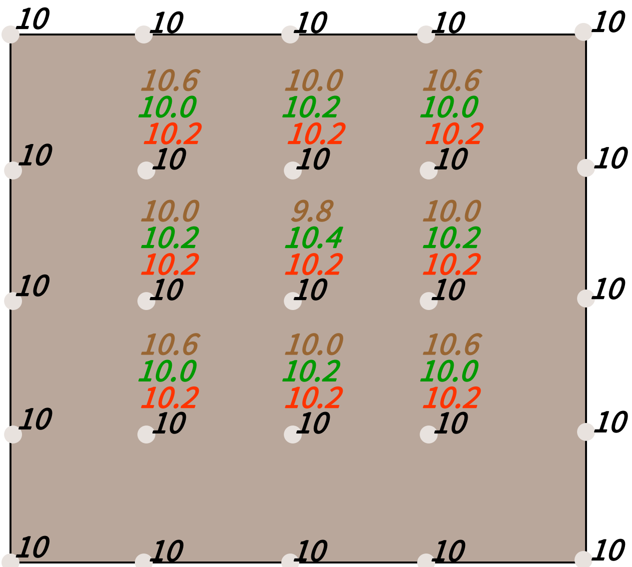
… computation:

- *1st time step;*
- *2nd time step;*
- *3rd time step;*
- *…*

… no problem

Explicit method

$$h_{ij}(t+\Delta t)=h_{ij}(t)+\frac{I.\Delta t}{S}+\frac{T.\Delta t}{(\Delta m)^2.S}.\left(h_{i+1\,j}(t)+h_{i-1\,j}(t)+h_{i\,j+1}(t)+h_{i\,j-1}(t)-4h_{ij}(t)\right)$$



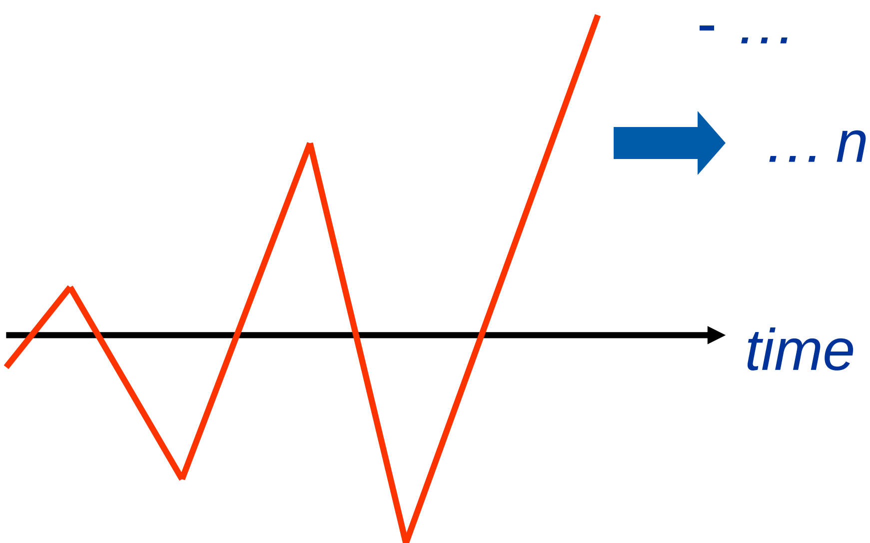
… now with a Δt = 40 days

$$\frac{I.\Delta t}{S}=0.2 \qquad \frac{T.\Delta t}{(\Delta m)^2.S}=1$$

… computation:

- *1st time step;*
- *2nd time step;*
- *3rd time step;*
- *…*

… numerically not stable



Explicit method: stability criterion (example)

$$h_{ij}(t+\Delta t) = h_{ij}(t) + \frac{I.\Delta t}{S} + \frac{T.\Delta t}{(\Delta m)^2.S}.\left(h_{i+1\,j}(t) + h_{i-1\,j}(t) + h_{i\,j+1}(t) + h_{i\,j-1}(t) - 4h_{ij}(t)\right)$$

… worst case

$$\frac{I.\Delta t}{S} = 0 \qquad \frac{T.\Delta t}{(\Delta m)^2.S} = \alpha$$

10 + ε

10 + ε ***10 - ε*** ***10 + ε***

10 + ε

$$h_{ij}(t) = (10-\varepsilon)$$

$$\Rightarrow h_{ij}(t+\Delta t) = (10-\varepsilon) + 0 + \alpha(8\varepsilon)$$

$$h_{ij}(t+\Delta t) = 10 + (8\alpha - 1)\varepsilon$$

$\Rightarrow$ *… for obtaining the stability :*

$$(8\alpha - 1)\varepsilon \le \varepsilon$$

$$\Rightarrow \alpha \le 1/4$$

$$\Rightarrow \frac{T.\Delta t}{(\Delta m)^2.S} = \alpha \le 1/4$$

Explicit method: stability criterion

- *the stability of the computations depends on the size of the time step with regards to the size of the grid cells and of the parameters values*

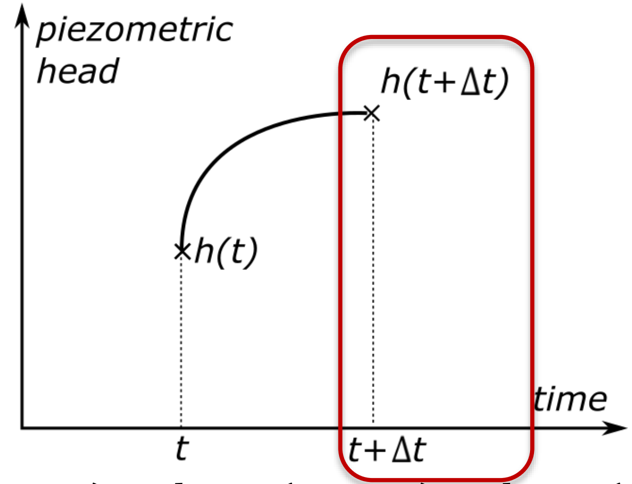
  stability criterion :

$$\frac{T.\Delta t}{S.(\Delta m)^2} \leq \frac{1}{4}$$

$$\frac{T}{S}.\left(\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right) \leq \frac{1}{2} \qquad \Delta x \neq \Delta y$$

- *Additional drawbacks:*
  - *physical propagation of rounding errors*
  - *numerical errors*
  - *long CPU time*

Time integration scheme



Implicit

… at the time $t + \Delta t$

$$h_{ij}(t+\Delta t)[1+4\alpha] = h_{ij}(t) + \frac{Q_{ij}\Delta t}{S} + \frac{T\Delta t}{(\Delta m)^2 S}\left(h_{i+1j}(t+\Delta t) + h_{i-1j}(t+\Delta t) + h_{ij+1}(t+\Delta t) + h_{ij-1}(t+\Delta t)\right)$$

implicit equation

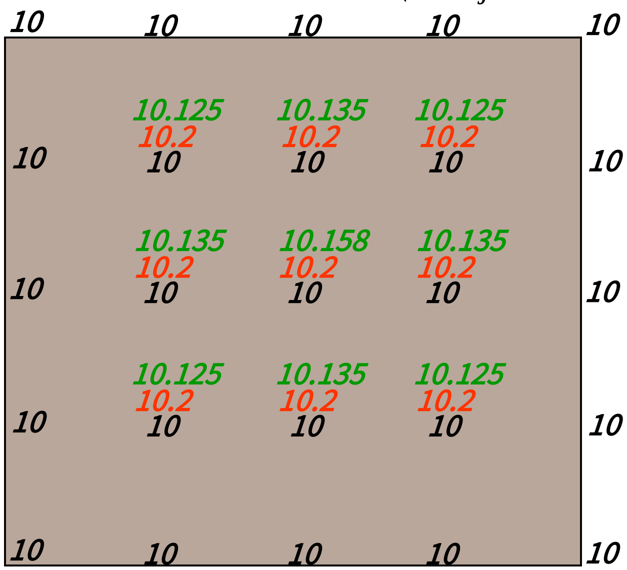
the unknown cannot be deduced from one equation
you need the whole system to be solved

- *physically: not so accurate (error increases with time step)*
- *numerically: unconditional stability*
- *mathematically: more complex/heavy*

(Bear and Cheng 2010)

Implicit method

$$h_{ij}(t+\Delta t).\left[1+4.\alpha\right] = h_{ij}(t) + \frac{I.\Delta t}{S}$$
$$+\alpha.\left(h_{i+1\,j}(t+\Delta t) + h_{i-1\,j}(t+\Delta t) + h_{i\,j+1}(t+\Delta t) + h_{i\,j-1}(t+\Delta t)\right)$$



… even with a Δt = 40 days

$$\frac{I.\Delta t}{S} = 0.2 \qquad \frac{T.\Delta t}{(\Delta m)^2.S} = 1$$

… computation:

- *1st time step;*
- *2nd time step;*
- *…*

… numerical stability

Implicit method: stability can be proven

$$h_{ij}(t+\Delta t).\left[1+4.\alpha\right] = h_{ij}(t) + \frac{I.\Delta t}{S}$$
$$+\alpha.\left(h_{i+1\,j}(t+\Delta t) + h_{i-1\,j}(t+\Delta t) + h_{i\,j+1}(t+\Delta t) + h_{i\,j-1}(t+\Delta t)\right)$$

… the worst case

$$\frac{I.\Delta t}{S} = 0 \qquad \frac{T.\Delta t}{(\Delta m)^2.S} = \alpha$$

$10+\varepsilon$

$10+\varepsilon$ $\quad$ $10-\varepsilon$ $\quad$ $10+\varepsilon$

$10+\varepsilon$

$$h_{ij}(t) = (10-\varepsilon)$$

$$\Rightarrow h_{ij}(t+\Delta t)(1+4\alpha) = (10-\varepsilon) + 0 + 4\alpha(10+\varepsilon)$$

$$h_{ij}(t+\Delta t) = \frac{(10-\varepsilon)+4\alpha(10+\varepsilon)}{(1+4\alpha)}$$

$\Rightarrow$ *for obtaining stability :*

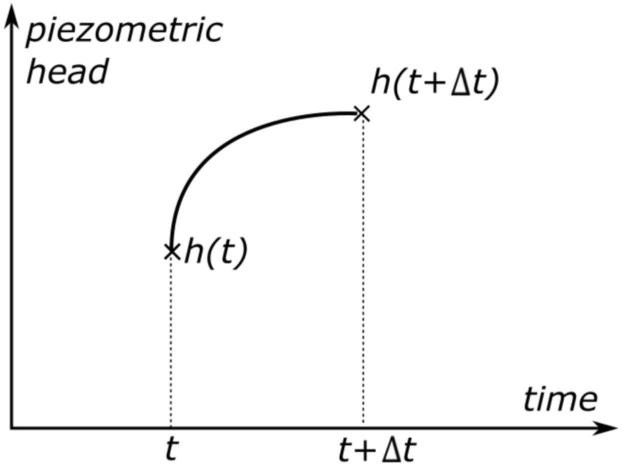
$$\boxed{h_{ij}(t+\Delta t) - 10 \leq \varepsilon}$$

$$\Rightarrow \frac{(10-\varepsilon)+4\alpha(10+\varepsilon)}{(1+4\alpha)} - 10 \leq \varepsilon$$

$$\Rightarrow 10-\varepsilon+40\alpha+4\alpha\varepsilon < 10+\varepsilon+40\alpha+4\alpha\varepsilon$$

…always the case

Time integration scheme



Crank-Nicholson method

… at the time $t+\Delta t/2$

- *physically: more accurate*
- *numerically: implicit procedure, unconditional stability*

Galerkin method

… at the time $t+2\Delta t/3$

- *physically: most accurate*
- *numerically: implicit procedure, unconditional stability*

Time integration scheme

$$\frac{T}{(\Delta m)^2}\left(h_{i+1j}+h_{i-1j}+h_{ij+1}+h_{ij-1}-4h_{ij}\right)+Q_{ij}=S\frac{h_{ij}(t+\Delta t)-h_{ij}(t)}{\Delta t}$$

$$\boxed{\begin{aligned}&\frac{T}{(\Delta m)^2}(1-\theta)\left(h_{i+1j}(t)+h_{i-1j}(t)+h_{ij+1}(t)+h_{ij-1}(t)-4h_{ij}(t)\right)\\&+\frac{T}{(\Delta m)^2}\theta\left(h_{i+1j}(t+\Delta t)+h_{i-1j}(t+\Delta t)+h_{ij+1}(t+\Delta t)+h_{ij-1}(t+\Delta t)-4h_{ij}(t+\Delta t)\right)\end{aligned}}$$

$\theta=0$ → *Full explicit time integration*

$\theta=1$ → *Full implicit time integration*

$\theta=1/2$ → *Crank-Nicholson implicit*

$\theta=2/3$ → *Galerkin implicit*

→ *stability criterion only for explicit schemes* $\theta<1/2$

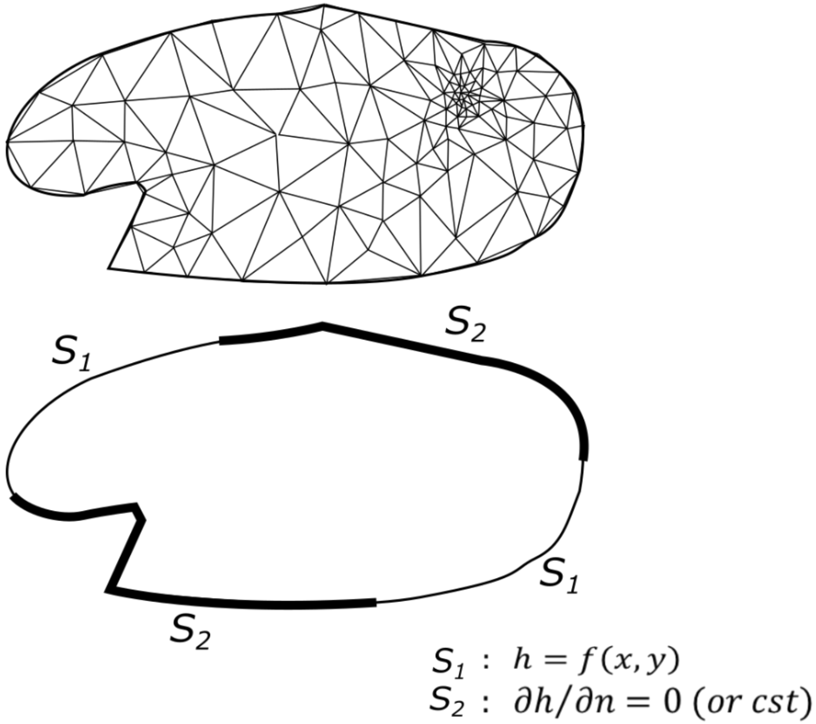
time integration schemas used in all numerical techniques

Introduction to solving methods: FD practical recommendations

- *an initial field of values for the main unknown variable (piezometric head) needed for initiating the iterative solving*
- *accuracy increases with the number of cells but portability (i.e. computing efficiency) decreases*
- *use smaller cells where a steep gradient of the main variable is expected*
- *spatial discretization: nodes located at pumping wells and observation piezometers*
- *avoid distances between nodes greater than 1.5 the previous one*
- *avoid ratios greater than 1/10 for the cell dimensions (bad numerical conditions for solving the system of equations)*
- *boundaries with a prescribed head should correspond to nodes (central points of the cells, if BCFD)*
- *boundaries with a prescribed flux should correspond to sides of the cells (where the flux condition is calculated) if BCFD*
- *…*

Introduction to solving methods: Finite Elements

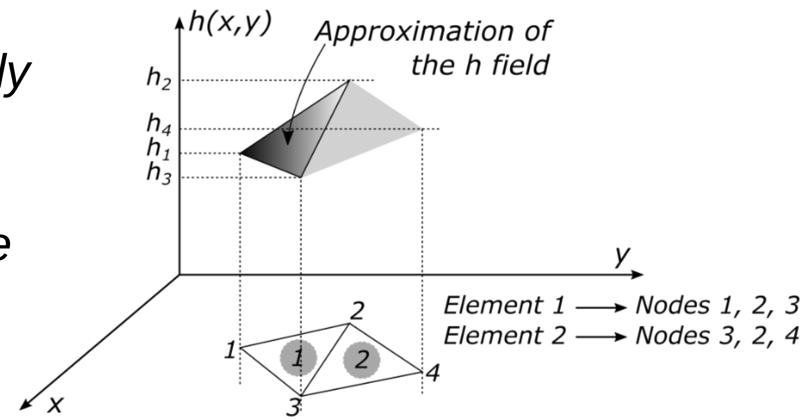
- *discrete elements, unstructured FE mesh*
- *better for irregular boundaries, spatial variations, and exact locations for stress-factors and observation measurements*
- *optimized mesh generation to reduce the needed memory space*



(refs among others: Narasimhan et al. 1978, Huyakorn and Pinder 1983, Bear and Verruijt 1987, Wang and Anderson 1982, Fitts 2002, Rausch et al. 2005, Bear and Cheng 2010, Anderson et al. 2015, Diersch 2014, Pinder and Celia 2006, Dassargues 2018 and 2020)

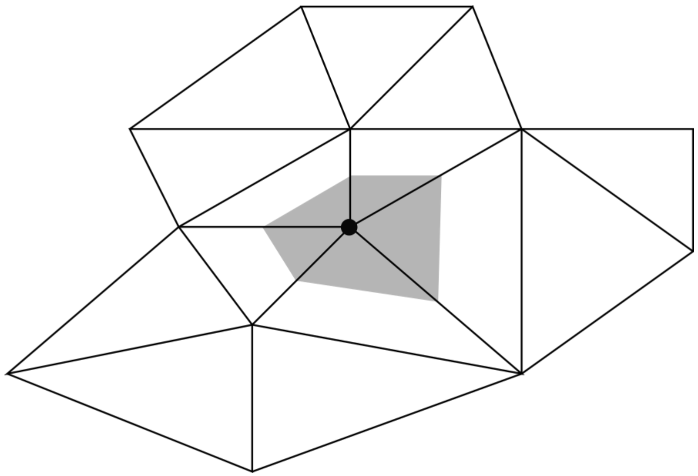
Introduction to solving methods: FE

- *the continuous field of the variable (i.e. piezometric head) approximated typically by interpolation functions (here also referred to as basis functions)*
- *piezometric field described in each finite element by a plane*
- *the discrete unknowns are the nodal values*
- *an integral approach expressing the weak formulation (i.e. a variational form integrating the governing partial differential equation of the process with its BCs and initial conditions) for obtaining a global continuum balance statement*
- *two ways:*
  (1) minimum of a natural variational functional (when it exists)
  (2) method of weighted residuals (applicable to all types of partial differential equations)



Introduction to solving methods: FV

- *common features with FD and FE*
- *FD for unstructured grids*
- *if triangles: similarities with triangle FE*
- *as for FE, FV approximates the main variable using basis functions in the triangular element*
- *Finite Volume refers to the volume surrounding each node point in a mesh with nodal basis function = 1 only at the considered node and 0 at all others*
- *conservation law is satisfied locally for a given control volume with respect to its neighboring volumes (similar to FD not to FE)*
- *balance relies on evaluation of surface integrals on the boundaries (i.e. the conservation must be satisfied across the boundaries of the adjoining control volumes)*



(refs among others: Patankar 1980, Baliga and Patankar 1983, Chung 2002, Diersch 2014, Narasimhan and Witherspoon 1976, Rausch et al. 2005, Fletcher 1988, Idelsohn and Onate 1994, Forsyth et al. 1995, Therrien and Sudicky 1996, Pinder and Celia 2006, Therrien et al. 2010)

References

- *Anderson,M.P., Woessner, W.W. and R.J. Hunt. 2015. Applied groundwater modeling – Simulation of flow and advective transport. Academic Press Elsevier.*
- *Baliga, B.R. and S.V. Patankar. 1983. A control volume finite-element method for two-dimensional fluid flow and heat transfer. Numerical Heat Transfer 6(3): 245-261.*
- *Bear, J. and A. Verruijt. 1987. Modeling groundwater flow and pollution. Dordrecht: Reidel Publishing Company.*
- *Bear, J. and A.H.D. Cheng. 2010. Modeling groundwater flow and contaminant transport. Springer.*
- *Brouyère, S., Carabin, G. and Dassargues, A., 2004. Climate change impacts on groundwater reserves: modelled deficits in a chalky aquifer, Geer basin, Belgium. Hydrogeology Journal 12(2), pp.123-134.*
- *Brouyère, S., Orban, P., Wildemeersch, S., Couturier, J., Gardin, N. and Dassargues, A., 2009. The Hybrid Finite Element Mixing Cell Method: A New Flexible Method for Modelling Mine Groundwater Problems. Mine Water & the Environment 28(2): 102-114.*
- *Carabin, G. and A. Dassargues. 1999. Modeling groundwater with ocean and river interaction. Water Resources Research 35(8): 2347-2358.*
- *Carabin G. and Dassargues A., 2000. Coupling of parallel river and groundwater models to simulate dynamic groundwater boundary conditions (Proc. of Computational Methods in Water Resources 2000), Bentley L.R., Sykes J.F., Brebbia C.A., Gray W.G. & Pinder G.F., vol.2,1107-1113, Balkema.*
- *Castany, G. 1963. Traité pratique des eaux souterraines (in French). Paris, Bruxelles, Montréal : Dunod.*
- *César E, Wildemeersch S., Orban P., Carrière S., Brouyère S. and Dassargues A., 2014. Simulation of spatial and temporal trends in nitrate concentrations at the regional scale in the Upper Dyle basin, Belgium. Hydrogeology Journal 22: 1087 – 1100.*
- *Chung, T. 2002. Computational fluid dynamics. Cambridge University Press.*
- *de Marsily, G. 1986. Quantitative hydrogeology : groundwater hydrology for engineers. Academic Press.*
- *Dagan, G. 1989. Flow and transport in porous formations, New York: Springer.*
- *Dassargues A., Radu J.P., Charlier R., 1988. Finite elements modelling of a large water table aquifer in transient conditions. Advances in Water Resources, Volume 11(2): 58-66.*
- *Dassargues A., 1991. Water table aquifers and finite element method: analysis and presentation of a case study, in Computational Modelling of Free and Moving Boundary Problems, Vol. 1, Fluid Flow, Computational Mechanics Publications, Southampton, 63-72.*
- *Dassargues, A. 1995. Modélisation en hydrogéologie, Programme Tempus JEP 3801 Sciences de l'Eau et Environnement, Ed. Didac. et Pédag. RA, Bucarest, Romania.*
- *Dassargues, A. 1995. On the necessity to consider varying parameters in land subsidence computations, in Proc. of the 5th Int. Symp. on Land Subsidence, ed. Barends, B.J., Brouwer F.J.J. and F.H. Schröder, IAHS 234: 259-268.*
- *Dassargues, A. 1997. Vers une meilleure fiabilité dans le calcul des tassements dus aux pompages d'eau souterraine, A) Première partie: prise en compte de la variation au cours du temps des paramètres hydrogéologiques et géotechniques (in French), Annales de la Société Géologique de Belgique, 118 (1995)(2) : 95-115.*
- *Dassargues A., 1997. Théorie de l'approche hydrogéologique des écoulements et transports en zone partiellement saturée, Annales de la Société Géologique de Belgique, T. 119(1) 1996, pp. 71-89.*
- *Dassargues, A., 1997. Modeling baseflow from an alluvial aquifer using hydraulic-conductivity data obtained from a derived relation with apparent electrical resistivity. Hydrogeology Journal 5(3): 97-108.*
- *Dassargues, A. 1998. Prise en compte des variations de la perméabilité et du coefficient d'emmagasinement spécifique dans les simulations hydrogéologiques en milieux argileux saturés (in French), Bull. Soc. Géol. France, 169(5) : 665-673.*

References (2)

- *Dassargues A., 2018. Hydrogeology: groundwater science and engineering, 472p. Taylor & Francis CRC press, Boca Raton.*
- *Dassargues A. 2020. Hydrogéologie appliquée : science et ingénierie des eaux souterraines, 512p. Dunod. Paris.*
- *Dassargues, A., Biver, P. and A. Monjoie. 1991. Geotechnical properties of the Quaternary sediments in Shanghai. Engineering Geology 31(1): 71-90.*
- *Dassargues, A., Schroeder Ch. and X.L. Li. 1993. Applying the Lagamine model to compute land subsidence in Shanghai, Bulletin of Engineering Geology (IAEG) 47: 13-26.*
- *Dassargues, A. and A. Monjoie. 1993. Chalk as an aquifer in Belgium. In Hydrogeology of the Chalk of North-West Europe, ed. R.A. Downing, M. Price and G.P. Jones, 153-169. Oxford University Press.*
- *Delleur, J.W. 1999. The handbook of groundwater engineering. Boca Raton: CRC Press.*
- *Diersch, H-J.G. 2014. Feflow – Finite element modeling of flow, mass and heat transport in porous and fractured media. Springer.*
- *Dupuit, J. 1863. Estudes théoriques et pratiques sur le mouvement des eaux dans les canaux découverts et à travers les terrains perméables (in French) (2nd Ed.). Paris Dunod.*
- *Eckis, R. 1934. Geology and ground-water storage capacity of valley fill, South Coastal Basin Investigation: California Dept. Public Works, Div. Water Resources Bull. 45, 273 p.*
- *Fitts, Ch. R. 2002. Groundwater science. Academic Press.*
- *Forsyth, P.A., Wu, Y.S. and K. Pruess. 1995. Robust numerical methods for saturated-unsaturated flow with dry initial conditions in heterogeneous media, Advances in Water Resources 18(1) : 25-38.*
- *Fletcher, C. 1988. Computational techniques for fluid dynamics. Vol.1 and Vol.2, New York: Springer.*
- *Goderniaux, P., Brouyère, S., Fowler, H.J., Blenkinsop, S., Therrien, R. Orban, Ph. and Dassargues, A., 2009. Large scale surface – subsurface hydrological model to assess climate change impacts on groundwater reserves. Journal of Hydrology 373: 122-138.*
- *Goderniaux, P., Brouyère, S., Blenkinsop, S., Burton, A., Fowler, H.J., Orban, P. and Dassargues, A., 2011. Modelling climate change impacts on groundwater resources using transient stochastic climatic scenarios. Water Resources Research 47(12): W12516*
- *Hadley, P.W. and Ch. Newell. 2014. The new potential for understanding groundwater contaminant transport. Groundwater 52(2): 174-186.*
- *Hoffmann R., Goderniaux P., Jamin P., Orban Ph., Brouyère S. and A. Dassargues. 2021. Differentiated influence of the double porosity of the chalk on solute and heat transport. In The Chalk Aquifers of Northern Europe. Farrell, R. P., Massei, N., Foley, A. E., Howlett, P. R. and West, L. J. (eds), Geological Society, London, Special Publications, 517, https://doi.org/10.1144/SP517-2020-170*
- *Huyakorn, P.S. and G.F. Pinder. 1983. Computational methods in subsurface flow. Academic Press.*
- *Idelsohn, S. and E. Onate. 1994. Finite volumes and finite elements: two 'good friends'. International Journal for Numerical Methods in Engineering 37(19) : 3323-3341.*
- *Jusseret, S., Vu Thanh, T. and Dassargues, A., 2009. Groundwater flow modelling in the central zone of Hanoi, Vietnam, Hydrogeology Journal 17(4): 915-934.*
- *Narasimhan, T.N. and P.A. Witherspoon. 1976. An integrated finite difference method for analyzing fluid flow in porous media, Water Resources Research 12(1): 57-64.*
- *Narasimhan, T.N., Neuman, S.P. and P.A. Witherspoon. 1978. Finite element method for subsurface hydrology using a mixed explicit-implicit scheme, Water Resources Research 14(5) : 863-877.*
- *Orban, P., Brouyère,S., Batlle-Aguilar, J., Couturier, J., Goderniaux, P., Leroy, M., Malozewski, P. and Dassargues, A., 2010. Regional transport modelling for nitrate trend assessment and forecasting in a chalk aquifer. Journal of Contaminant Hydrology 118: 79-93.*
- *Patankar, S. 1980. Numerical heat transfer and fluid flow. CRC Press.*

References (3)

- *Payne, F.C., Quinnan, A. and S.T. Potter. 2008. Remediation hydraulics. Boca Raton: CRC Press/ Taylor & Francis.*
- *Peeters, L., Haerens, B., Van Der Sluys, J. and Dassargues, A., 2004. Modelling seasonal variations in nitrate and sulphate concentrations in a threatened alluvial aquifer, Environmental Geology 46(6-7): 951-961.*
- *Pinder, G.F. and M.A. Celia. 2006. Subsurface hydrology. Hoboken, New Jersey: Wiley & Sons.*
- *Pujades E., Jurado A., Carrera J. Vázque-Sunè E. and Dassargues A., 2016. Hydrogeological assessment of non-linear underground enclosures. Engineering Geology 207: 91-102.*
- *Rausch, R., Schäfer, W., Therrien, R. and Chr. Wagner. 2005. Solute transport modelling – An introduction to models and solution strategies. Berlin-Stuttgart: Gebr.Borntraeger Verlagsbuchhandlung Science Publishers.*
- *Rocha, D., Feyen, J. and Dassargues, A. 2007. Comparative analysis between analytical approximations and numerical solutions describing recession flow in unconfined hillslope aquifers. Hydrogeology Journal 15: 1077-1091.*
- *Rojas, R. & Dassargues, A., 2007. Groundwater flow modelling of the regional aquifer of the Pampa del Tamarugal, northern Chile, Hydrogeology Journal 15: 537-551.*
- *Rojas, R., Feyen, L. and Dassargues, A., 2008. Conceptual model uncertainty in groundwater modeling: Combining generalized likelihood uncertainty estimation and Bayesian model averaging, Water Resources Research 44: W12418*
- *Rojas, R., Batelaan, O., Feyen, L., and Dassargues, A., 2010. Assessment of conceptual model uncertainty for the regional aquifer Pampa del Tamarugal – North Chile. Hydrol. Earth Syst. Sci. 14: 171-192.*
- *Stefanescu Ch. and Dassargues A., 1996. Simulation of pumping and artificial recharge in the phreatic aquifer near Bucarest (Romania). Hydrogeology Journal 4(3): 72-83.*
- *Terzaghi, K. 1943. Theoretical soil mechanics, London: Chapman and Hall.*
- *Therrien, R. and E.A. Sudicky. 1996. Three-dimensional analysis of variably-saturated flow and solute transport in discretely-fractured porous media, Journal of Contaminant Hydrology 23(1-2) : 1-44.*
- *Therrien, R., McLaren, R.G., Sudicky, E.A. and S.M. Panday. 2010. HydroGeoSphere: A three-dimensional numerical model describing fully-integrated subsurface and surface flow and solute transport. User manual. Université Laval & University of Waterloo.*
- *Wang, H.F. and M.P. Anderson. 1982. Introduction to groundwater modelling: finite difference and finite element methods, San Diego (CA): Academic Press.*
- *Wildemeersch, S., Brouyère, S., Orban, P., Couturier, J., Dingelstadt, C., Veschkens, M. and Dassargues, A., 2010. Application of the Hybrid Finite Element Mixing Cell method to an abandoned coalfield in Belgium. Journal of Hydrology 392 (3-4): 188-200.*
- *Wildemeersch S., Goderniaux P., Orban P., Brouyère S. and Dassargues A., 2014. Assessing the effects of spatial discretization on large-scale flow model performance and prediction uncertainty. Journal of Hydrology 510: 10-25.*