Generalized Interpolation: a Functorial Point of View

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Boyd functions

A function $\phi : (0, \infty) \to (0, \infty)$ is a *Boyd function* if it is continuous, $\phi(1) = 1$ and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(st)}{\phi(s)} < \infty,$$

for all $t \in (0, \infty)$. The *lower* and *upper Boyd indices* of a Boyd function ϕ are defined by

$$\underline{b}(\phi) := \sup_{t<1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t \to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

and

$$\overline{b}(\phi) := \inf_{t>1} \frac{\log \bar{\phi}(t)}{\log t} = \lim_{t \to \infty} \frac{\log \bar{\phi}(t)}{\log t},$$

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Admissible sequences

A sequence $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ of positive real numbers is admissible if there exists a constant $C > 0$ such that $C^{-1}\sigma_j \le \sigma_{j+1} \le C\sigma_j$ for all j. Let $\underline{\sigma}_j := \inf_{k\ge1} \sigma_{j+k}/\sigma_k$ and $\overline{\sigma}_j := \sup_{k\ge1} \sigma_{j+k}/\sigma_k$. The lower and upper Boyd indices of σ are defined by

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Classical link

Given an admissible sequence σ, the function

$$\phi_\sigma(t) := \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \sigma_0 & \text{if } t \in (0,1) \end{cases},$$

with $\sigma_0 = 1$ is a Boyd function.

Properties of the Boyd functions

- The indices $\underline{b}(\phi)$ and $\overline{b}(\phi)$ are two numbers such that $\underline{b}(\phi) \leq \overline{b}(\phi)$.
- Given $\varepsilon > 0$ and $R > 0$, there exists $C > 0$ such that

$$C^{-1} t^{\overline{b}(\phi)+\varepsilon} \leq \phi(t) \leq C t^{\underline{b}(\phi)-\varepsilon},$$

  for any $t \leq R$.
- In the same way, we also have

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1 germ versus 2 germs

We will denote by $\mathcal{B}^\infty$ the set of continuous functions $\phi : [1,\infty) \to I$ such that $\phi(1) = 1$ and

$$0 < \underline{\phi}(t) := \inf_{s\geq 1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s\geq 1} \frac{\phi(ts)}{\phi(s)} < \infty,$$

for any $t \geq 1$. Given $\phi \in \mathcal{B}$, we denote by ϕ_∞ the restriction of ϕ to $[1,\infty)$ and by ϕ_0 the restriction of ϕ to $(0,1]$.

Proposition

The application

$$\tau : \mathcal{B} \to \mathcal{B}^\infty \times \mathcal{B}^\infty \quad \phi \mapsto (t \mapsto \frac{1}{\phi_0(1/t)}, \phi_\infty)$$

is a bijection.

A representation theorem

Theorem

A function $\phi : [1, \infty) \to I$ *belongs to* $\mathcal{B}^\infty$ *if and only if* $\phi(1) = 1$ *and there exist two bounded continuous functions* $\eta, \xi : [1, \infty) \to I$ *such that*

$$\phi(t) = e^{\eta(t) + \int_1^t \xi(s) \frac{ds}{s}}.$$

Corollary

A function $\phi : I \to I$ *belongs to* $\mathcal{B}$ *if and only if* $\phi(1) = 1$ *and there exist four bounded continuous functions* $\eta_0, \xi_0 : (0, 1] \to I$ *and* $\eta_\infty, \xi_\infty : [1, \infty) \to I$ *such that*

$$\phi(t) = \begin{cases} e^{\eta_0(t) + \int_1^{1/t} \xi_0(s) \frac{ds}{s}} & \text{if } t \in (0, 1] \\ e^{\eta_\infty(t) + \int_1^t \xi_\infty(s) \frac{ds}{s}} & \text{if } t \in [1, \infty) \end{cases}.$$

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One has $SV \subset \mathfrak{B}^\infty \subset R$.

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If σ is an admissible sequence, then there exist two bounded continuous functions $\eta, \xi : [1, \infty) \to I$ such that

$$\sigma_j = e^{\eta(2^j) + \int_1^{2^j} \xi(s) \frac{ds}{s}},$$

for all $j \in \mathbb{N}$.

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Properties of the admissible sequences

If σ is an admissible sequence, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$C^{-1}2^{(\underline{s}(\sigma)-\varepsilon)j} \leq \underline{\sigma}_j \leq \frac{\sigma_{j+k}}{\sigma_k} \leq \overline{\sigma}_j \leq C2^{(\overline{s}(\sigma)+\varepsilon)j},$$

for any $j, k \in \mathbb{N}$.

Remark

The previous inequalities are not necessarily valid for $\varepsilon = 0$.

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Some instructive examples

Consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define the admissible sequence σ by

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ 2^{j_{2n}} 4^{j - j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases} .$$

The sequence oscillates between $(j)_j$ and $(2^j)_j$ and we have $\underline{s}(\sigma) = 0$ and $\overline{s}(\sigma) = 1$.

Some instructive examples

Let $\sigma_0 = 1$, $\alpha > 0$ and σ be defined by

$$\sigma_{j+1} := \begin{cases} \sigma_j & \text{if } j_{2n} \leq j \leq j_{2n+1} \\ \sigma_j 2^\alpha & \text{if } j_{2n+1} \leq j < j_{2n+2} \end{cases} .$$

We have $\underline{s}(\sigma) = 0$, $\overline{s}(\sigma) = 1$ and for all $\varepsilon > 0$, there exists $C > 0$ such that $\sigma_j \leq C\, 2^{j\varepsilon}$ for all j.

Relations between Boyd functions and admissible sequences

Proposition

If $\phi \in \mathcal{B}$ *and* $\sigma_j = \phi(2^j)$ *or* $\sigma_j = 1/\phi(2^{-j})$ *then we have* $\underline{b}(\phi) \leq \underline{s}(\sigma) \leq \overline{s}(\sigma) \leq \overline{b}(\phi)$.

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ *and* $\theta_j = 1/\phi(2^{-j})$ *then*

$$\underline{b}(\phi) = \min\{\underline{s}(\sigma), \underline{s}(\theta)\} \quad \text{and} \quad \overline{b}(\phi) = \max\{\overline{s}(\sigma), \overline{s}(\theta)\}.$$

Corollary

If ϕ *belongs to* $\mathcal{B}$, *then we have* $\underline{b}(\phi) = \min\{\underline{s}(\tau_1(\phi)), \underline{s}(\tau_2(\phi))\}$ *and* $\overline{b}(\phi) = \max\{\overline{s}(\tau_1(\phi)), \overline{s}(\tau_2(\phi))\}$.

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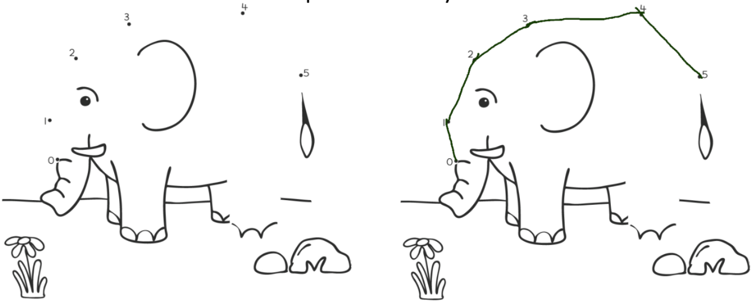
Boyd function obtained from one admissible sequence

Admissible sequences vs Boyd functions



Boyd function obtained from one admissible sequence

Some elementary examples :

$$\phi_\sigma(t) = \begin{cases} \dfrac{\sigma_{j+1} - \sigma_j}{2^j}(t - 2^j) + \sigma_j & \text{if } t \in [2^j, 2^{j+1}), \\ \dfrac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} & \text{if } t \in (2^{-j-1}, 2^{-j}]. \end{cases}$$

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where s satisfies $\underline{s}(\sigma) \leq s \leq \overline{s}(\sigma)$.

Constructing a regular Boyd function from an admissible sequence

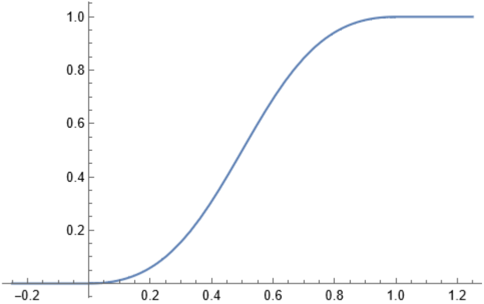
Let

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$$

to define

$$g : x \mapsto \frac{f(x)}{f(x) + f(1-x)}$$

on $[0, 1]$.



Constructing a regular Boyd function from an admissible sequence

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For $j \in \mathbb{N}$, we set

$$\begin{cases} X_j = 2^j \cos\alpha + \sigma_j \sin\alpha \\ Y_j = -2^j \sin\alpha + \sigma_j \cos\alpha \end{cases},$$

$$\xi^{(j)}(X) = \frac{X - X_j}{X_{j+1} - X_j}$$

and

$$\tau^{(j)}(X) = Y_j + (Y_{j+1} - Y_j)X$$

to consider the curve

$$Y = \tau^{(j)}(g(\xi^{(j)}(X)))$$

on $[X_j, X_{j+1}]$.

Constructing a regular Boyd function from an admissible sequence

It gives rise to

$$Y(y) = \tau^{(j)}(g(\xi^{(j)}(X(x))))$$

on the original Euclidean plane.

Let $\eta_j^{(\alpha)}$ be the function $x \mapsto y$ on $[2^j, 2^{j+1}]$.

We can construct $\phi \in \mathcal{B}$ by setting

$$\phi(t) = \begin{cases} \eta_j^{(\alpha)}(t) & \text{if } t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0 \\ \frac{1}{\phi(1/t)} & \text{if } t \in (0,1) \end{cases}.$$

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Constructing a regular Boyd function from an admissible sequence

For $\alpha = 0$, we explicitly get

$$\eta_j^{(0)}(t) = \sigma_j + \frac{\sigma_{j+1} - \sigma_j}{1 + \left(\frac{t-2^{j+1}}{t-2^j}\right)^2}.$$

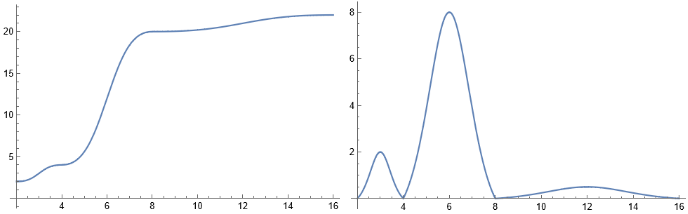
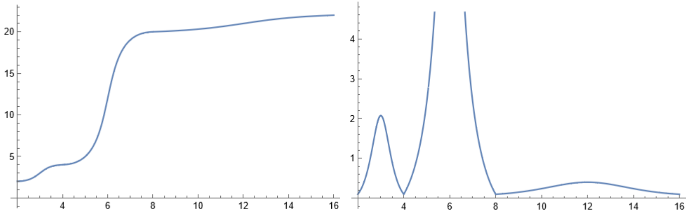


Figure: The function $\eta^{(\alpha)}$ (left panel) and its derivative (right panel) for $\alpha = 0$ and σ such that $\sigma_1 = 2$, $\sigma_2 = 4$, $\sigma_3 = 20$ and $\sigma_4 = 22$.

Constructing a regular Boyd function from an admissible sequence

If $\alpha > 0$ is small enough, we get a function $\eta_j^{(\alpha)}$ whose explicit form is far more complicated.



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Constructing a regular Boyd function from an admissible sequence

Let $\mathcal{B}'$ denote the set of functions $f : I \to I$ that belong to $C^1(I)$ with $f(1) = 1$ and satisfy

$$0 < \inf_{t>0} t\frac{|f'(t)|}{f(t)} \leq \sup_{t>0} t\frac{|f'(t)|}{f(t)} < \infty.$$

One can show that $\mathcal{B}'$ is a subset of $\mathcal{B}$. If $\phi \in \mathcal{B}$ with $\underline{b}(\phi) > 0$ (resp. $\overline{b}(\phi) < 0$), then there exists a non-decreasing bijection (resp. a non-increasing bijection) $\psi \in \mathcal{B}'$ such that $\phi \sim \psi$ and $\psi^{-1} \in \mathcal{B}'$

Constructing a regular Boyd function from an admissible sequence

Proposition

If σ is an admissible sequence such that either $\underline{s}(\sigma) > 0$ or $\overline{s}(\sigma) < 0$, then there exists $\xi \in \mathcal{B}' \cap C^\infty(I)$ such that $(\xi(2^j))_j \sim \sigma$.

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An handy way to define generalized spaces

- Let $\mathcal{J}$ be Bessel operator of order s:

$$\mathcal{J}^s f = \mathcal{F}^{-1}((1 + |\cdot|^2)^{s/2} \mathcal{F} f),$$

for $f \in \mathcal{S}'$ and $s \in \mathbb{R}$.

- The fractional Sobolev space H_p^s ($p \in [1, \infty]$) is defined by

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$$x^m |\phi^{(m)}(x)| < C_m \phi(x),$$

for all $m \in \mathbb{N}$ and $x \geq 1$.

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- Given $\phi \in \mathcal{B}''$, the generalized Bessel operator $\mathcal{J}^\phi$ is given by

$$\mathcal{J}^\phi f = \mathcal{F}^{-1}(\phi(\sqrt{1+|\cdot|^2})\mathcal{F}f),$$

for $f \in \mathcal{S}'$.

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An handy way to define generalized spaces

- Let $q \in [1, \infty]$ and $\phi \in \mathcal{B}$; given a Banach space X, the space $l^q_\phi(X)$ is the space of sequences $(a_j)_j$ of X such that

$$(\phi(2^j)\|a_j\|_X) \in l^q.$$

  We set $l^q_\phi = l^q_\phi(\mathbb{C})$.

- This space is equipped with the norm

$$\|(a_j)_j\|_{l^q_\phi(X)} = \|\phi(2^j)\|a_j\|_X\|_{l^q}.$$

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- This space is equipped with the norm

$$\|(a_j)_j\|_{l^q_\phi(X)} = \|\phi(2^j)\|a_j\|_X\|_{l^q}.$$

- If $\phi(t) = t^s$, we write $l^q_s(X) = l^q_\phi(X)$.

An handy way to define generalized spaces

- Let $\varphi \in \mathcal{S}$ be a function with support in $\{x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2\}$ such that $\varphi(x) > 0$ and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}x) = 1$ for $x \neq 0$.
- Let us define ψ_0 and Φ_j ($j \in \mathbb{Z}$) such that

$$\mathcal{F}\psi_0(x) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}x)$$

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- We set $\varphi_0 = \psi_0$ and $\varphi_j = \Phi_j$ for $j \in \mathbb{N}$; these functions are called test functions.

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- Let $p, q \in [1, \infty]$ and $\phi \in \mathcal{B}$; the generalized Besov space $B^{\phi}_{p,q}$ is defined by

$$B^{\phi}_{p,q} = \{f \in \mathcal{S}' : (\varphi_k * f)_k \in l^q_{\phi}(L^p)\}.$$

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- An object B in a given category is a retract of an object A if there exist morphisms $u : B \to A$ and $v : A \to B$ in the category such that $v \circ u$ is the identity on B.
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Some conventions

- Two topological vector spaces A_0 and A_1 are *compatible* if there is a Hausdorff topological vector space H such that A_0 and A_1 are sub-spaces of H.
- $\mathscr{N}$ denotes the category of all normed vector spaces (a sub-category of all topological vector spaces).
- $\mathscr{C}$ denotes any sub-category of the category $\mathscr{N}$ that is closed under the operations sum and intersection
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- The morphisms $T : (A_0, A_1) \to (B_0, B_1)$ in $\mathscr{C}_1$ are all bounded linear mappings from $A_0 + A_1$ to $B_0 + B_1$ such that

$$T_{A_0} : A_0 \to B_0, \quad T_{A_1} : A_1 \to B_1$$

are morphisms in $\mathscr{C}$. In the sequel, T will stand for the restrictions to the various subspaces of $A_0 + A_1$.

- Two basic functors from $\mathscr{C}_1$ to $\mathscr{C}$: $\Sigma(T) = \Delta(T) = T$ and

$$\Delta(\overline{A}) = A_0 \cap A_1,$$

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Interpolation spaces

- Let $\overline{A} = (A_0, A_1)$ be a given couple in $\mathscr{C}_1$. Then a space A in $\mathscr{C}$ will be called an *intermediate space* between A_0 and A_1 (or with respect to $\overline{A}$) if

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with continuous inclusions.

- The space A is called an *interpolation space* between A_0 and A_l (or with respect to $\overline{A}$) if in addition

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- More generally, let $\overline{A}$ and $\overline{B}$ be two couples in $\mathscr{C}_1$. Then we say that two spaces A and B in $\mathscr{C}$ are *interpolation spaces* with respect to $\overline{A}$ and $\overline{B}$ if A and B are intermediate spaces with respect to A and B respectively, and if

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- If $\|T\|_{A,B} \leq \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$ holds, we shall say that A and B are *exact* interpolation spaces.
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- The interpolation spaces A and B are of *exponent* $\theta \in [0, 1]$ if

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Generalized interpolation spaces

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- An *interpolation functor* on $\mathscr{C}$ is a functor F from $\mathscr{C}_1$ into $\mathscr{C}$ such that if $\overline{A}$ and $\overline{B}$ are couples in $\mathscr{C}_1$, then $F(\overline{A})$ and $F(\overline{B})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{B}$ and

$$F(T) = T \text{ for all } T : \overline{A} \to \overline{B}.$$

- F is a *uniform* (*exact*) interpolation functor if $F(\overline{A})$ and $F(\overline{B})$ are uniform (exact) interpolation spaces with respect to $\overline{A}$ and $\overline{B}$. Similarly, F is (*exact*) of *exponent* θ ($\phi \in \mathcal{B}$) if $F(\overline{A})$ and $F(\overline{B})$ are (*exact*) of exponent θ ($\phi \in \mathcal{B}$).

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The K-method

The K-operator of interpolation is defined for $t > 0$ and $a \in \Sigma(\overline{A})$ by

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1\}.$$

If $\theta \in (0, 1)$ and $q \in [1, \infty]$, then a belongs to the interpolation space $K_{\theta,q}(A_0, A_1)$ if $a \in \Sigma(\overline{A})$ and

$$(2^{-\theta j} K(2^j, a))_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}).$$

This last condition is equivalent to $t \mapsto t^{-\theta} K(t, a) \in L^q_*$.

For example, $B^s_{p,q} = K_{\alpha,q}(H^t_p, H^u_p)$ for $s = (1 - \alpha)t + \alpha u$.

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Let $\phi \in \mathcal{B}$ and $q \in [1, \infty]$, we let $K_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{\phi,q,K} := \int_0^\infty (\frac{1}{\phi(t)} K(t,a))^q \frac{dt}{t} < \infty$$

holds.

Theorem

$K_{\phi,q}$ is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$. Moreover, we have

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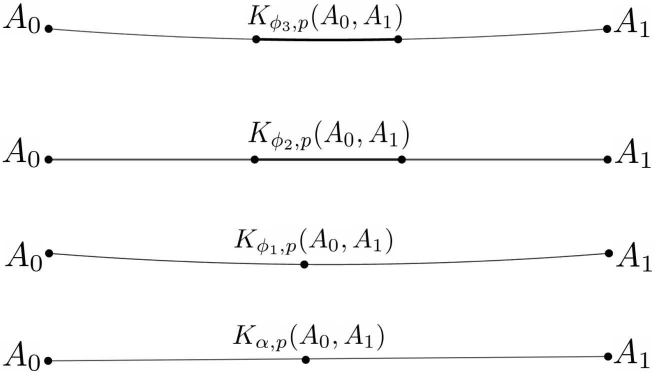

Figure: Differents interpolation spaces where for example $\phi_1(t) = t^\alpha \log(1/t)$, $\phi_2(t) = t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}$ and $\phi_3(t) = (t^\alpha \chi_{]0,1]} + t^\beta \chi_{]1,\infty[}) \log(1/t)$.

The K-method

Given $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, a belongs to the generalized interpolation space $[A_0, A_1]^{\gamma}_{\phi,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\gamma}_{\phi,q}} := \|\phi(t)^{-1} K(\gamma(t), a)\|_{L^q_*} < \infty.$$

Proposition

If $\phi, \gamma \in \mathcal{B}$ and $q \in [1, \infty]$, then a belongs to $[A_0, A_1]^{\gamma}_{\phi,q}$ if and only if $\sum_{j \in \mathbb{Z}} \left(\frac{1}{\phi(2^j)} K(\gamma(2^j), a)\right)^q < \infty$.

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Let σ be an admissible sequence and $q \in [1, \infty]$; a belongs to the upper generalized interpolation space $[A_0, A_1]^{\wedge}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\wedge}_{\sigma,q}} := \sum_{j=1}^{\infty} \frac{1}{\sigma_j} K(2^j, a) < \infty.$$

In the same way, a belongs to the lower generalized interpolation space $[A_0, A_1]^{\vee}_{\sigma,q}$ if $a \in A_0 + A_1$ and

$$\|a\|_{[A_0,A_1]^{\vee}_{K,\sigma,q}} := \sum_{j=1}^{\infty} \sigma_j K(2^{-j}, a) < \infty.$$

Proposition

If $\phi \in \mathcal{B}$, $\sigma_j = \phi(2^j)$ and $\theta_j = 1/\phi(2^{-j})$ then

$$K_{\phi,q}(A_0, A_1) = [A_0, A_1]^{\vee}_{\delta,q} \cap [A_0, A_1]^{\wedge}_{\sigma,q}.$$

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Proposition

Let $\phi \in \mathcal{B}$, $1 \le p \le q \le +\infty$, one has

$$K_{\phi,p}(\overline{A}) \subseteq K_{\phi,q}(\overline{A}),$$

with continuous embedding.

Proposition

Let $q, q_0, q_1 \in [1, +\infty]$ and $\phi, \phi_0, \phi_1 \in \mathcal{B}$ such that

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Then,

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Proposition

If A_0 and A_1 are complete, then so is $K_{\phi,q}(\overline{A})$.

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If A_0 *and* A_1 *are complete, then so is* $K_{\phi,q}(\overline{A})$.

The J-method

Let $\phi \in \mathcal{B}$, $q \in [1, \infty]$, and let $J_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ which can be represented by $a = \int_0^\infty b(t) \frac{dt}{t}$, where $b(t) \in \Delta(\overline{A})$ for all $t > 0$ and

$$t \mapsto \phi(t)^{-1} J(\gamma(t), b(t)) \in L^q_\star.$$

This space is equipped with the norm

$$\|a\|_{\phi,q,J} := \inf_b \left\| \phi(t)^{-1} J(\gamma(t, b(t)) \right\|_{L^q_\star},$$

the infimum being taken on all b such that $a = \int_0^\infty b(t) \frac{dt}{t}$.

The J-method

Let $\phi \in \mathcal{B}$, $q \in [1, \infty]$, and let $J_{\phi,q}(\overline{A})$ denote the space of all $a \in \Sigma(\overline{A})$ which can be represented by $a = \int_0^\infty b(t) \frac{dt}{t}$, where $b(t) \in \Delta(\overline{A})$ for all $t > 0$ and

$$t \mapsto \phi(t)^{-1} J(\gamma(t), b(t)) \in L^q_\star.$$

This space is equipped with the norm

$$\|a\|_{\phi,q,J} := \inf_b \left\| \phi(t)^{-1} J(\gamma(t, b(t)) \right\|_{L^q_\star},$$

the infimum being taken on all b such that $a = \int_0^\infty b(t) \frac{dt}{t}$.

The J-method

Theorem

$J_{\phi,q}$ is an exact interpolation functor of exponent $\phi \in \mathcal{B}$ on the category $\mathscr{N}$. Moreover, we have

$$\|a\|_{\phi,q,J} \leq C\,\frac{1}{\phi(t)}J(t,a)$$

for $a \in \Delta(\overline{A})$.

Equivalence Theorem

Theorem

Let $\phi \in \mathcal{B}$ *such that* $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$, $q \in [1, \infty]$, *then*

$$J_{\phi,q}(\overline{A}) = K_{\phi,q}(\overline{A}).$$

A density result

Proposition

- *If* $q < \infty$, *then* $\Delta(\overline{A})$ *is dense in* $K_{\phi,q}(\overline{A})$,
- *The closure of* $\Delta(\overline{A})$ *in* $K_{\phi,\infty}(\overline{A})$ *is the space* $\overline{A}^0_{\phi,\infty}$ *of all a such that*

$$\frac{1}{\phi(t)}K(t,a) \to 0 \quad \textit{as} \quad t \to 0 \quad \textit{or} \quad t \to \infty.$$

Reiteration theorem

Let $\overline{A}$ be a given couple of normed vector spaces and $\phi \in \mathcal{B}$ and suppose that X is an intermediate space with respect to $\overline{A}$.
Then,

- X is *of class* $\mathscr{C}_K(\phi; \overline{A})$ if $K(t, a) \leq C\, \phi(t) \|a\|_X$ for all $a \in X$;
- X is *of class* $\mathscr{C}_J(\phi; \overline{A})$ if $\|a\|_X \leq C\, \frac{1}{\phi(t)} J(t, a)$ for all $a \in \Delta(\overline{A})$.

Furthermore, X is *of class* $\mathscr{C}(\phi; \overline{A})$ if X is of class $\mathscr{C}_K(\phi; \overline{A})$ and $\mathscr{C}_J(\phi; \overline{A})$.
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Reiteration theorem

Theorem

If $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$, *then*

- *X is of class* $\mathscr{C}_K(\phi; \overline{A})$ *iif*

$$\Delta(\overline{A}) \subset X \subset K_{\phi,\infty}(\overline{A}).$$

- *a Banach space X is of class* $\mathscr{C}_J(\phi; \overline{A})$ *iif*

$$K_{\phi,1}(\overline{A}) \subset X \subset \Sigma(\overline{A}).$$

Reiteration theorem

Theorem

For $j \in \{0,1\}$, assume that X_j are complete and of class $\mathscr{C}(\phi_j; \overline{A})$, where $0 \leq \underline{b}(\phi_j) \leq \overline{b}(\phi_j) \leq 1$.
Let $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$ and set $f = \phi_1/\phi_0$ and $\psi = \phi_0 \,.\, (\phi \circ f)$.
If $\underline{b}(f) > 0$ or $\overline{b}(f) < 0$, then

$$K_{\phi,q}(\overline{X}) = K_{\psi,q}(\overline{A}).$$

In particular, if $0 < \underline{b}(\phi_j) \leq \overline{b}(\phi_j) < 1$ and $K_{\phi_j,q_j}(\overline{A})$ are complete, then

$$K_{\phi,q}((K_{\phi_0,q_0}(\overline{A}), K_{\phi_1,q_1}(\overline{A}))) = K_{\psi,q}(\overline{A}).$$

There exists a version for semi-normed spaces.

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There exists a version for semi-normed spaces.

The duality theorem

Theorem

Let $\overline{A}$ be a couple of Banach spaces such that $\Delta(\overline{A})$ is dense in A_0 and A_1.
Assume that $1 \leq q < \infty$ and $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$.
Then,

$$K_{\phi,q}(\overline{A})' = K_{1/\phi(1/\cdot),q'}(\overline{A'}),$$

where $1/q + 1/q' = 1$.

For the case $q = \infty$, one has $(\overline{A}^0_{\phi,\infty})' = K_{1/\phi(1/\cdot),1}(\overline{A'})$.

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First compactness theorem

Theorem

Let T be a bounded linear operator between a couple of Banach spaces (A_0, A_1) and a Banach space B.
If $T : A_0 \to B$ is compact and if E is of class $\mathscr{C}_K(\phi; \overline{A})$ for some $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$,
then $T : E \to B$ is a compact operator.

Second compactness theorem

Theorem

Let T be a bounded linear operator between a Banach space B and a couple of Banach spaces (A_0, A_1).
If $T : B \to A_0$ is compact and if E is of class $\mathscr{C}_J(\phi; \overline{A})$ for some $\phi \in \mathcal{B}$ such that $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$,
then $T : B \to E$ is a compact operator.

Compactness theorems

Corollary

If A_0 *and* A_1 *are Banach spaces,* $A_1 \subset A_0$ *compactly and* $0 < \underline{b}(\phi_0) \leq \overline{b}(\phi_0) < \underline{b}(\phi_1) \leq \overline{b}(\phi_1) < 1$*, then*

$$K_{\phi_1,q_1}(\overline{A}) \subset K_{\phi_0,q_0}(\overline{A}),$$

with compact inclusion.

An extremal property

Theorem

Suppose that F is an interpolation functor of exponent ϕ, where $0 < \underline{b}(\phi) \leq \overline{b}(\phi) < 1$.
Then, for any compatible Banach couple $\overline{A}$, one has

$$J_{1/\overline{1/\phi},1}(\overline{A}) \subset F(\overline{A}).$$

Moreover, if $\Delta(\overline{A})$ is dense in A_0 and A_1, then

$$F(\overline{A}) \subset K_{\overline{\phi},\infty}(\overline{A}).$$

Power theorem

Theorem

$$K_{\phi,q}((A_0^p, A_1^p))^{1/p} = K_{\psi,pq}(A_0, A_1),$$

where $\psi(t) = \phi(t^p)^{1/p}$.

Generalized interpolation for triples

Let $\overline{A} = (A_0, A_1, A_2)$ be a Banach or quasi-Banach triple.
We define

$$K(t_1, t_2, a) = \inf\{\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + t_2\|a_2\|_{A_2} : a = a_0 + a_1 + a_2\}.$$

Let
$H = \{(\phi, \psi) \in \mathcal{B} \times \mathcal{B} : \underline{b}(\phi) > 0,\ \underline{b}(\psi) > 0 \text{ and } \overline{b}(\phi) + \overline{b}(\psi) < 1\}$.
Given $(\phi, \psi) \in H$, we define $K_{\phi,\psi,p}(\overline{A})$ as the space of all $a \in \Sigma(\overline{A})$ such that

$$\|a\|_{K_{\phi,\psi,p}(\overline{A})} := \left(\iint_{(0,\infty)^2} \left(\frac{1}{\phi(t_1)} \frac{1}{\psi(t_2)} K(t_1, t_2) \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/p} < \infty.$$

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Reiteration formula

Theorem

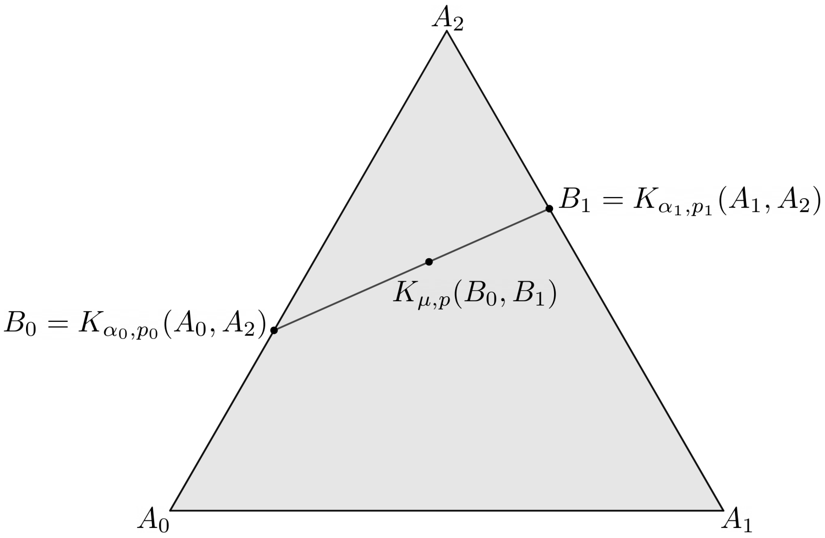
Let $\overline{A}$ be a triple of Banach or quasi-Banach function lattices. If $p, p_0, p_1 \in [1, \infty]$ and

$$\frac{1}{p} = \frac{1-\mu}{p_0} + \frac{\mu}{p_1},$$

then

$$K_{\mu,p}\left(K_{\alpha_0,p_0}(A_0, A_2), K_{\alpha_1,p_1}(A_1, A_2)\right) = K_{\theta_1,\theta_2,p}(A_0, A_1, A_2),$$

where $\theta_1 = (1-\alpha_1)\mu$ and $\theta_2 = \alpha_0(1-\mu) + \alpha_1\mu$.


A_2
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A_0
A_1

Generalized Reiteration formula

Theorem

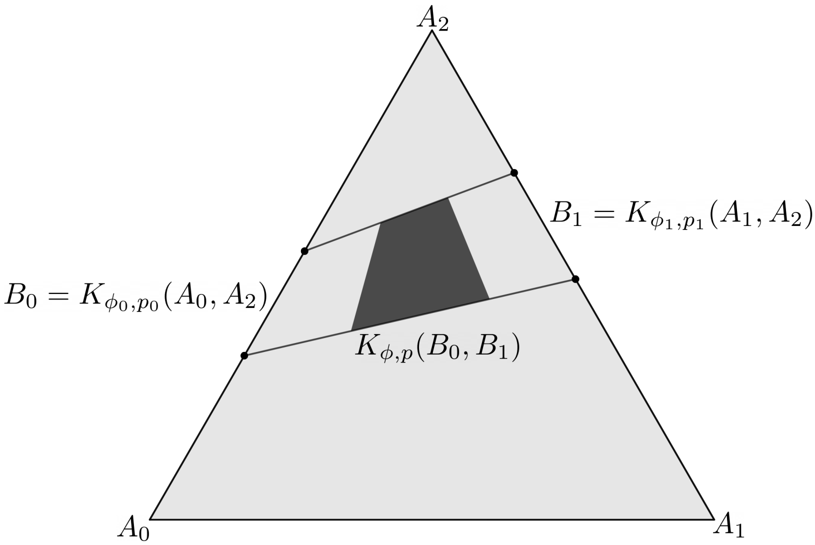
Let $\overline{A}$ be a triple of Banach or quasi-Banach function lattices. If $p, p_0, p_1 \in [1, \infty]$ and

$$t^{\frac{1}{p}} = \left(\frac{t}{\phi(t)}\right)^{\frac{1}{p_0}} + \phi(t)^{\frac{1}{p_1}}$$

for all $t > 0$, then

$$K_{\phi,p}\left(K_{\phi_0,p_0}(A_0, A_2), K_{\phi_1,p_1}(A_1, A_2)\right) = K_{\psi_1,\psi_2,p}(A_0, A_1, A_2),$$

where $\psi_1(t) = \phi\left(\frac{t}{\phi_1(t)}\right)$ and $\psi_2(t) = \frac{\phi_0(t)}{\phi(\phi_0(t))}\phi(\phi_1(t))$.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Thank you for your attention !