S-adic characterization of minimal dendric shifts: an example

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Notations

- (unidimensional) minimal shift spaces: X, Y, ...
- language of the shift space X:

$$\mathcal{L}(X) = \bigcup_{x \in X} \mathcal{L}(x)$$

- image of a X under σ:

$$\sigma(X) = \left\{ S^k \sigma(x) \mid x \in X, 0 \leq k < |\sigma(x_0)| \right\}$$

Definitions

Extensions

Left and right extensions:

$$E_X^L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(X)\}, \quad E_X^R(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}(X)\}$$

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Bi-extensions:

$$E_X(w) = \{(a, b) \in E_X^L(w) \times E_X^R(w) \mid awb \in \mathcal{L}(X)\}$$

Extension graphs

Definition

The *extension graph* of $w \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}_X(w)$ with vertices $E^L_X(w) \sqcup E^R_X(w)$ and edges $E_X(w)$.

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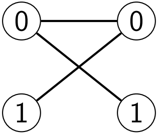
If X is the Fibonacci shift space,

$\mathcal{E}_X(\varepsilon)$



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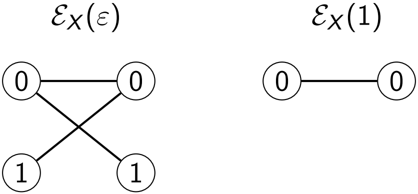


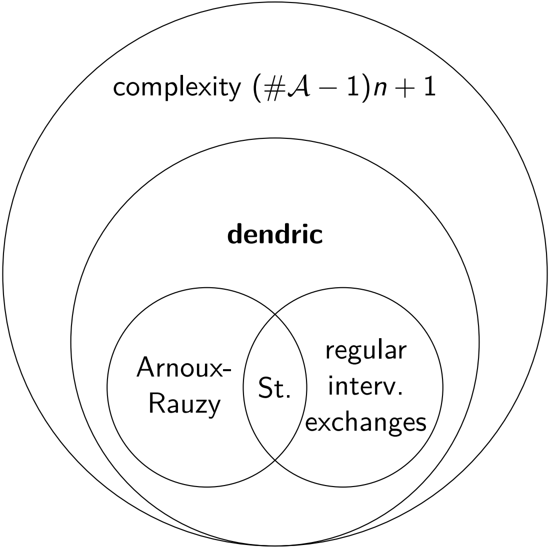
Dendric words

Definition (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

A word $w \in \mathcal{L}(X)$ is *dendric* if its extension graph $\mathcal{E}_X(w)$ is a tree.

A shift space X is *dendric* if all the words $w \in \mathcal{L}(X)$ are.

Relation with other families



Derived shift spaces: example

$$x = \ldots 0102010102010010201 \ldots$$

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$$x = \ldots 01020101020100102 01 \ldots$$

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$$x = \ldots 0 \mid 1020 \mid 10 \mid 1020 \mid 100 \mid 1020 \mid 1 \ldots$$

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If

$$\sigma : \begin{cases} a \mapsto 10 \\ b \mapsto 100 \\ c \mapsto 1020 \end{cases}$$

Derived shift spaces: example

$$x = \dots 0 \mid 1020 \mid 10 \mid 1020 \mid 100 \mid 1020 \mid 1 \dots$$

If

$$\sigma : \begin{cases} a \mapsto 10 \\ b \mapsto 100 \\ c \mapsto 1020 \end{cases}$$

then $x = S^k\sigma(y)$ where $0 \leq k < |\sigma(y_0)|$ and

$$y = \dots cacbc \dots$$

Derived shift spaces

Definition

A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is *strongly left proper* (slp) if there exists a letter $\ell \in \mathcal{B}$ such that

$$\sigma(a) \in \ell(\mathcal{B} \setminus \{\ell\})^*, \quad \forall a \in \mathcal{A}.$$

Definition

The *derived shift* of a minimal shift space $X \subseteq \mathcal{B}^{\mathbb{Z}}$ with respect to $\ell \in \mathcal{B}$ is 'the' shift space $Y \subseteq \mathcal{A}^{\mathbb{Z}}$ such that $X = \sigma(Y)$ for some injective morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ slp for the letter ℓ.

Case of dendric shifts

If $X \subseteq \mathcal{B}^{\mathbb{Z}}$ is a minimal dendric shift space and $Y \subseteq \mathcal{A}^{\mathbb{Z}}$ is its derived shift with respect to ℓ, then...

Theorem (Balková, Pelantová, Steiner)

... $\#\mathcal{A} = \#\mathcal{B}$.

Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

... Y is also a minimal dendric shift space.

S-adic representations

Definition

A primitive *S-adic representation* of a minimal shift space X is a primitive sequence of morphisms $(\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_n$ such that

$$\mathcal{L}(X) = \bigcup_N \text{Fac}(\sigma_0 \dots \sigma_N(\mathcal{A}_{N+1})).$$

A sequence $(\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_n$ is *primitive* if, for all N, there exists $m \geq 0$ such that, for all $a \in \mathcal{A}_{N+m+1}$, $\sigma_N \dots \sigma_{N+m}(a)$ contains all the letters of $\mathcal{A}_N$.

S-adic representations of dendric shifts

Every minimal dendric shift over $\mathcal{A}$ has an S-adic representation such that

- morphisms are injective and slp,

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- morphisms are injective and slp,
- intermediary shift spaces are dendric shifts over $\mathcal{A}$,
- the letter for which the k^{th} morphism is slp is right special in the k^{th} intermediary shift space.

Dendric shifts having one right special factor

Dendric shifts and right special factors

Proposition

If X is a dendric shift over $\mathcal{A}$, then for all $n \in \mathbb{N}$,

$$\sum_{w \in \mathcal{L}_n(X)} \left(\# E_X^R(w) - 1 \right) = \#\mathcal{A} - 1.$$

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Corollary

If X is a dendric shift over $\mathcal{A}$, then the following are equivalent:

1. *for each length, X has a unique right special factor,*
2. *for each length, X has a right special factor w such that $E_X^R(w) = \mathcal{A}$,*
3. *X has an infinite number of right special factors w such that $E_X^R(w) = \mathcal{A}$.*

Family $\mathcal{F}$

Definition

The family $\mathcal{F}$ is the family of minimal dendric shifts over $\mathcal{A}_4 := \{0, 1, 2, 3\}$ satisfying the properties of the previous slide.

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Every shift in $\mathcal{F}$ has an S-adic representation such that

- morphisms are injective slp,
- intermediary shift spaces are dendric shifts over $\mathcal{A}_4$,
- the letter for which the k^{th} morphism is slp is the unique right special letter in the k^{th} intermediary shift space.

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$$\sigma(X) \in \mathcal{F} \overset{?}{\Longrightarrow} X \in \mathcal{F}$$

Unique antecedent: example

$$\sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 032 \end{cases}$$

2020010

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2 | 02 | 0 | 01 | 0

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$$2 \mid 02 \mid 0 \mid 01 \mid 0 = 2\sigma(201)0$$

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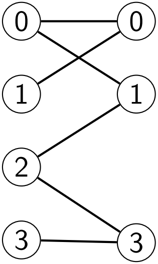
$\mathcal{E}_X(201)$



$\mathcal{E}_{\sigma(X)}(2020010)$

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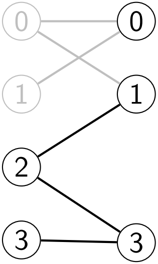

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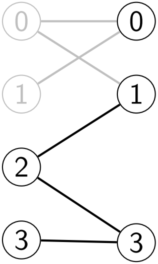

$$\varphi_2^L : \begin{cases} 2 \mapsto 0 \\ 3 \mapsto 3 \end{cases}$$

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$$\sigma : \begin{cases} 0 \mapsto 00 \\ 1 \mapsto 010 \\ 2 \mapsto 020 \\ 3 \mapsto 0320 \end{cases}$$

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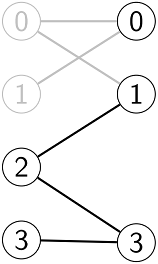
$\mathcal{E}_X(201)$



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$$\varphi_0^R : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \end{cases}$$

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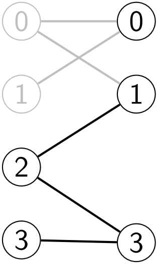

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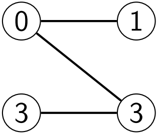

Unique antecedent

Proposition (G., Lejeune, Leroy)

Let $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ be an injective slp morphism for ℓ. If $u \in \mathcal{L}(\sigma(X))$ is such that $|u|_\ell \geq 1$, it has a unique decomposition $(s, v, p) \in \mathcal{A}^* \times \mathcal{L}(X) \times \ell\mathcal{A}^*$ such that

- $u = s\sigma(v)p$,
- *s is a proper suffix of a word of $\sigma(\mathcal{A})$,*
- *p is a proper prefix of a word of $\sigma(\mathcal{A})\ell$.*

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We then have

$$E_{\sigma(X)}(u) = (\varphi_s^L \times \varphi_p^R) E_X(v)$$

where

$$\varphi_s^L : a \mapsto b \text{ st. } \sigma(a) \in \mathcal{A}^* bs \qquad \varphi_p^R : a \mapsto b \text{ st. } \sigma(a)\ell \in pb\mathcal{A}^*.$$

S-adic representations of shifts in $\mathcal{F}$

Every shift in $\mathcal{F}$ has an S-adic representation such that

- all the morphisms are injective and slp,
- all the intermediary shift spaces are in $\mathcal{F}$,
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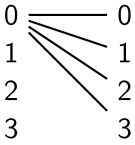
Possible extension graphs

If $X \in \mathcal{F}$, then the extension graph of ε is, up to a permutation, one of



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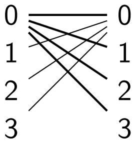

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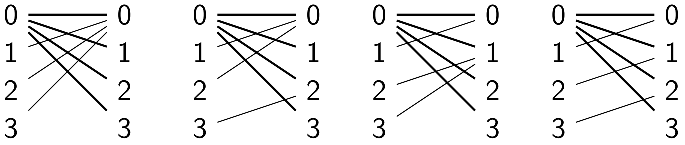

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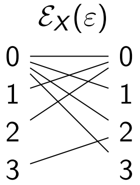

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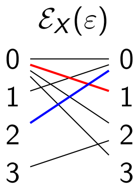

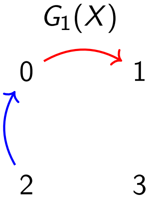

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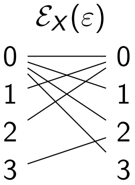

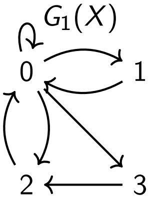

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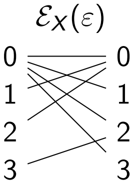
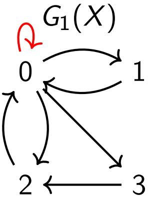
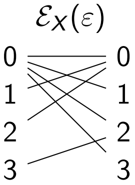
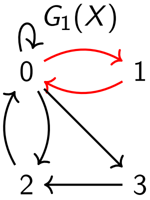




0

Possible morphisms: example

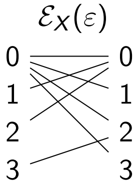


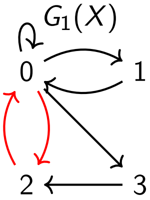


0

01

Possible morphisms: example



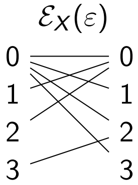


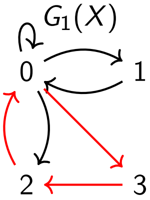
0

01

02

Possible morphisms: example





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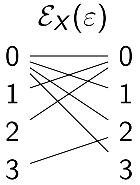
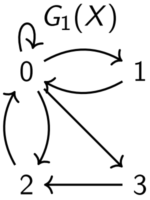
01

02

032

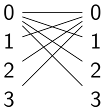
Possible morphisms: example

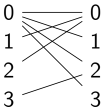


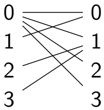


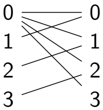
$$\beta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 032 \end{cases}$$

Possible morphisms



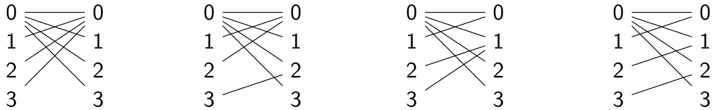






$$\alpha : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 03 \end{cases} \quad \beta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 032 \end{cases} \quad \gamma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \\ 3 \mapsto 031 \end{cases} \quad \delta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \\ 3 \mapsto 0321 \end{cases}$$

Possible morphisms



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If

$$\mathcal{S}_{\mathcal{F}} = \Sigma_4\{\alpha, \beta, \gamma, \delta\}\Sigma_4,$$

every shift in $\mathcal{F}$ has an $\mathcal{S}_{\mathcal{F}}$-adic representation.

Not a characterization

If $\boldsymbol{\sigma} \in \beta\delta\mathcal{S}_{\mathcal{F}}^{\mathbb{N}}$ is an $\mathcal{S}_{\mathcal{F}}$-adic representation of X, then 0 is not dendric in X.

$$\beta \circ \delta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 001 \\ 2 \mapsto 00201 \\ 3 \mapsto 00320201 \end{cases}$$

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$\mathcal{E}_X(0)$

0	0
1	1
2	2
	3

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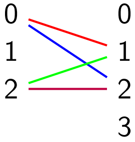
$$\beta \circ \delta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 001 \\ 2 \mapsto 00201 \\ 3 \mapsto 00320201 \end{cases}$$

$\mathcal{E}_X(0)$



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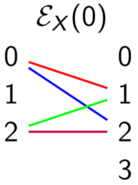

If $\boldsymbol{\sigma}$ is an $\mathcal{S}_{\mathcal{F}}$-adic representation of X such that $\sigma_n = \beta$ and $\sigma_{n+1} = \delta$ for some n, then $X \notin \mathcal{F}$.

How to avoid that?

Question: given $X \in \mathcal{F}$ and $\sigma \in \mathcal{S}_{\mathcal{F}}$, when is $\sigma(X)$ in $\mathcal{F}$?

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Idea: if $u = s\sigma(v)p$ is right special, then $p = \ell \in \mathcal{A}$ and

$$\mathcal{E}_X(v) \xrightarrow{\text{rm. useless vertices}} \mathcal{E}_{X,s}(v) \xrightarrow{\varphi_s^L \times \varphi_\ell^R} \mathcal{E}_{\sigma(X)}(u)$$

Stability of images

Theorem (G., Lejeune, Leroy)

Let $X \in \mathcal{F}$ and $\sigma \in \mathcal{S}_{\mathcal{F}}$. The image $\sigma(X)$ is in $\mathcal{F}$ if and only if, for all $s \in \mathcal{A}_4^*$ and for all $v \in \mathcal{L}(X)$, the graph $\mathcal{E}_{X,s}(v)$ is connected.

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$$\mathcal{A}_s = \{a \in \mathcal{A}_4 \mid \sigma(a) \notin \mathcal{A}_4^* s\}$$

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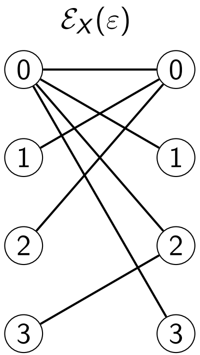
- difficult to check...

Graph $G_n(X)$: example



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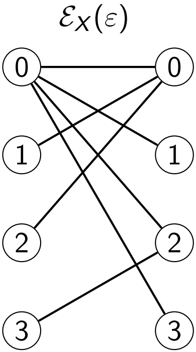
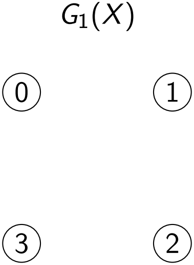
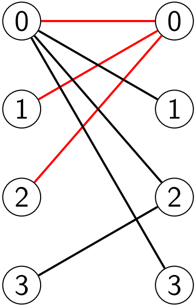


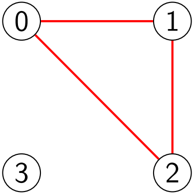


Graph $G_n(X)$: example

$\mathcal{E}_X(\varepsilon)$



$G_1(X)$



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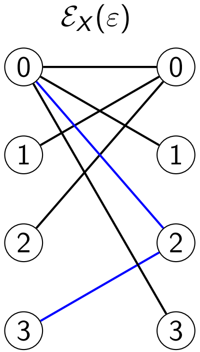
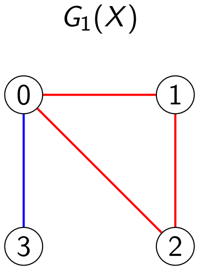
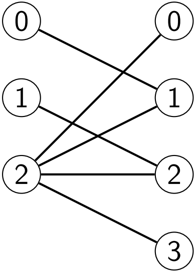




Graph $G_n(X)$: example (II)

$\mathcal{E}_X(0)$



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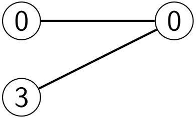

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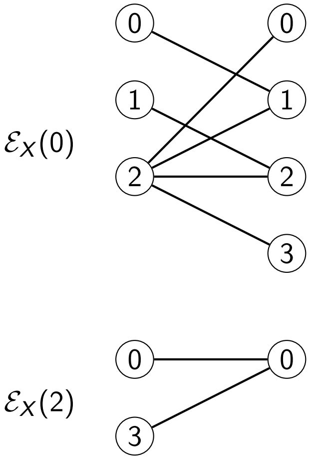

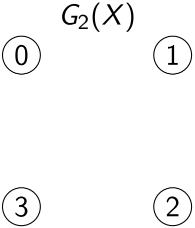

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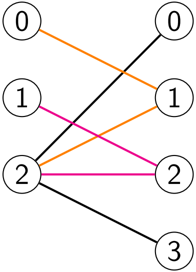

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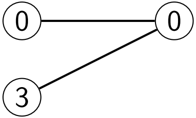
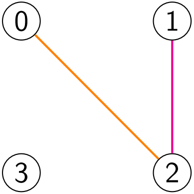


$G_2(X)$



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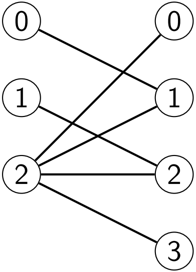

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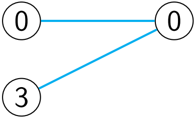

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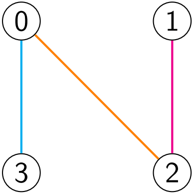

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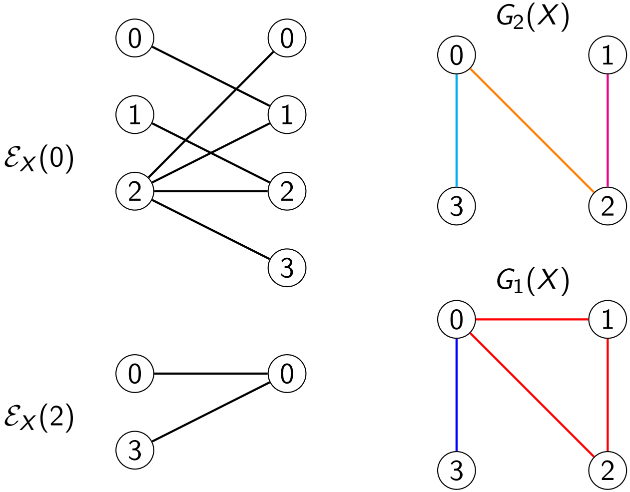

Graph $G(X)$

Definition

If X is in $\mathcal{F}$, $G_n(X)$ is the union of the cliques given by the sets of vertices $E_X^L(v)$ for all $v \in \mathcal{L}(X) \cap \mathcal{A}_4^n$.

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We define

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Proposition

Let $\mathcal{B} \subseteq \mathcal{A}_4$. The following are equivalent.

1. *The graph $G(X) \setminus \mathcal{B}$ is connected.*
2. *For all $v \in \mathcal{L}(X)$, the subgraph of $\mathcal{E}_X(v)$ where we removed, on the left, the vertices in $\mathcal{B}$ (and the isolated vertices) is connected.*

Stability of images (II)

Theorem

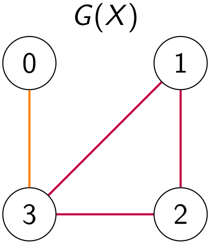
Let $X \in \mathcal{F}$ and $\sigma \in \mathcal{S}_{\mathcal{F}}$. The image $\sigma(X)$ is in $\mathcal{F}$ if and only if

$\mathcal{C}(G(X), \sigma)$: *for all $s \in \mathcal{A}_4^*$, the graph $G(X) \setminus \mathcal{A}_s$ is connected, where*

$$\mathcal{A}_s = \{a \in \mathcal{A}_4 \mid \sigma(a) \notin \mathcal{A}_4^* s\}.$$

Stability of images (II): example

$X \in \mathcal{F}$ st.



$$\beta : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 02 \\ 3 \mapsto 032 \end{cases}$$

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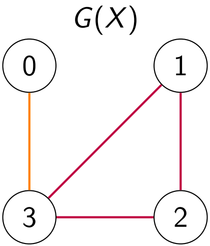

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ε

2

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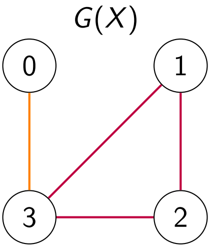

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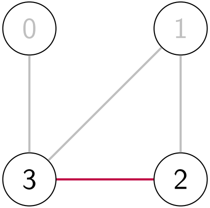
$G(X)$



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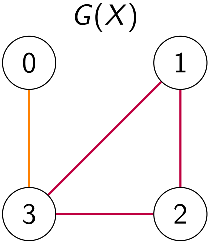
$X \in \mathcal{F}$ st.  $\implies \beta(X) \in \mathcal{F}$

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Image graph

Recall that

$$E^L_{\sigma(X)}(s\sigma(v)\ell) = \varphi^L_s(E^L_X(v)).$$

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Proposition

If X and $\sigma(X)$ are in $\mathcal{F}$ and if $G(X)$ is built with the cliques $C_1, \ldots, C_k$, then $G(\sigma(X))$ is built with the cliques

$$\varphi^L_s(C_i) \qquad s \in \mathcal{A}^*_4, i \leq k.$$

For a graph G, we denote $\sigma(G)$ the graph built with this technique.

Image graph: example

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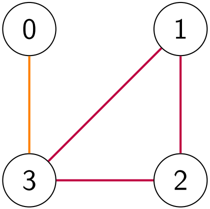

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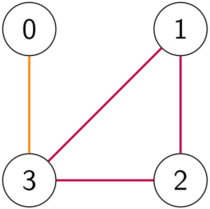

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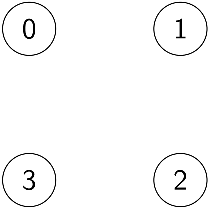

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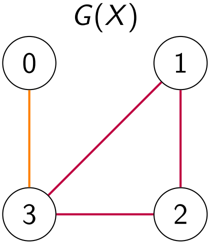

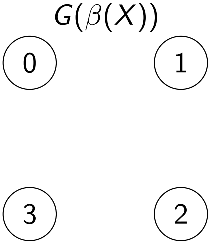

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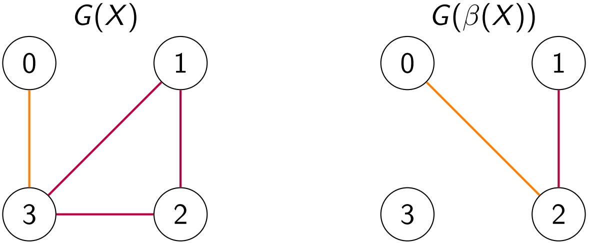

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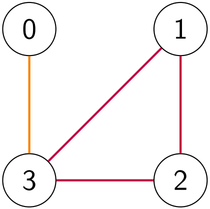
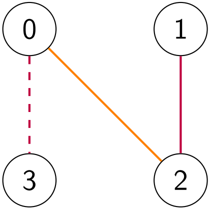


$G(\beta(X))$



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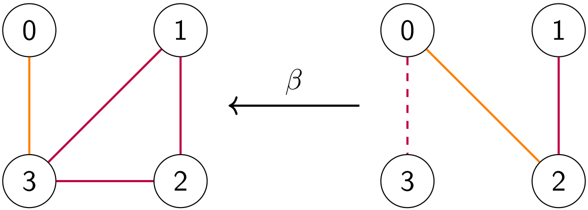

$\mathcal{S}_{\mathcal{F}}$-adic characterization

Theorem

A shift space X is in $\mathcal{F}$ if and only if it has a primitive $\mathcal{S}_{\mathcal{F}}$-adic representation labeling an infinite path in $\mathcal{G}(\mathcal{S}_{\mathcal{F}})$.

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The graph $\mathcal{G}_t(\mathcal{S}_{\mathcal{F}})$ is defined as follows:

- its vertices are the trees over $\mathcal{A}_4$,
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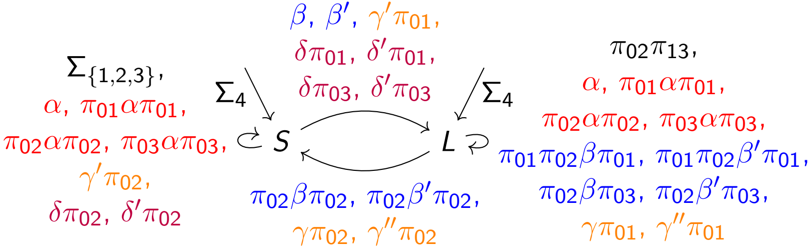

where

$$\beta' = \beta\pi_{23}, \quad \delta' = \delta\pi_{23}, \quad \gamma' = \gamma\pi_{12}, \quad \gamma'' = \pi_{23}\gamma\pi_{23}.$$

General case

What changes?

1. Bigger set of morphisms

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2. So far, we only had to consider the left side

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$$\mathcal{A}_s$$

$$G(X)$$

$$\mathcal{C}(G,\sigma)$$

$$\sigma(G)$$

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$$\mathcal{E}_{X,s} \longrightarrow \mathcal{E}_{X,s}^{L}$$

$$\mathcal{A}_s \longrightarrow \mathcal{A}_s^{L}$$

$$G(X) \longrightarrow G^{L}(X)$$

$$\mathcal{C}(G,\sigma) \longrightarrow \mathcal{C}^{L}(G,\sigma)$$

$$\sigma(G) \longrightarrow \sigma^{L}(G)$$

Set of morphisms

How to find the set $\mathcal{S}_{\mathcal{A}}$:

1. list all possible extension graphs,
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Set of morphisms

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3. problem: often too many paths (sometimes, infinite number of paths),
4. choose all subsets of compatible paths.

Dendric images

Theorem (G., Lejeune, Leroy)

Let X be a dendric shift over $\mathcal{A}$ and $\sigma \in \mathcal{S}_{\mathcal{A}}$. The image $\sigma(X)$ is dendric if and only if the following conditions are satisfied:

- 'L' *for all $s \in \mathcal{A}^*$ and for all $v \in \mathcal{L}(X)$, the graph $\mathcal{E}^L_{X,s}(v)$ is connected,*

Dendric images

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- 'R' *for all $p \in \mathcal{A}^*$ and for all $v \in \mathcal{L}(X)$, the graph $\mathcal{E}^R_{X,p}(v)$ is connected.*

where

$$\mathcal{A}^R_p = \{a \in \mathcal{A} \mid \sigma(a) \notin p\mathcal{A}^*\}$$

and $\mathcal{E}^R_{X,p}(v)$ is the subgraph of $\mathcal{E}_X(v)$ where we removed the right vertices in $\mathcal{A}^R_p$ (and the isolated vertices).

Graphs $G^L(X)$ and $G^R(X)$

Definition

If X is dendric, $G^R_n(X)$ is the union of the cliques given by the sets of vertices $E^R_X(v)$ for all $v \in \mathcal{L}(X) \cap \mathcal{A}^n$ and we define

$$G^R(X) = \lim_n G^R_n(X).$$

Graphs $G^L(X)$ and $G^R(X)$

Definition

If X is dendric, $G_n^R(X)$ is the union of the cliques given by the sets of vertices $E_X^R(v)$ for all $v \in \mathcal{L}(X) \cap \mathcal{A}^n$ and we define

$$G^R(X) = \lim_n G_n^R(X).$$

Theorem (G., Leroy)

Let X be a dendric shift over $\mathcal{A}$ and $\sigma \in \mathcal{S}_{\mathcal{A}}$. The image $\sigma(X)$ is dendric if and only if the following conditions are satisfied:

$\mathcal{C}^L(G^L(X), \sigma)$: *for all $s \in \mathcal{A}^*$, the graph $G^L(X) \setminus \mathcal{A}_s^L$ is connected,*

$\mathcal{C}^R(G^R(X), \sigma)$: *for all $p \in \mathcal{A}^*$, the graph $G^R(X) \setminus \mathcal{A}_p^R$ is connected.*

S-adic characterization of dendric shifts

The graph $\mathcal{G}_t^L(\mathcal{S}_\mathcal{A})$ is defined as follows:

- its vertices are the trees over $\mathcal{A}$,
- there is an edge labeled by $\sigma \in \mathcal{S}_\mathcal{A}$ from G to G' if the condition $\mathcal{C}^L(G', \sigma)$ is satisfied and if $G = \sigma^L(G')$.

S-adic characterization of dendric shifts

The graph $\mathcal{G}_t^R(\mathcal{S}_\mathcal{A})$ is defined as follows:

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Theorem (G., Leroy)

A shift space X over $\mathcal{A}$ is minimal dendric if and only if it has a primitive $\mathcal{S}_\mathcal{A}$-adic representation labeling infinite paths in both $\mathcal{G}_t^L(\mathcal{S}_\mathcal{A})$ and $\mathcal{G}_t^R(\mathcal{S}_\mathcal{A})$.

Thank you for your attention!