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RELATIVE PARACOMPACTNESS AS TAUTNESS CONDITION IN SHEAF THEORY

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RESUME : Nous introduisons la paracompacité relative. Cette notion nous permet d'obtenir un critère de raideur qui unifie et généralise les résultats classiques de [2].

INTRODUCTION

Let X be a topological space, S a subset of X, Φ a family of supports in X and $\mathcal{V}_S$ the set of the open neighborhoods of S in X, ordered by the relation $\supset$. In this paper, we consider only sheaves of abelian groups. We say that S is Φ-taut in X if the canonical morphism

$$(r_S^\cdot : \varinjlim_{V\in\mathcal{V}_S} H^\cdot_{\Phi\cap V}(V,\mathcal{F}|_V) \longrightarrow H^\cdot_{\Phi\cap S}(S,\mathcal{F}|_S))$$

is an isomorphism whenever $\mathcal{F}$ is a sheaf on X. In [2] G.E. Bredon proves that it is equivalent to say that the canonical morphism

$$(r_{SX} : \Gamma_\Phi(X,\mathcal{F}) \longrightarrow \Gamma_{\Phi\cap S}(S,\mathcal{F}|_S))$$

is onto and that $\mathcal{F}|_S$ is $\Phi\cap S$-acyclic whenever $\mathcal{F}$ is a flabby sheaf on X. The tautness appears in the hypothesis of many important theorems of sheaf theory. So, for pratical use, we need criteria stating that S is Φ-taut in X under more explicit topological assumptions on S and Φ. For example, it is trivial to see that an open subset of X is Φ-taut. In [2] it is proved that S is Φ-taut in X if one of the following conditions is satisfied :

a) Φ is paracompactifying for the pair (X,S)

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b) Φ is paracompactifying, X is completely paracompact, S is arbitrary.

c) Φ is paracompactifying, S is closed in X.

d) Φ is maximum, S is compact and relatively Hausdorff in X.
The purpose of this paper is to prove a tautness criterion wich unifies and generalizes the preceding ones. For this reason, we introduce in definition 1 the notion of relative paracompactness of S in X. We say that Φ is S-paracompactifying if every element of Φ has a neighorhood belonging to Φ and if $S \cap F$ is relatively paracompact in F whenever F belongs to Φ. Our main result states that S is Φ-taut in X if Φ is S-paracompactifying.

RELATIVE PARACOMPACTNESS

In order to avoid confusions, let us recall the following definitions.

An *open covering of* S *in* X is a set $\mathcal{U}$ of open subsets of X, such that $\cup\mathcal{U} \supset S$. For a set $\mathcal{U}$ of subsets of X we write $\mathcal{U}(S) = \{U : U \in \mathcal{U}, U \cap S \neq \emptyset\}$. We say then that :

a) $\mathcal{U}$ *is punctually finite on* S if $\mathcal{U}(\{s\})$ is finite for every element s of S.

b) $\mathcal{U}$ *is locally finite on* S if each element of S has a neighborhood V such that $\mathcal{U}(V)$ is finite.

c) *A* S-*refinement* of an open covering $\mathcal{U}$ of S in X is an open covering $\mathcal{V}$ of S in X such that every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$.

Now let us introduce the following

DEFINITION 1. The subset S of X is

a) *relatively Hausdorff in* X if two distinct points of S have disjoint neighborhoods in X.

b) *relatively normal in* X if two disjoint closed subsets of S have disjoint neighborhoods in X.

c) *relatively paracompact in* X if every covering of S in X has a S-refinement which is locally finite on S and if moreover S is relatively Hausdorff in X.

REMARK 2. It is clear that X is relatively Hausdorff (resp. normal; paracompact) in X if and only if X is Hausdorff (resp. normal; paracompact).

Slight modifications of classical proofs give the following three results.

PROPOSITION 3. *If* S *is relatively normal in* X *and if* $\mathcal{U}$ *is an open covering of* S *in* X *which is punctually finite on* S *then there exists a family* $(V_U)_{U\in\mathcal{U}}$ *of open subsets of* X, *covering* S *and such that* $\bar{V}_U$ *is contained in* U *for every* U *belonging to* $\mathcal{U}$.

PROPOSITION 4. *If* F *is a closed subset of* S *and if* S *is relatively paracompact in* X *then every open covering of* F *in* X *has a* F*-refinement which is locally finite on* S. *In particular* F *is relatively paracompact in* X.

PROPOSITION 5. *If* S *is relatively paracompact in* X *then* S *is relatively normal in* X.

The following easy results are also usefull.

PROPOSITION 6. *If* S *is relatively paracompact in* X *and if* Y *is a subset of* X *containing* S *then* S *is relatively paracompact in* Y. *In particular* S *is paracompact.*

Proof : Let $\mathcal{U}$ be an open covering of S in Y. It is clear that there exists an open covering $\mathcal{V}$ of S in X such that $\mathcal{V} \cap Y = \mathcal{U}$. Thus there exists a S-refinement $\mathcal{W}$ of $\mathcal{V}$ which is locally finite on S. We see directly that $\mathcal{W} \cap Y$ is a S-refinement of $\mathcal{U}$ in Y which is locally finite on S. To conclude, we just have to note that S is relatively Hausdorff in Y.///

PROPOSITION 7. *If* S *has a fundamental system of paracompact neighborhoods in* X *then* S *is relatively paracompact in* X.

Proof : Let $\mathcal{U}$ be an open covering of S in X. Let us choose a paracompact neighbourhood V of S in X contained in $\cup\mathcal{U}$. Since $\mathcal{U} \cap V$ is an open covering of V in V, there exists a V-refinement $\mathcal{V}$ of $\mathcal{U} \cap V$ in V which is locally finite on V. Thus $\mathcal{V} \cap \overset{\circ}{V}$ is a S-refinement of $\mathcal{U}$ in X which is locally finite on S. To conclude, it remains to prove that S is relatively Hausdorff in X. Let x,y be two distincts elements of S and W a paracompact neighborhood of S in X. Since W is a Hausdorff space, there exist neighborhoods V_x, V_y of x and y in W, such that $V_x \cap V_y = \emptyset$. But W is a neighborhood of x (resp. y) so that V_x (resp. V_y) is a neighborhood of x (resp. y) in X. Thus x and y have disjoint neighborhoods in X.///

COROLLARY 8.

a) A subset S *of a completely paracompact space (e.g. a metric space)* X *is relatively paracompact in* X.

b) A closed subset S *of a paracompact space is relatively paracompact in* X.

<u>Proof</u> : a) Since X is completely paracompact, every open subset of X is paracompact and we may apply proposition 7.

b) Since X is paracompact we know that X is normal and the closed neighborhoods of S in X form a fundamental system of paracompact neighborhoods of S in X. So we may apply proposition 7.///

PROPOSITION 9. *If* S *is compact and relatively Hausdorff in* X *then* S *is relatively paracompact in* X.

<u>Proof</u> : Let $\mathcal{U}$ be an open covering of S in X. Since $\mathcal{U} \cap S$ is an open covering of S, there exists a finite subset $\mathcal{V}$ of $\mathcal{U} \cap S$ which covers S. Let us choose a finite subset $\mathcal{W}$ of $\mathcal{U}$ such that $\mathcal{W} \cap S = \mathcal{V}$. Clearly $\mathcal{W}$ is a S-refinement of $\mathcal{U}$ which is locally finite on S. Since S is relatively Hausdorff in X, the proof is complete.///

A TAUTNESS CRITERION

PROPOSITION 10. *If* S *is relatively paracompact in* X *then the canonical morphism*

$$(r_S : \varinjlim_{U \in \mathcal{V}_S} \Gamma(U, F_{|U}) \longrightarrow \Gamma(S, F_{|S}))$$

is an isomorphism for every sheaf F *on* X.

<u>Proof</u> : It is clear that r_S is injective, thus we just have to prove that it is onto. Let σ be a section of F over S. For every $x \in S$, let us choose a neighborhood U_x of x in X and a section s_x of F over U_x such that $s_{x|U_x \cap S} = \sigma_{|U_x \cap S}$.

We know that S is relatively paracompact in X and that $\mathcal{U} = \{U_x : x \in S\}$ is an open covering of S in X, thus there exists a S-refinement $\mathcal{V}$ of $\mathcal{U}$ which is locally finite on S. For every V belonging to $\mathcal{V}$, let us choose a an element x_V of S, such that $V \subset U_{x_V}$. Let s'_V denote the section $s_{x_V|V}$. Since S is relatively normal in X, the proposition 3 gives us a family $(W_V)_{V\in\mathcal{V}}$ of open subsets of X covering S and such that $\bar{W}_V \subset V$ whenever $V \in \mathcal{V}$. For every V belonging to $\mathcal{V}$ we denote by s''_V the section $s'_{V|\bar{W}_V}$. Let us set

$$B = \bigcap_{U,V\in\mathcal{V}} \{y : (y \in \bar{W}_U \cap \bar{W}_V) \Rightarrow (s''_U(y) = s''_V(y))\}.$$

We shall establish that B is a neighborhood of S. Let x be an element of S. Since $\mathcal{V}$ is locally finite on S there exists an open neighborhood ω of x in X such that $\mathcal{V}(\omega)$ is finite. Let us set

$$\omega' = \omega \setminus \Big(\bigcup_{\substack{V\in\mathcal{V}(\omega) \\ x\notin\bar{W}_V}} \bar{W}_V \Big)$$

Clearly ω' is still a neighborhood of x in X and $x \in \bar{W}_V$ if $\bar{W}_V \cap \omega' \neq \emptyset$. Let us set

$$\omega'' = \omega' \cap \bigcap_{\substack{\bar{W}_V\cap\omega'\neq\emptyset \\ \bar{W}_U\cap\omega'\neq\emptyset}} \{y : y \in V \cap U, s'_V(y) = s'_U(y)\}.$$

We see immediately that ω'' is an open neighborhood of x in X and that

$$(y \in \omega'' \cap \bar{W}_V \cap \bar{W}_U) \Rightarrow (s''_V(y) = s''_U(y))$$

if U, V belong to $\mathcal{V}$. Thus ω'' is contained in B and B is a neighborhood of **x** in X. Since x is an arbitrary point of S, this proves that S is contained in $\overset{\circ}{B}$. Now, since $\mathcal{V}$ is locally finite on S, we know that $\{W_V : V \in \mathcal{V}\}$ is locally finite on an open neighborhood Ω of S in X. Let Ω' be the set $\Omega \cap (\bigcup_{V\in\mathcal{V}} W_V) \cap \overset{\circ}{B}$, clearly, Ω' is an open neighborhood of S in X. For overy $V \in \mathcal{V}$ let F_V be the set $\bar{W}_V \cap \Omega'$ and let s'''_V be the section $s''_{V|F_V}$. The family $(F_V)_{V\in\mathcal{V}}$ defines a closed locally finite covering of Ω' and $s'''_V|_{F_V \cap F_U}$ equals $s'''_U|_{F_V \cap F_U}$ if $U, V \in \mathcal{V}$. Thus there exists a section s of $\mathcal{F}$ over Ω' such that $s_{|F_V} = s'''_V$ if $V \in \mathcal{V}$. This shows that $s_{|S} = \sigma$. Since σ is an arbitrary section of $\mathcal{F}$ over S, we have proved that r_S is onto.///

DEFINITION 11. The family of supports Φ is A-*paracompactifying* if every element of Φ has a neighborhood which belongs to Φ and if $F \cap A$ is relatively paracompact in F for every F belonging to Φ.

PROPOSITION 12. *If* Φ *is* S-*paracompactifying and if* F *is a closed subset of* S *then* Φ *is* F-*paracompactifying.*

Proof : Let F' be an element of Φ. We know that $F' \cap S$ is relatively paracompact in F' and that $F \cap F'$ is closed in $F' \cap S$, thus, by proposition 4, $F \cap F'$ is relatively paracompact in F'.///

PROPOSITION 13. *If* Φ *is* S-*paracompactifying, then*

a) $\Phi \cap S$ *is paracompactifying in* S,

b) *the canonical morphism,*

$$(r_S : \varinjlim_{U \in \mathcal{V}_S} \Gamma_{\Phi \cap U}(U, \mathcal{F}_{|U}) \longrightarrow \Gamma_{\Phi \cap S}(S, \mathcal{F}_{|S}))$$

is an isomorphism for every sheaf $\mathcal{F}$ *on* X,

c) $\mathcal{F}_{|S}$ *is* $\Phi \cap S$ - *soft for every flabby sheaf* $\mathcal{F}$ *on* X.

Proof : a) If $F \in \Phi$, $F \cap S$ relatively paracompact in F and proposition 6 shows that $F \cap S$ is paracompact.

b) Let $\mathcal{F}$ be a sheaf on X. It is clear that r_S is injective, so we just have to prove that it is onto. Let σ be a section of $\mathcal{F}$ over S with support belonging to $\Phi \cap S$. Let us choose an element F of Φ such that $\text{supp}(\sigma) = F \cap S$ and a neighborhood F' of F belonging to Φ. Since $F' \cap S$ is relatively paracompact in F', proposition 10 shows that the section σ extends to a section σ' of $\mathcal{F}$ over a neighborhood V of $S \cap F'$ in F'. Let G be the set $\text{supp}(\sigma')$. By construction, we know that $G \cap S \subset \overset{\circ}{F'} \cap S$ and that $\overset{\circ}{F'} \cap S \subset \overset{\circ}{V}$. Thus $S \setminus \overset{\circ}{V}$ is contained in $S \setminus G$. This proves that S is contained in the open set $(X \setminus G) \cup \overset{\circ}{V}$. Let us denote by Ω this open set and by σ'' the section of $\mathcal{F}$ over Ω which is equal to 0 on $X \setminus G$ and to $\sigma'_{|\overset{\circ}{V}}$ on $\overset{\circ}{V}$. We see immediately that $\sigma_{|S} = \sigma$ and that $\text{supp}(\sigma'') \subset G$. Since $G \subset F'$,

σ'' is an element of $\Gamma_{\Phi\cap\Omega}(\Omega, F_{|\Omega})$, such that $r_{S\Omega}(\sigma'') = \sigma$. Since σ is an arbitrary element of $\Gamma_{\Phi\cap S}(S, F_{|S})$ we have proved that r_S is onto.

c) Let F be a flabby sheaf on X, B, B' two elements of $\Phi \cap S$ such that $B \subset B'$ and σ a section of F over B. Since B is closed in S, we know, by proposition 12, that Φ is B - paracompactifying and what is proved above shows that there exists an open neighborhood Ω of B in X and an element σ' of $\Gamma_{\Phi\cap\Omega}(\Omega, F_{|\Omega})$ such that $\sigma'_{|B} = \sigma$. Since F is flabby, there exists a section σ'' of F over X, such that $\sigma_{|\Omega} = \sigma'$. Let us denote by σ''' the section $\sigma_{|B'}$. It is clear that $\sigma_{|B} = \sigma$. Since σ is an arbitrary section of F over B, we have proved that $F_{|S}$ is $\Phi \cap S$ - soft.///

CRITERION 14. *If Φ is S-paracompactifying then S is Φ-taut.*

Proof : It is an easy consequence of the preceding proposition if we remember that an open subset of X is Φ-taut and that a Φ-soft sheaf is Φ - acyclic if Φ is paracompactifying.///

REMARK 15. If S satisfies the condition a) (resp. b); c); d)) then proposition 7 (resp. 7; 7; 9) shows that Φ is S-paracompactifying and the preceding result shows that S is Φ-taut.///

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