Dendric words and strongly left proper morphisms

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Joint work with Julien Leroy

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Combinatorics on words background

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- finite alphabets: $\mathcal{A}$, $\mathcal{B}$

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  - u is a factor of v if there exist w, w' such that $v = wuw'$
- recurrent factorial languages: $\mathcal{L}$
  - recurrent: if u and v are in $\mathcal{L}$, there exists w such that uwv is in $\mathcal{L}$
  - factorial: if u is in $\mathcal{L}$, so are its factors

Examples

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- words: ε, 10, 010, 01001
- ε, 10 and 010 are factors of 01001
  ε and 010 are also prefixes of 01001 and ε is a suffix
- language: $\{0, 1\}^* \setminus \{0, 1\}^* 11 \{0, 1\}^*$ the set of words that do not have the factor 11

Morphisms

Definition

A *morphism* $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is a monoid morphism, i.e.

$$\sigma(w) = \sigma(w_1) \ldots \sigma(w_n).$$

$$\sigma : \{0, 1, 2\}^* \to \{0, 1\}^*, \quad \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

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Definition

Let $\mathcal{L} \subseteq \mathcal{A}^*$ be a language and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be a morphism. The image of $\mathcal{L}$ under σ is

$$\sigma \cdot \mathcal{L} = \bigcup_{u \in \mathcal{L}} \mathrm{Fac}(\sigma(u)).$$

Definitions

Extension graphs

$$E^-_{\mathcal{L}}(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}\}, \quad E^+_{\mathcal{L}}(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}\}$$

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Definition

The *extension graph* of $w \in \mathcal{L}$ is the bipartite graph $\mathcal{E}_{\mathcal{L}}(w)$ with vertices $E^-_{\mathcal{L}}(w) \sqcup E^+_{\mathcal{L}}(w)$ and such that, for all $a \in E^-_{\mathcal{L}}(w)$, $b \in E^+_{\mathcal{L}}(w)$, it contains the edge (a, b) iff $(a, b) \in E_{\mathcal{L}}(w)$.

Example

If $\mathcal{L} = \{0,1\}^* \setminus \{0,1\}^* 11 \{0,1\}^*$, then

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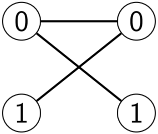
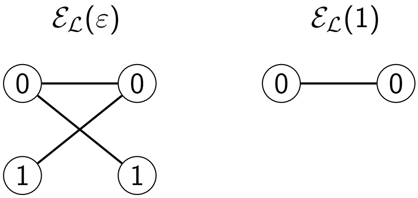


Example

If $\mathcal{L} = \{0,1\}^* \setminus \{0,1\}^*11\{0,1\}^*$, then

$$E^-_{\mathcal{L}}(\varepsilon) = \{0,1\} \qquad E^+_{\mathcal{L}}(1) = \{0\}$$

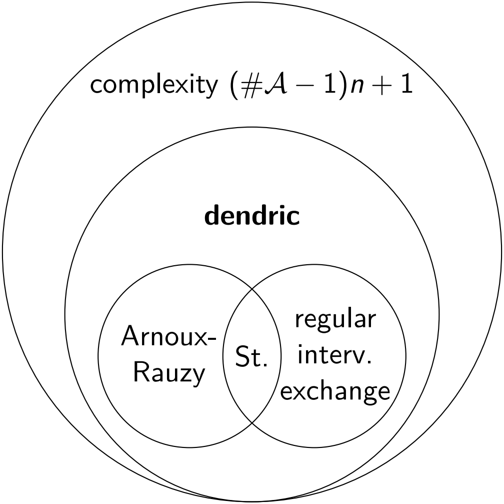


Dendric

A word $w \in \mathcal{L}$ is *dendric* if its extension graph in $\mathcal{L}$ is a tree.

A language $\mathcal{L}$ is *dendric* if the words $w \in \mathcal{L}$ are.

Hierarchy


complexity $(\#\mathcal{A}-1)n+1$
dendric
Arnoux-
Rauzy
St.
regular
interv.
exchange

Slp morphisms

Definition

A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is *strongly left proper* (slp) if there exists a letter $\ell \in \mathcal{B}$ such that

$$\sigma(a) \in \ell(\mathcal{B} \setminus \{\ell\})^*, \quad \forall a \in \mathcal{A}.$$

$$\sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \end{cases}$$

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$$\begin{array}{ccccc} 021001 & \xleftarrow{fac.} & 021001\textcolor{gray}{0} & \xleftarrow{\sigma} & 201 \\ 10001 & \xleftarrow{fac.} & 0210001\textcolor{gray}{0} & \xleftarrow{\sigma} & 2001 \end{array} \qquad \sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \end{cases}$$

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Theorem (Berthé, De Felice, Dolce, Leroy, Perrin, Reutenauer, Rindone)

Let $\mathcal{L} \subseteq \mathcal{A}^*$ be a language and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be an slp morphism. If $\sigma \cdot \mathcal{L}$ is recurrent dendric, then $\mathcal{L}$ is recurrent dendric. Moreover, $\mathrm{Card}(\mathcal{A}) = \mathrm{Card}(\mathcal{B})$.

S-adic representations

Definition

A *primitive S-adic representation* of a language $\mathcal{L}$ is a primitive sequence of morphisms $\boldsymbol{\sigma} = (\sigma_n : \mathcal{A}_{n+1} \to \mathcal{A}_n)_n$ such that

$$\mathcal{L} = \lim_n (\sigma_0 \circ \cdots \circ \sigma_n) \cdot \mathcal{A}_{n+1}.$$

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Corollary

Every recurrent dendric language has a primitive S-adic representation using slp morphisms.
Moreover, we can assume that all the intermediary alphabets are equal.

Dendric images under slp morphisms

Goal

Let $\mathcal{L} \subseteq \mathcal{A}^*$ be a language and $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ be an slp morphism for the letter ℓ.

$$\mathcal{L} \text{ dendric} \quad \Longleftarrow \quad \sigma \cdot \mathcal{L} \text{ dendric}$$

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When is a word $u \in \sigma \cdot \mathcal{L}$ dendric?

Initial factor: example

$$\sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \end{cases}$$

$$\mathcal{E}_{\sigma \cdot \mathcal{L}}(1)$$

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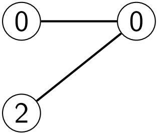

Initial factors

If $|u|_\ell = 0$, then

$$aub \in \sigma \cdot \mathcal{L} \implies \exists\, \alpha \in \mathcal{A} \text{ st. } aub \in \mathrm{Fac}(\sigma(\alpha)\ell).$$

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Let

$$\mathcal{L}_\sigma = \bigcup_{\alpha \in \mathcal{A}} \mathrm{Fac}(\sigma(\alpha)\ell).$$

We have

$$\mathcal{E}_{\sigma\cdot\mathcal{L}}(u) = \mathcal{E}_{\mathcal{L}_\sigma}(u).$$

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Definition

An slp morphism σ for ℓ is *dendric* if the words $u \in \mathcal{L}_\sigma$ such that $|u|_\ell = 0$ are dendric.

Extended images: example

$$\sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \end{cases}$$

$\mathcal{E}_{\sigma\cdot\mathcal{L}}(1021010)$

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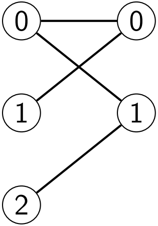

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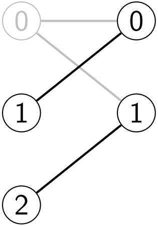

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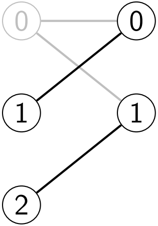

$$\varphi_1^- : \begin{cases} 1 \mapsto 0 \\ 2 \mapsto 2 \end{cases}$$

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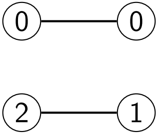

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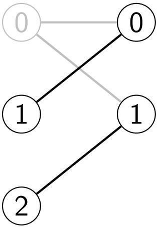

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Extended images

Proposition (G., Lejeune, Leroy)

If $u \in \sigma \cdot \mathcal{L}$ is such that $|u|_\ell \geq 1$, it has a unique decomposition $(s, v, p) \in \mathcal{B}^* \times \mathcal{L} \times \ell\mathcal{B}^*$ such that

- $u = s\sigma(v)p$,
- *s is a proper suffix of a word of $\sigma(\mathcal{A})$,*
- *p is a proper prefix of a word of $\sigma(\mathcal{A})\ell$.*

We then have

$$E_{\sigma\cdot\mathcal{L}}(u) = (\varphi_s^- \times \varphi_p^+)E_{\mathcal{L}}(v)$$

where

$$\varphi_s^- : a \mapsto b \text{ st. } \sigma(a) \in \mathcal{B}^* bs \qquad \varphi_p^+ : a \mapsto b \text{ st. } \sigma(a)\ell \in pb\mathcal{B}^*.$$

Dendric images

The graph $\mathcal{E}_{\mathcal{L},s,p}(v)$ is the subgraph of $\mathcal{E}_{\mathcal{L}}(v)$ where we have removed the vertices (and their edges) that are 'useless' to obtain $\mathcal{E}_{\sigma\cdot\mathcal{L}}(s\sigma(v)p)$.

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Theorem (G., Lejeune, Leroy)

Let $\mathcal{L}$ be a dendric language and σ an slp morphism. The image $\sigma \cdot \mathcal{L}$ is dendric if and only if σ is dendric and the following conditions are satisfied:

- '$-$' *for all $s \in \mathcal{B}^*$ and for all $v \in \mathcal{L}$, the graph $\mathcal{E}_{\mathcal{L},s,\varepsilon}(v)$ is connected,*
- '$+$' *for all $p \in \mathcal{B}^*$ and for all $v \in \mathcal{L}$, the graph $\mathcal{E}_{\mathcal{L},\varepsilon,p}(v)$ is connected.*

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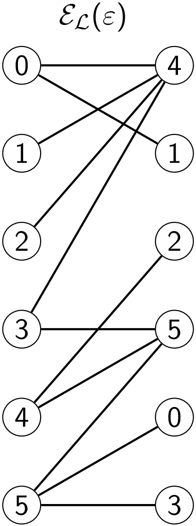
  the other left vertices remain connected,
- only a finite number of 'interesting' values of s,
- difficult to check

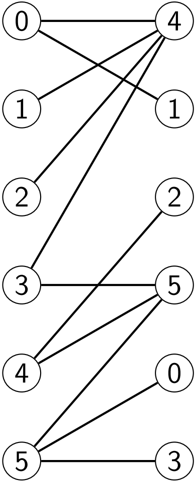
Graphs $G^{-}(\mathcal{L})$ et $G^{+}(\mathcal{L})$

Graph $G_n^-(\mathcal{L})$: example

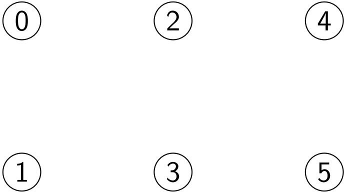


Graph $G_n^-(\mathcal{L})$: example

$\mathcal{E}_\mathcal{L}(\varepsilon)$



$G_1^-(\mathcal{L})$



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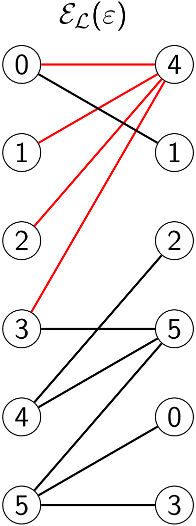

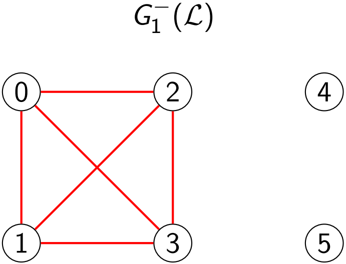

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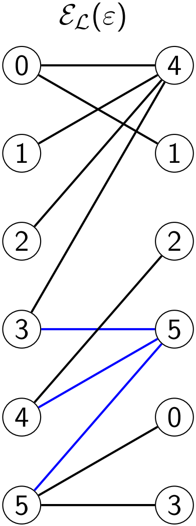
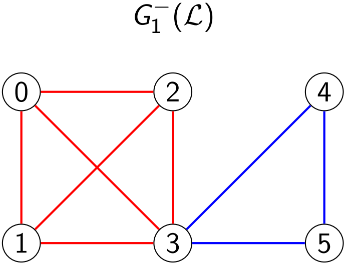
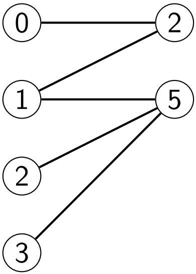


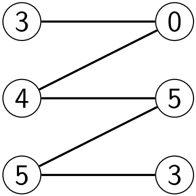


Graph $G_n^-(\mathcal{L})$: example (II)

$\mathcal{E}_\mathcal{L}(4)$



$\mathcal{E}_\mathcal{L}(5)$



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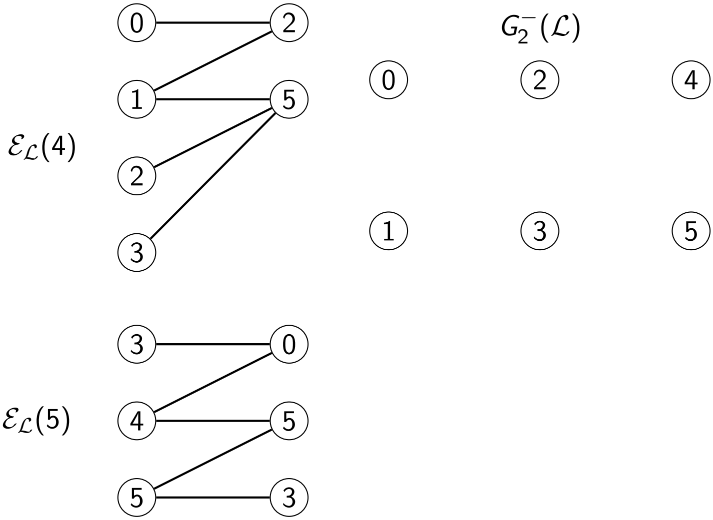

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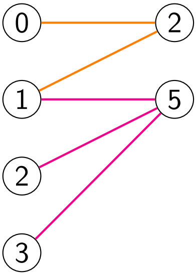
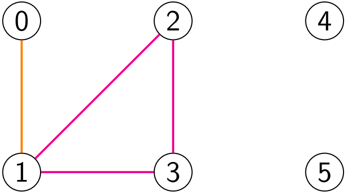


$G_2^-(\mathcal{L})$



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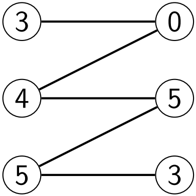
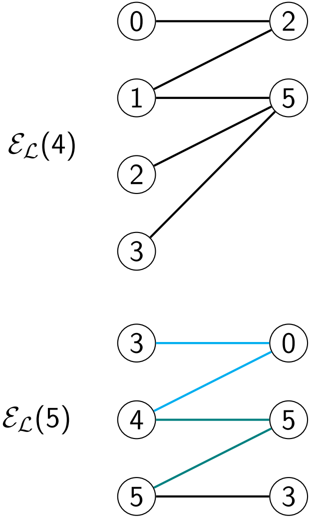
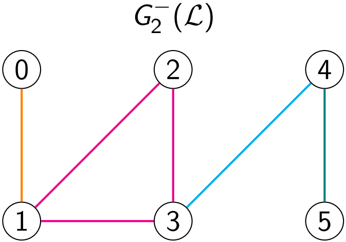


Graph $G_n^-(\mathcal{L})$: example (II)





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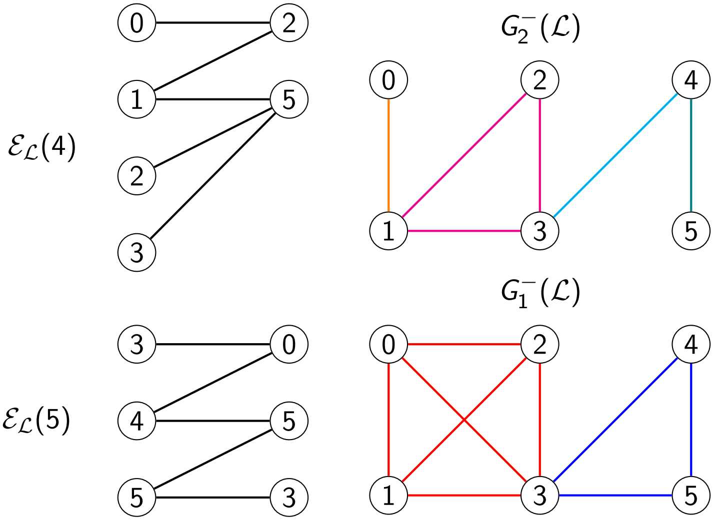

Graph $G^{-}(\mathcal{L})$

Definition

If $\mathcal{L}$ is dendric, $G_n^{-}(\mathcal{L})$ is the union of the cliques given by the sets of vertices $E_{\mathcal{L}}^{-}(v)$ for all $v \in \mathcal{L} \cap \mathcal{A}^n$.

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Proposition

Let $\mathcal{B} \subseteq \mathcal{A}$. The following are equivalent.

1. *For all $v \in \mathcal{L}$, the subgraph of $\mathcal{E}_{\mathcal{L}}(v)$ where we removed, on the left, the vertices in $\mathcal{B}$ (and the isolated vertices) is connected.*
2. *The graph $G^-(\mathcal{L}) \setminus \mathcal{B}$ is connected.*

Graph $G^+(\mathcal{L})$

Definition

If $\mathcal{L}$ is dendric, $G_n^+(\mathcal{L})$ is the union of the cliques given by the sets of vertices $E_{\mathcal{L}}^+(v)$ for all $v \in \mathcal{L} \cap \mathcal{A}^n$.
We define

$$G^+(\mathcal{L}) = \lim_n G_n^+(\mathcal{L}).$$

Proposition

Let $\mathcal{B} \subseteq \mathcal{A}$. The following are equivalent.

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2. *The graph $G^+(\mathcal{L}) \setminus \mathcal{B}$ is connected.*

Dendric images (II)

$$\mathcal{A}_s^- = \{a \in \mathcal{A} \mid \sigma(a) \notin \mathcal{B}^* s\}$$

$$\mathcal{A}_p^+ = \{a \in \mathcal{A} \mid \sigma(a)\ell \notin p\mathcal{B}^*\}$$

Theorem

Let $\mathcal{L}$ be a dendric language and σ an slp morphism. The image $\sigma \cdot \mathcal{L}$ is dendric if and only if σ is dendric and the following conditions are satisfied:

'$-$' *for all $s \in \mathcal{B}^*$, the graph $G^-(\mathcal{L}) \setminus \mathcal{A}_s^-$ is connected,*

'$+$' *for all $p \in \mathcal{B}^*$, the graph $G^+(\mathcal{L}) \setminus \mathcal{A}_p^+$ is connected.*

Image graph

Recall that

$$E_{\sigma\cdot\mathcal{L}}(s\sigma(v)p) = (\varphi_s^- \times \varphi_p^+)E_{\mathcal{L}}(v)$$

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Proposition

If $\mathcal{L}$ and $\sigma \cdot \mathcal{L}$ are dendric and if $G^-(\mathcal{L})$ is built with the cliques $C_1, \ldots, C_k$, then $G^-(\sigma \cdot \mathcal{L})$ is built with the cliques

$$\varphi_s^-(C_i) \qquad s \in \mathcal{B}^*, i \leq k.$$

Image graph

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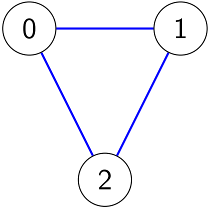
Proposition

If $\mathcal{L}$ and $\sigma \cdot \mathcal{L}$ are dendric and if $G^+(\mathcal{L})$ is built with the cliques $C_1, \dots, C_k$, then $G^+(\sigma \cdot \mathcal{L})$ is built with the cliques

$$\varphi_s^+(C_i) \qquad s \in \mathcal{B}^*, i \leq k.$$

$$\sigma : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 01 \\ 2 \mapsto 021 \end{cases}$$



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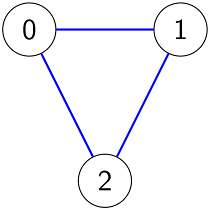

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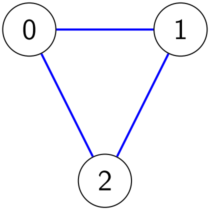

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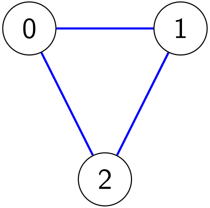

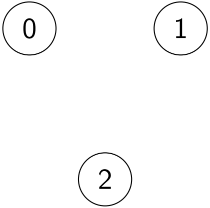

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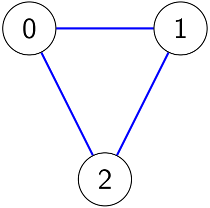

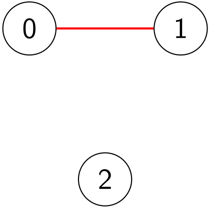

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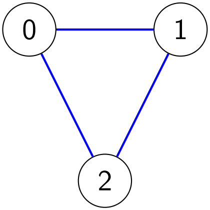

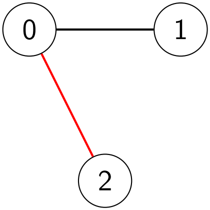

Back to S-adic representations

What we know

Given a dendric language $\mathcal{L}$, its graphs $G^-(\mathcal{L})$, $G^+(\mathcal{L})$ and slp morphisms $\sigma_0, \ldots, \sigma_n$, we can easily know if $(\sigma_0 \circ \cdots \circ \sigma_n) \cdot \mathcal{L}$ is dendric.

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Theorem

Let $\mathcal{L}$ be a language having a primitive $\mathcal{S}$-adic representation $\boldsymbol{\sigma}$. The language $\mathcal{L}$ is dendric if and only if $\boldsymbol{\sigma}$ labels infinite paths in both $\mathcal{G}^{-}(\mathcal{S})$ and $\mathcal{G}^{+}(\mathcal{S})$.

Simplification of the graph

Let $\mathcal{S}$ be a set of dendric slp morphisms over $\mathcal{A}$. The graph $\mathcal{G}_t^-(\mathcal{S})$ is defined as follows:

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Even more simplifications with the symmetries/permutations.

Sizes of the graphs

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cond. '$-$' and '$+$'	≥ 33	?	?

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with symmetries	2	14	?

End

Thank you for your attention!