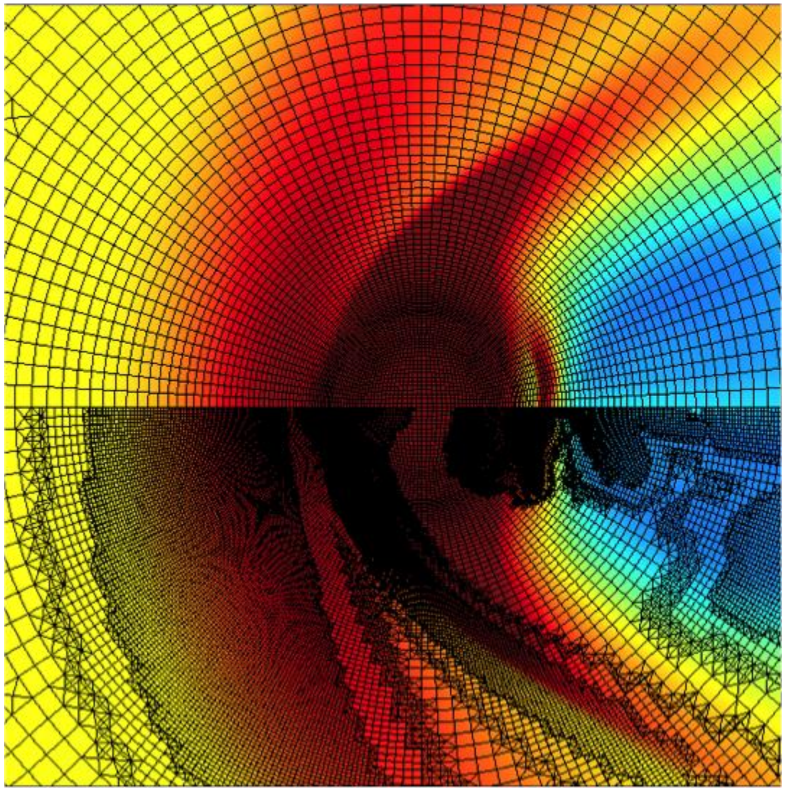
LIÈGE université
Sciences Appliquées

# COMPUTATIONAL FLUID DYNAMICS

## Discontinuous Galerkin method

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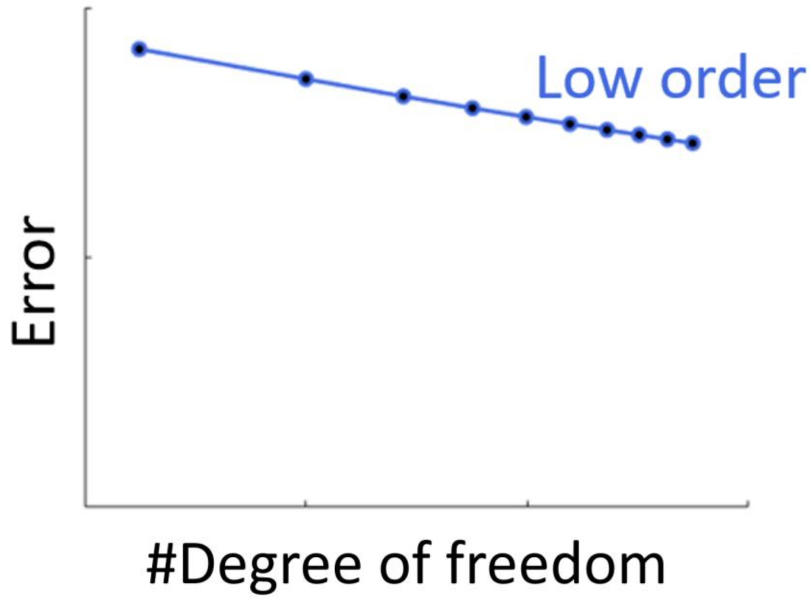
V.E. Terrapon
**A. Bilocq, N. Levaux**

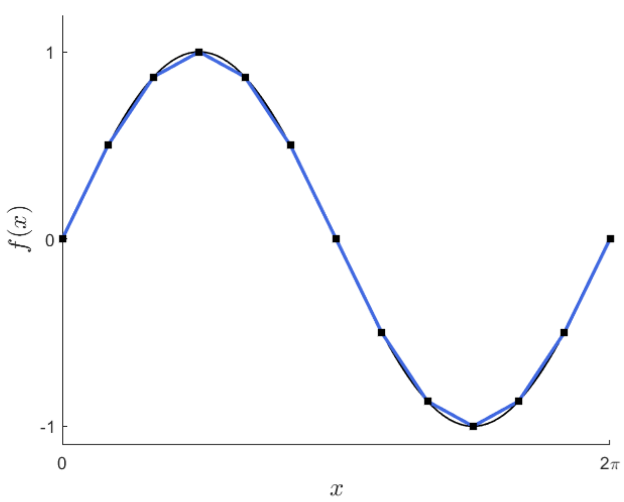
# Object of CFD

Solving numerically the governing laws of motion of fluids

From PDE to algebraic equations?

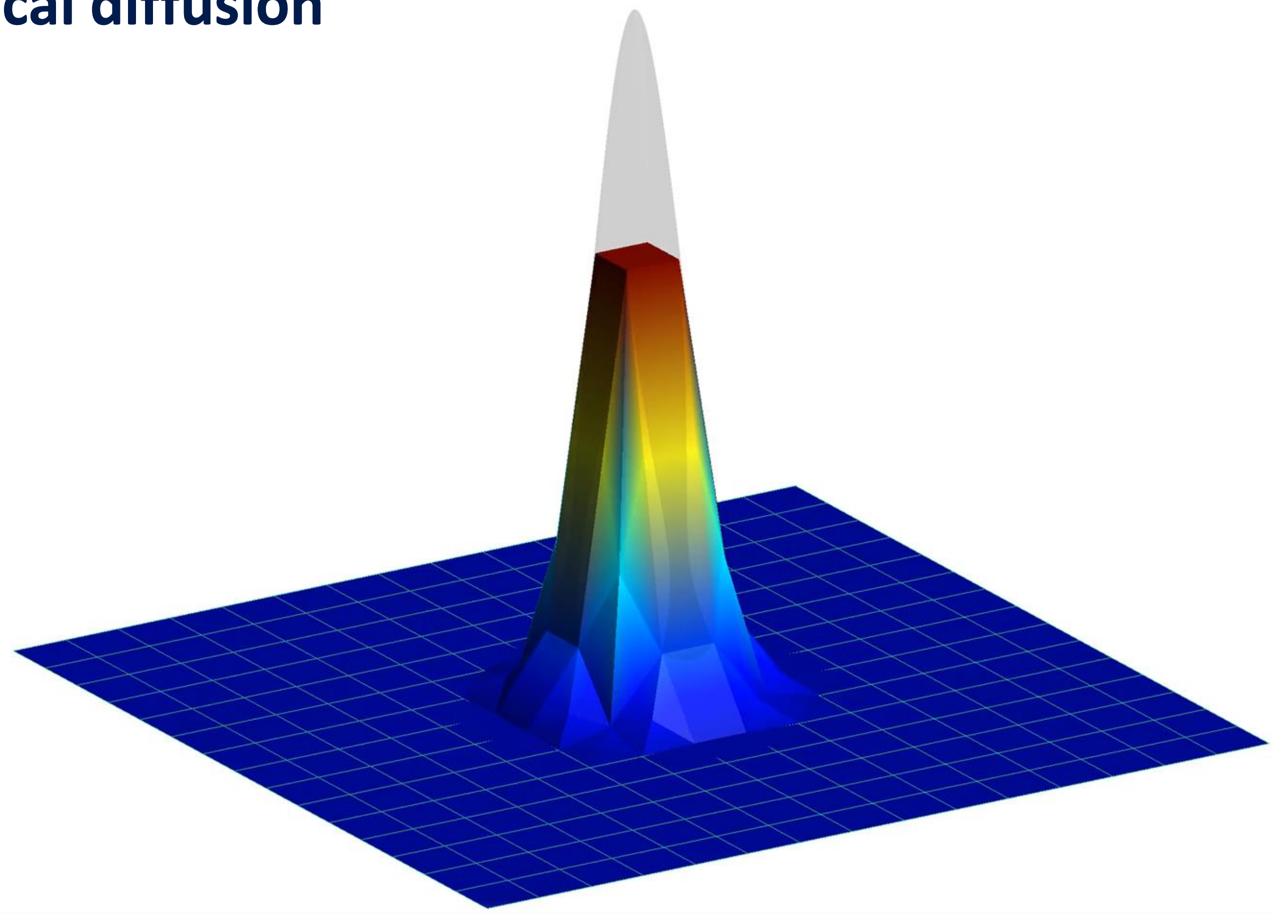
Finite difference method
Finite volume method



# Motivation

Low-order methods suffer from **numerical dispersion** and **numerical diffusion**



High-order methods can tackle these issues

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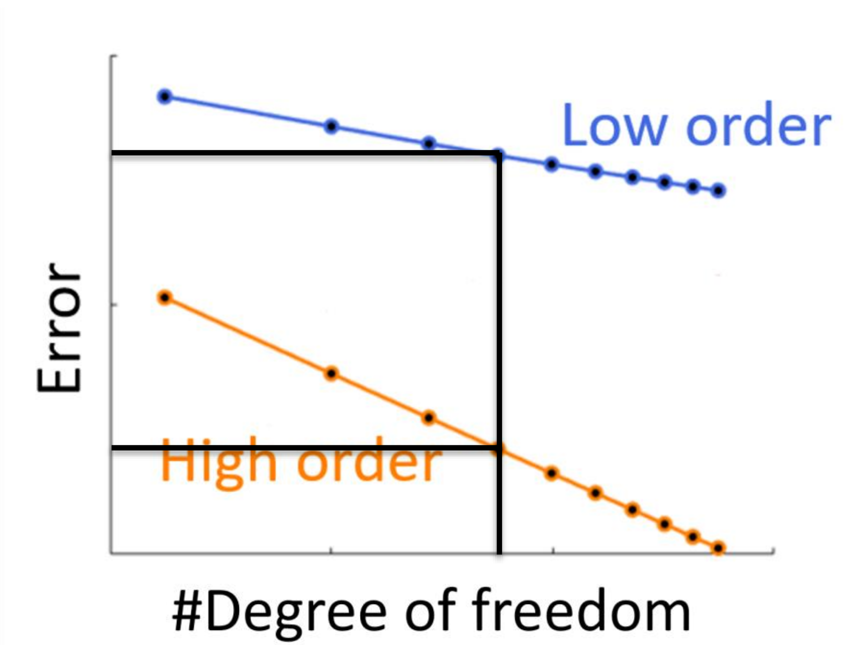
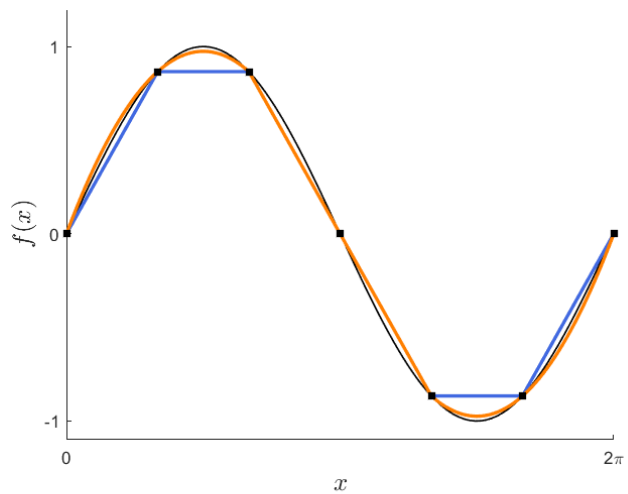
Solving numerically the governing laws of motion of fluids

From PDE to algebraic equations?

Finite difference method
Finite volume method



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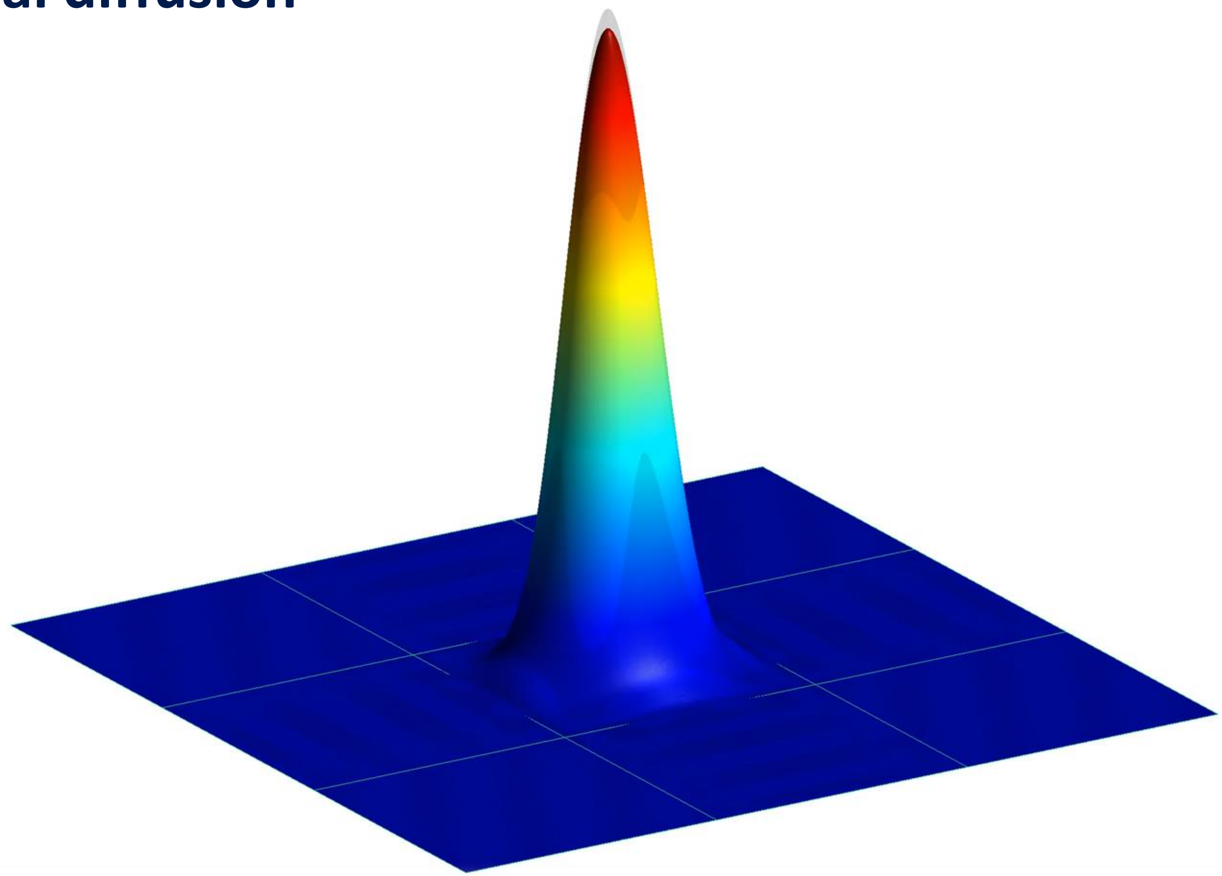
Low-order methods suffer from **numerical dispersion** and **numerical diffusion**

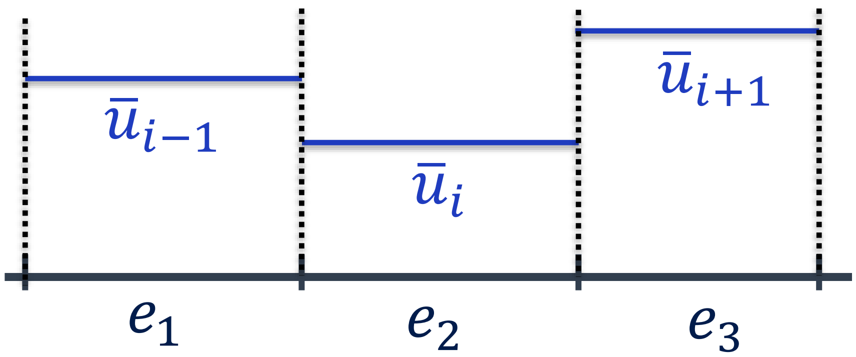


High-order methods can tackle these issues

# Outline

# What is Discontinuous Galerkin method?

**Finite volume method**



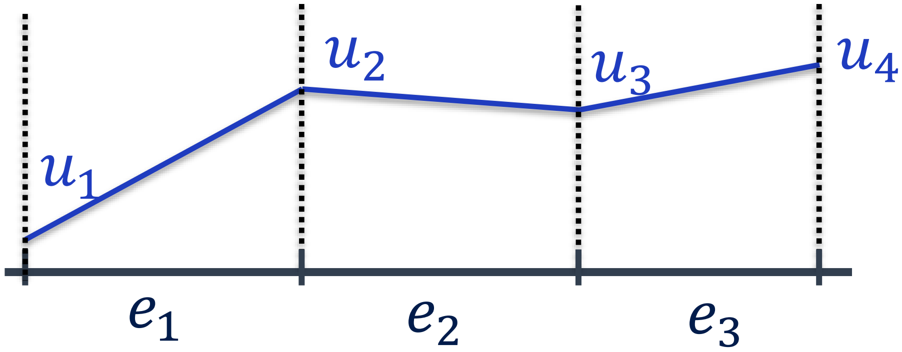

- Finite cells/volumes
- Cell averaged solution
- Discontinuous between volumes

+ Robust due to numerical conservation
+ Complex geometry
+ Well suited for convective problems

✗ Need large stencils to achieve high order

# What is Discontinuous Galerkin method?

**Finite element method**




- Finite cells/elements
- Solution at nodes
- Continuous between elements

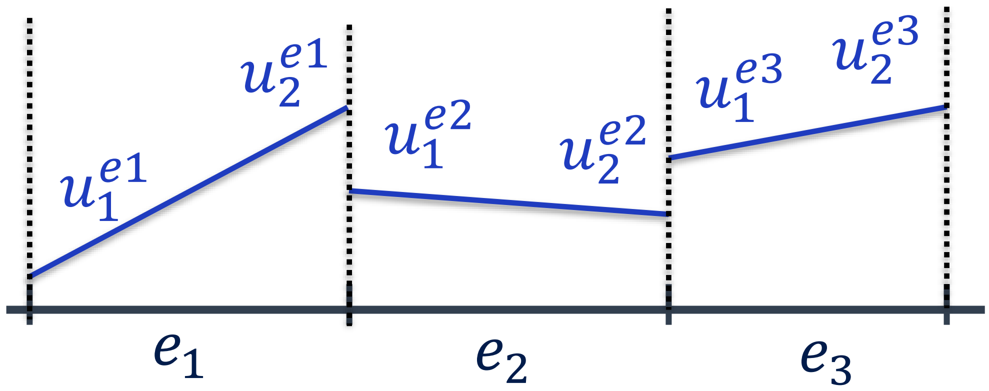
➕ High order accuracy
➕ Complex geometry

❌ Not well suited for convective problems
❌ Not conservative

# What is Discontinuous Galerkin method?

- **Numerical conservation** of the equations
- Suited for problem with a **direction**
- **High order** accuracy **without** large stencils

**Discontinuous Galerkin method**

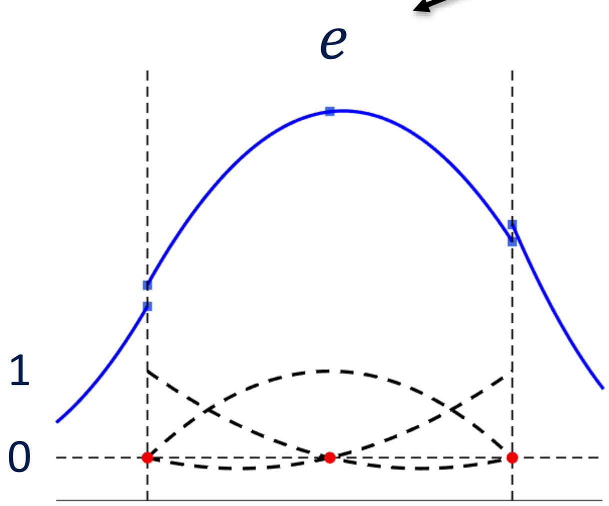



+ Better **accuracy** than **FVM**
+ Better **stability** than **FEM** for convective problem

# Outline

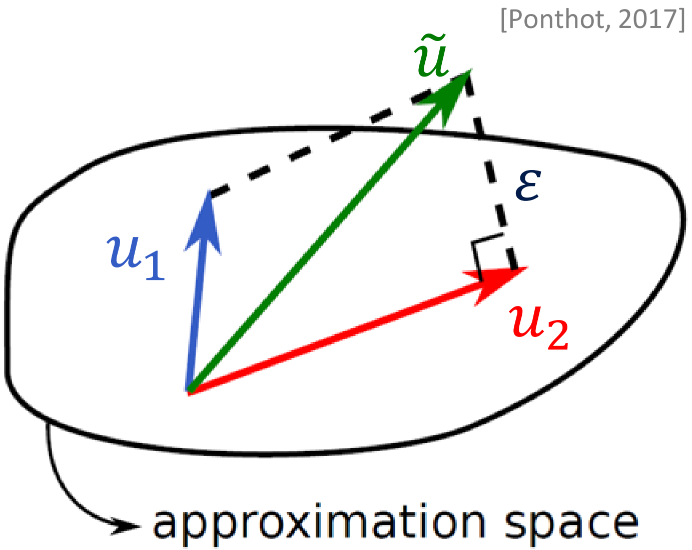
# Galerkin principle

Discontinuous Galerkin



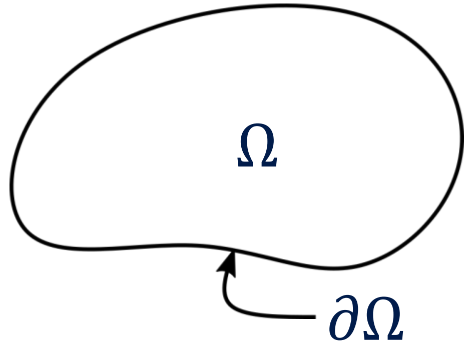

$$u^e(x) = \sum_{i=0}^{p} u_i^e \phi_i^e(x)$$

- Continuous within volume
- Doubling of DOF at interface
- Discontinuities allowed




- $\varepsilon = \tilde{u} - u$
- $\varepsilon \perp \mathcal{V}$
- $meas(\varepsilon) = R$

# Variational formulation




Generic system of PDE to solve on Ω

$$\frac{\partial \tilde{u}}{\partial t} + \nabla \cdot \vec{g}(\tilde{u}, \nabla \tilde{u}) = 0$$

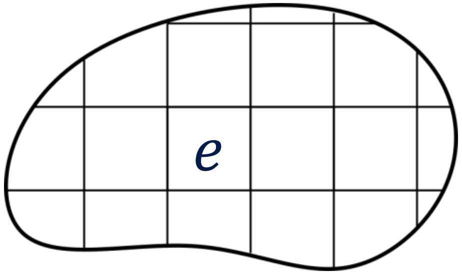
Galerkin variational form

$$\int_{\Omega} v \frac{\partial u}{\partial t} dV + \int_{\Omega} v \nabla \cdot \vec{g}(u, \nabla u)\, dV = 0, \qquad \forall\, v \in \mathcal{V}$$

Integration by part

$$\int_{\Omega} v \frac{\partial u}{\partial t} dV - \int_{\Omega} \nabla v \cdot \vec{g}\, dV + \int_{\partial\Omega} v \vec{g} \cdot \vec{n}\, dS = 0, \qquad \forall\, v \in \mathcal{V}$$

# Global to local formulation




Domain is meshed $\Omega \approx \mathcal{E} = \cup\, e$

$$\sum_e \left( \int_e v \frac{\partial u}{\partial t} dV - \int_e \nabla v \cdot \vec{g}\, dV + \int_{\partial e} v \vec{g} \cdot \vec{n}\, dS \right) = 0, \forall\, v \in \mathcal{V}$$

Set of SF spanning $\mathcal{V}$

Don't need to test every $v$, just shape functions!

$$\sum_e \left( \int_e \phi_i^e \frac{\partial u^e}{\partial t} dV - \int_e \nabla \phi_i^e \cdot \vec{g}^e dV + \int_{\partial e} \phi_i^e \vec{g}^e \cdot \vec{n}\, dS \right) = 0, \forall\, \phi_i^e$$

$\phi_i^e$ have elementwise support → from global to **local**

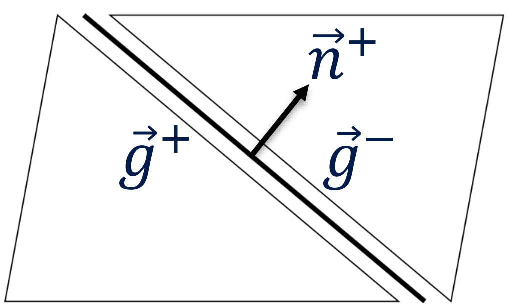
# Interface fluxes

Connect elements with each other's through faces integral

$$\sum_e \int_{\partial e} \phi_i^e \vec{g}^e \cdot \vec{n}\, dS = \sum_{f \in e} \int_f \phi_i^e \vec{g}^e \cdot \vec{n}\, dS$$

$$= \sum_f \int_f \underbrace{\left(\phi_i^{e^+} \vec{g}^+ \cdot \vec{n}^+ + \phi_i^{e^-} \vec{g}^- \cdot \vec{n}^-\right)} dS$$

$\gamma^e$: Suitable interface flux




$$[\![g]\!] = (g^+ - g^-)\, \vec{n}^+$$
$$\langle g \rangle = (g^+ + g^-)/2$$

# Outline

What is Discontinuous Galerkin method?

Variational formulation

Interface fluxes

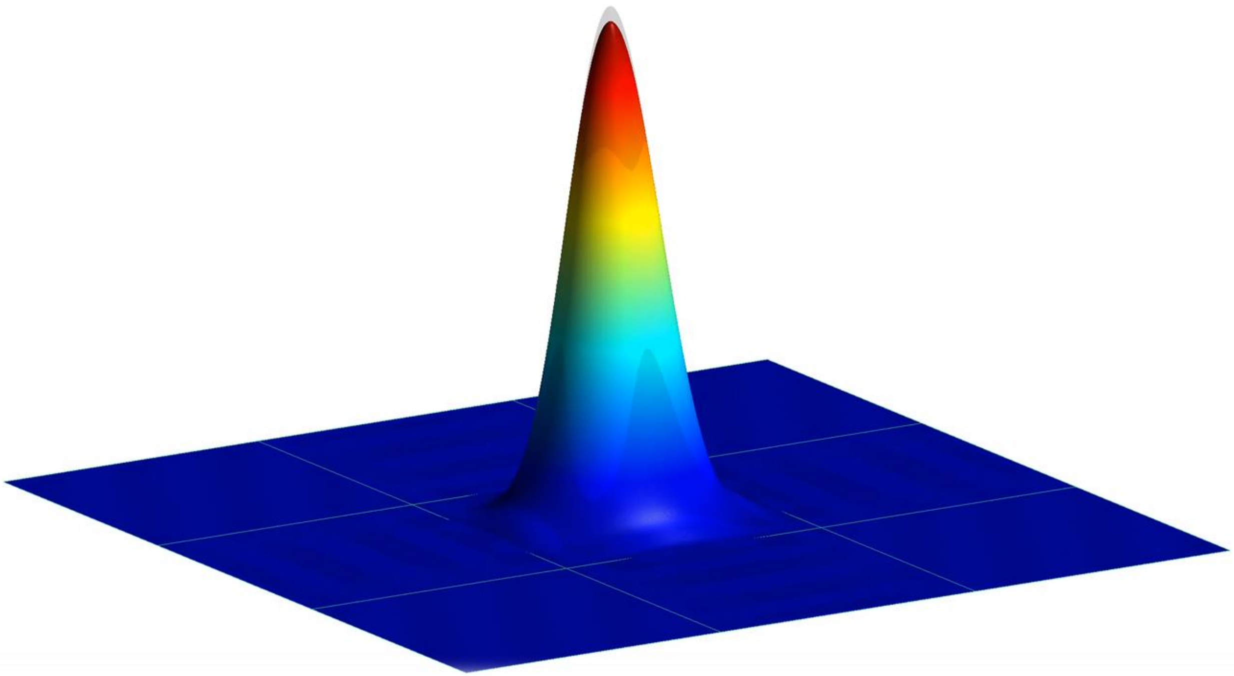
In practice

Our PhD theses

# Hyperbolic part

Convection

$$\frac{\partial u}{\partial t} + \boxed{\nabla \cdot \boldsymbol{f}(u)} + \nabla \cdot \boldsymbol{d}(u, \nabla u) = 0$$



# Hyperbolic part

Convection

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) + \nabla \cdot \boldsymbol{d}(u, \nabla u) = 0$$

$$\sum_e \int_e \phi_i^e \frac{\partial u}{\partial t} dV - \sum_e \int_e \nabla \phi_i^e \cdot \boldsymbol{f} dV + \sum_f \int_f \gamma^e dS = 0$$

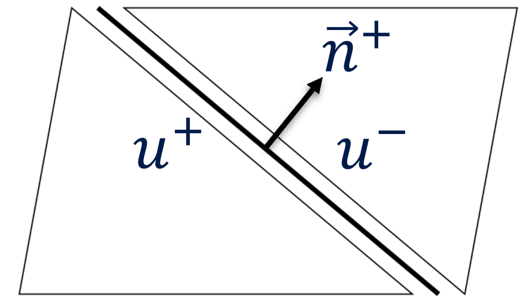
FVM theory

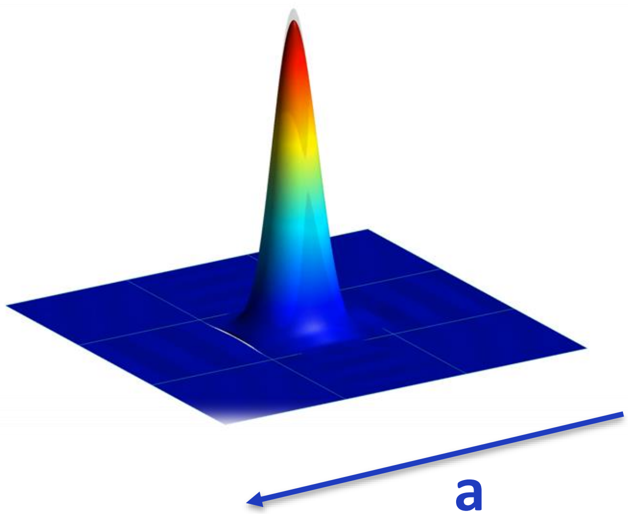
$$\gamma^e = [\![\phi_i^e]\!] H$$

upwind flux

# Hyperbolic part – Linear advection of a scalar

$$\frac{\partial u}{\partial t} + \boldsymbol{a} \cdot \nabla \mathrm{u} = 0$$




$$H = (\boldsymbol{a} \cdot \boldsymbol{n})\, u^+, \boldsymbol{a} \cdot \boldsymbol{n} \geq 0$$

$$= (\boldsymbol{a} \cdot \boldsymbol{n})\, u^-, \boldsymbol{a} \cdot \boldsymbol{n} \leq 0$$

$$H = (\boldsymbol{a} \cdot \boldsymbol{n})\, \frac{u^+ + u^-}{2} + |\boldsymbol{a} \cdot \boldsymbol{n}|\, \frac{u^+ - u^-}{2}$$

$$= \boxed{\frac{f^n(u^+) + f^n(u^-)}{2}} + \boxed{|\boldsymbol{a} \cdot \boldsymbol{n}|\, \frac{u^+ - u^-}{2}}$$

Conservation | Stabilization

# Hyperbolic part – Linear system

$$\frac{\partial u}{\partial t} + A^n \frac{\partial u}{\partial x} = 0$$

$n$ **real eigenvalues** $\rightarrow A^n = R\Lambda L$

$$H(u^+, u^-; \boldsymbol{n}) = \frac{f^n(u^+) + f^n(u^-)}{2} + |\boldsymbol{A^n}| \frac{u^+ - u^-}{2}$$

$$= \frac{f^n(u^+) + f^n(u^-)}{2} + \boldsymbol{R}|\boldsymbol{\Lambda}|\boldsymbol{L} \frac{u^+ - u^-}{2}$$

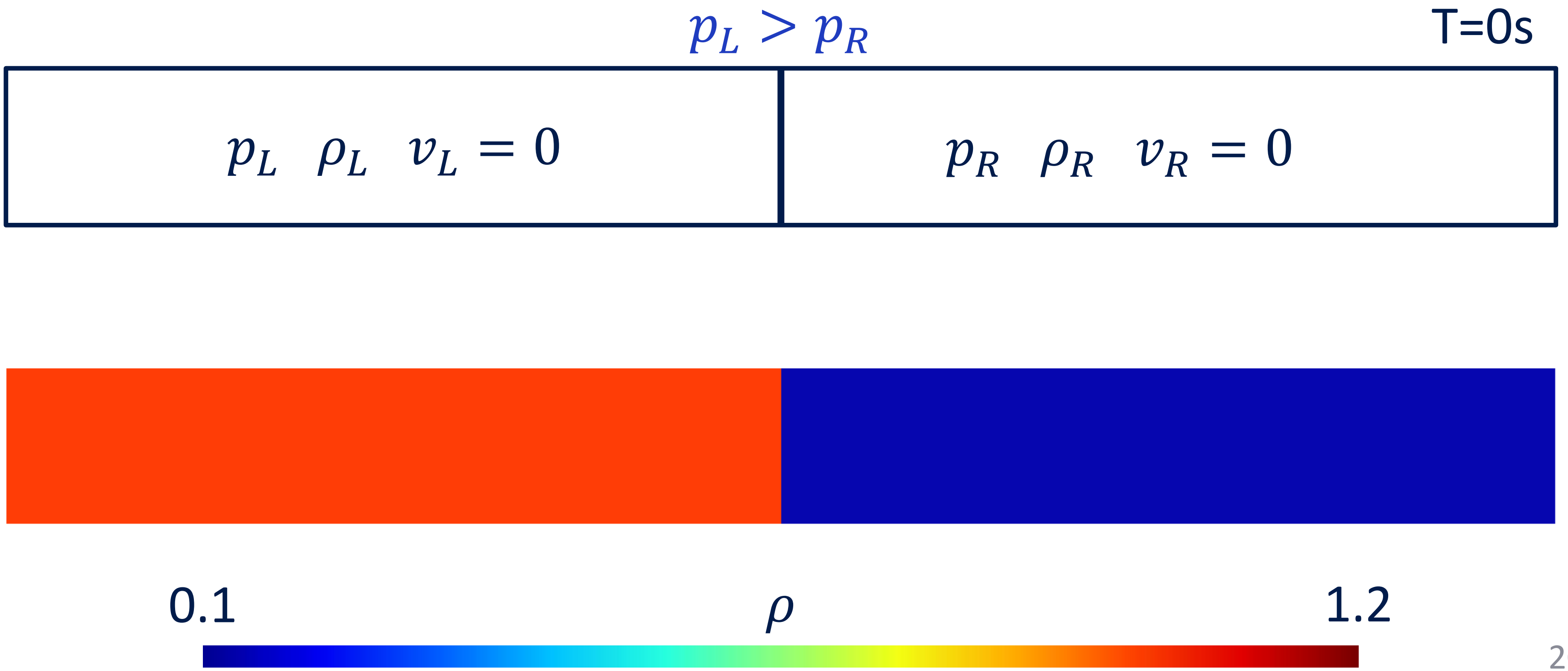
Characteristics

**Lax-Friedrichs:** $$H_{LX} = \frac{f^n(u^+) + f^n(u^-)}{2} + \max_i(|\lambda_i|) \frac{u^+ - u^-}{2}$$

# Hyperbolic part – Nonlinear system

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{F}(u) = 0$$

- The **numerical flux** is linked to the **characteristics**



# Hyperbolic part – Nonlinear system

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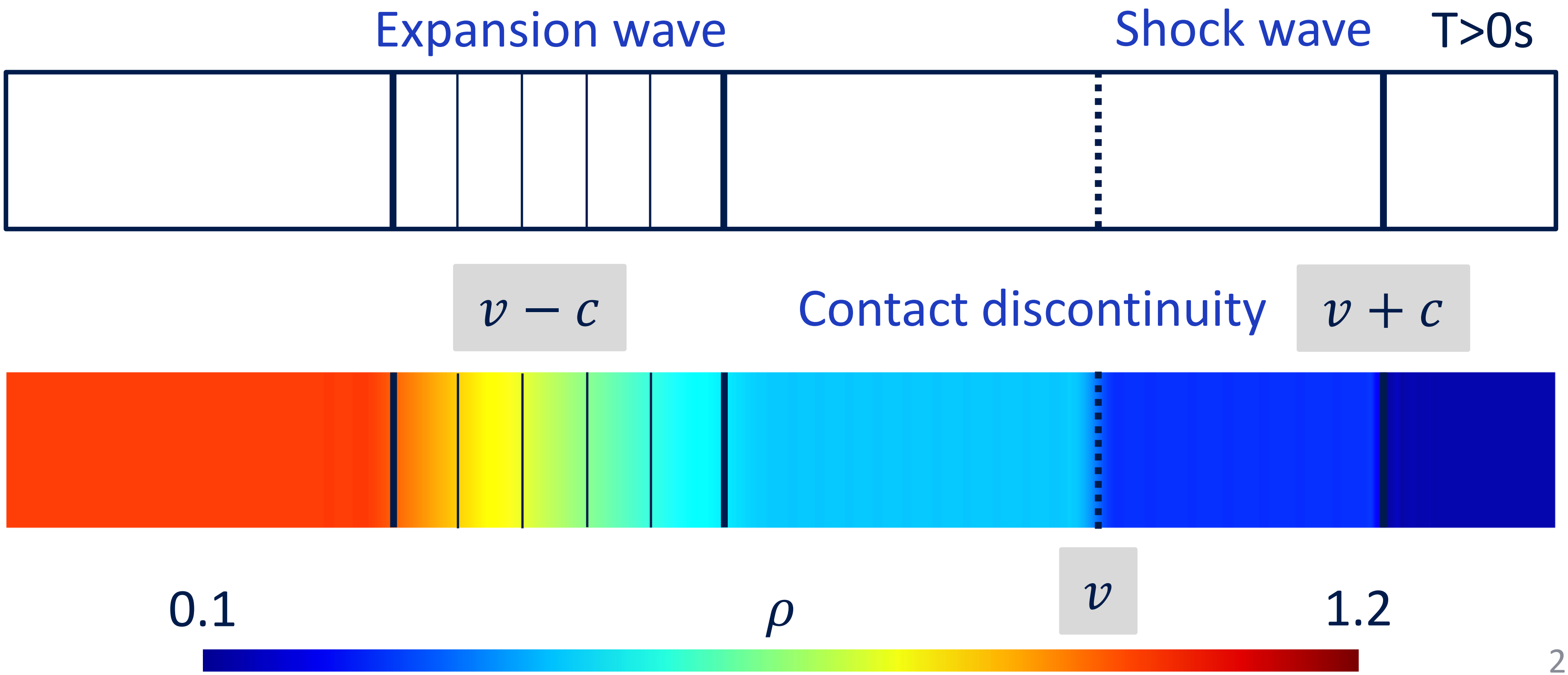
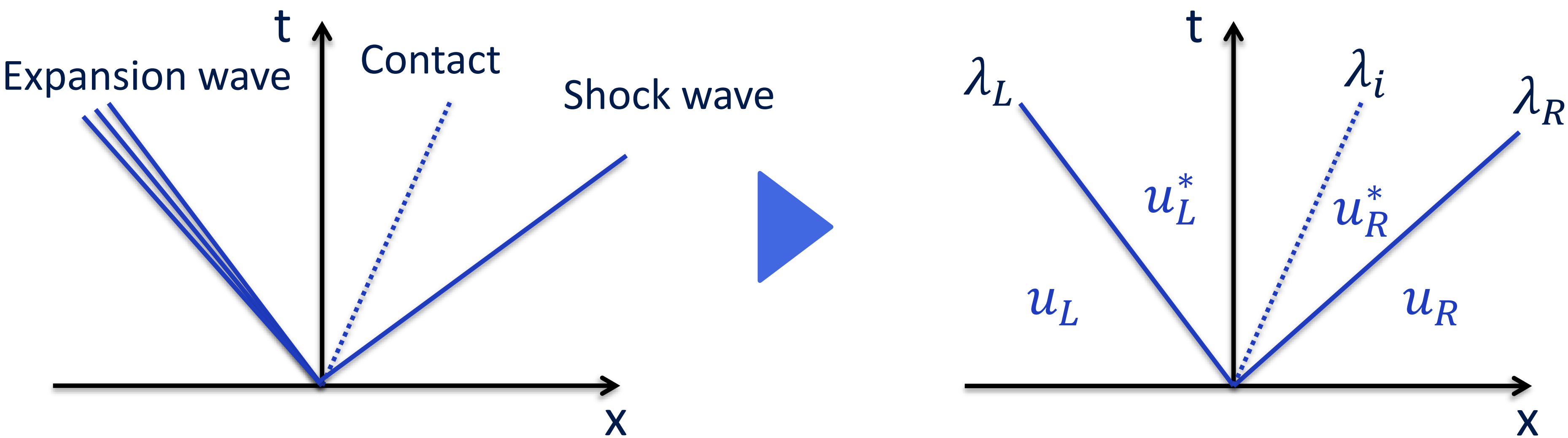


# Hyperbolic part – Riemann problem



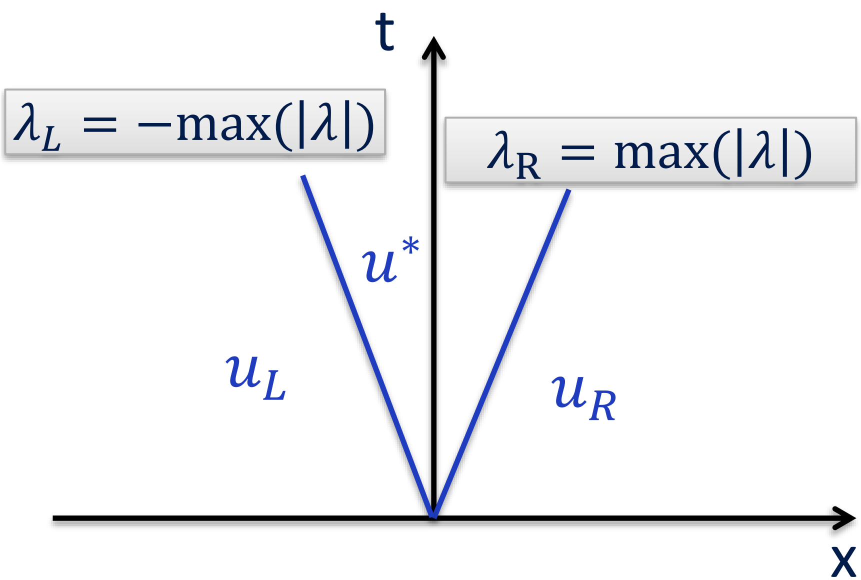

1. Exact Riemann solver: $H = \boldsymbol{F}(\tilde{u}_f) \cdot \boldsymbol{n}$ ← Too expensive
2. Approximate Riemann solver

$$\tilde{A}(u) = \frac{\partial F(u)}{\partial u}$$

# Hyperbolic part – Approximate Riemann problem

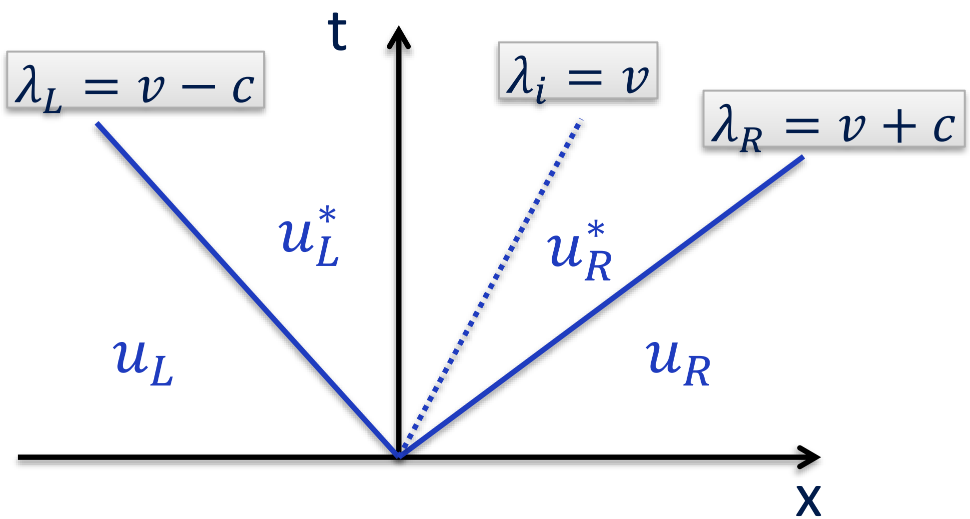
$$\frac{\partial u}{\partial t} + \tilde{A}(u)\,\frac{\partial u}{\partial x} = 0$$

**Lax-Friedrichs**




- ✚ Stable due to dissipation
- ✚ Easy to implement
- ✖ Lack of physics

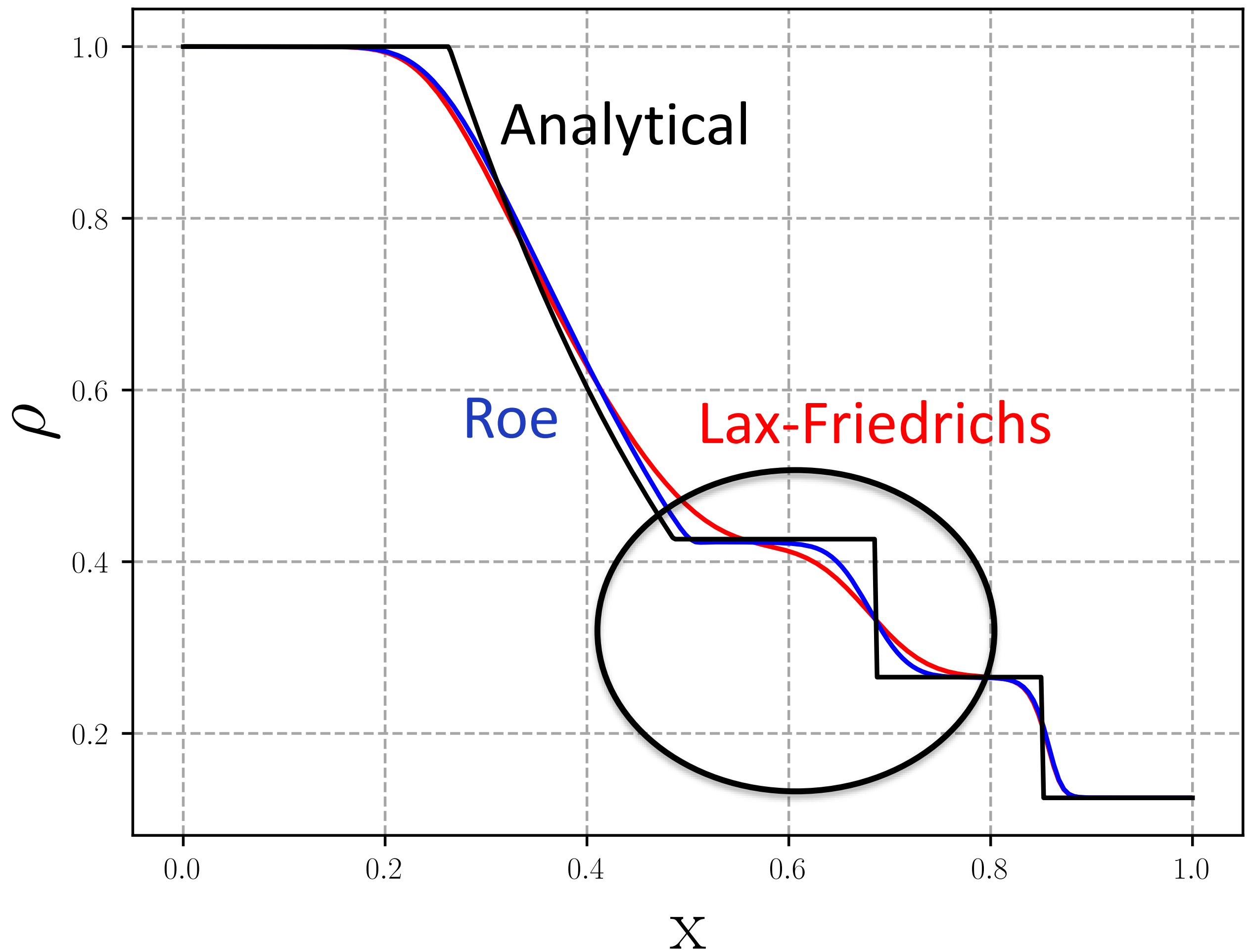
**Roe**




- ✚ More accurate
- ✖ Less stable

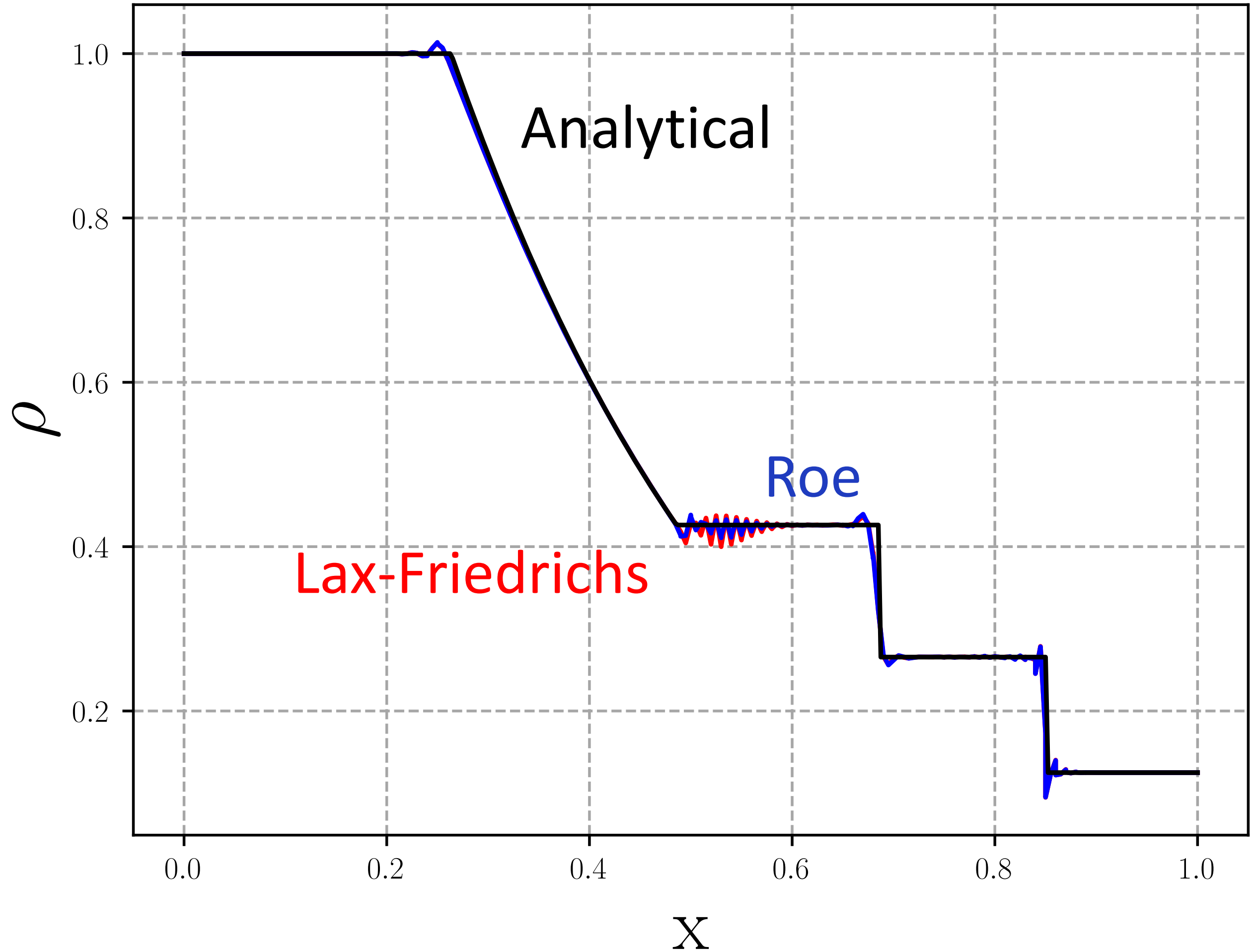
# Hyperbolic part – Finite volume

$$\sum_e \int_e \phi_i^e \frac{\partial u}{\partial t} dV - \sum_e \int_e \nabla \phi_i^e \cdot \boldsymbol{F} dV + \sum_f \int_f [\![ \phi_i^e ]\!] \boldsymbol{H}\, dS = 0$$



# Hyperbolic part – Discontinuous Galerkin

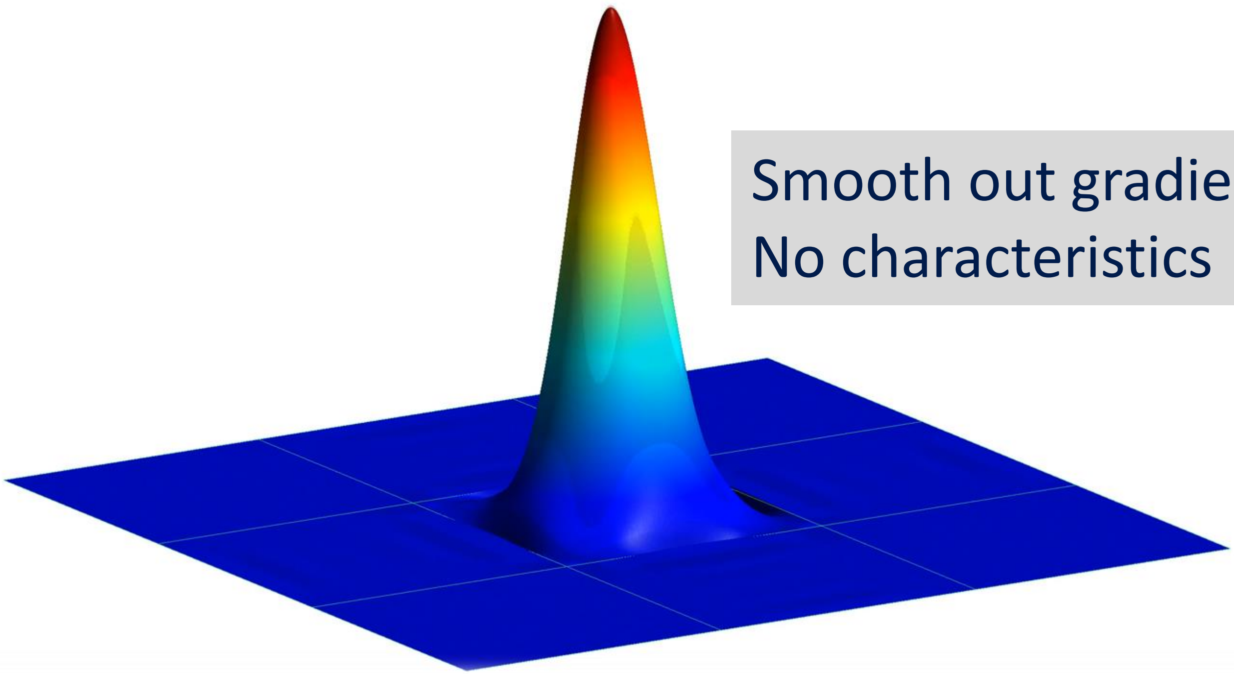
$$\sum_e \int_e \phi_i^e \frac{\partial u}{\partial t} dV - \sum_e \int_e \nabla \phi_i^e \cdot \boldsymbol{F} dV + \sum_f \int_f [\![\phi_i^e]\!] \boldsymbol{H}\, dS = 0$$



# Elliptic part

Diffusion

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) + \nabla \cdot \boldsymbol{d}(u, \nabla u) = 0$$



Smooth out gradients in time
No characteristics

# Elliptic part

Diffusion

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{f}(u) + \nabla \cdot \boldsymbol{d}(u, \nabla u) = 0$$

$$\sum_e \int_e \phi_i^e \frac{\partial u}{\partial t} dV - \sum_e \int_e \nabla \phi_i^e \cdot \boldsymbol{d} dV + \sum_f \int_f \gamma^e dS = 0$$

FEM theory

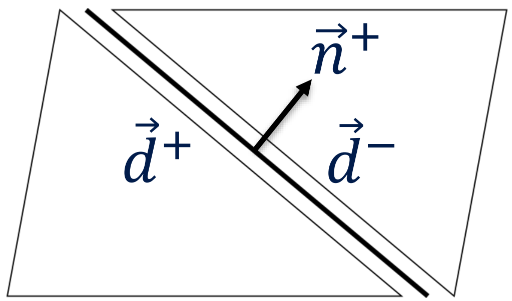
$\gamma^e$: central scheme

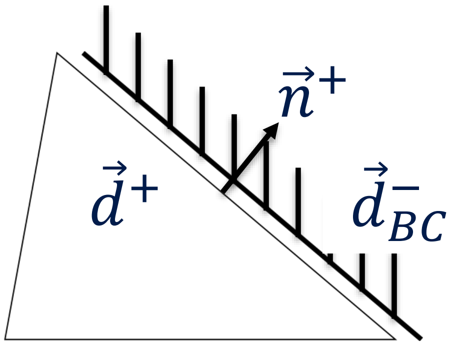
# Interior penalty method

How to construct appropriate **central** scheme?

$$\gamma^e = [\![\phi_i^e]\!] \cdot \{\vec{d}\}$$

We need boundary conditions! Neumann are dealt transparently.



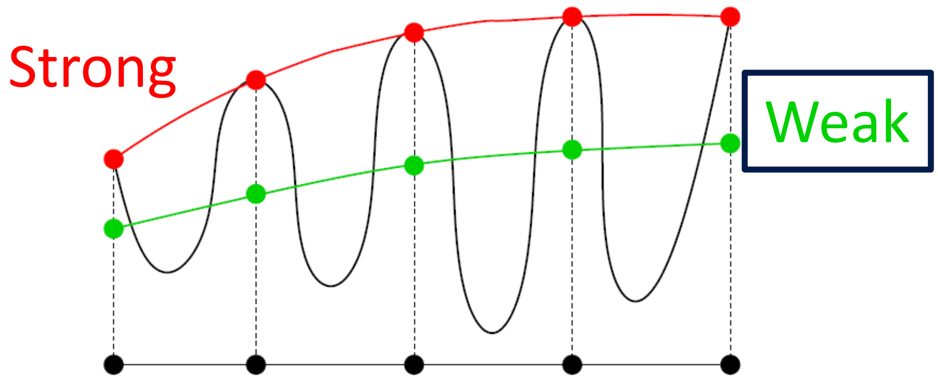



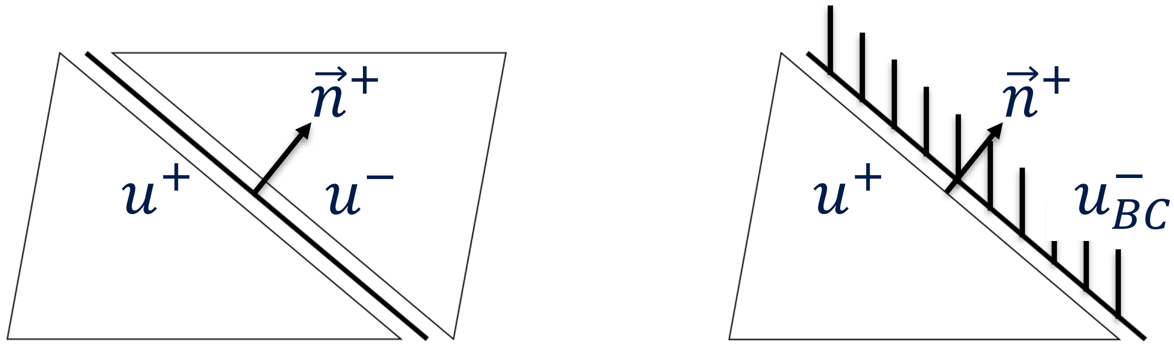
# Interior penalty method

How to construct appropriate **central** scheme?

$$\gamma^e = [\![\phi_i^e]\!] \cdot \{\vec{d}\}$$

We need boundary conditions! How to impose Dirichlet BC?



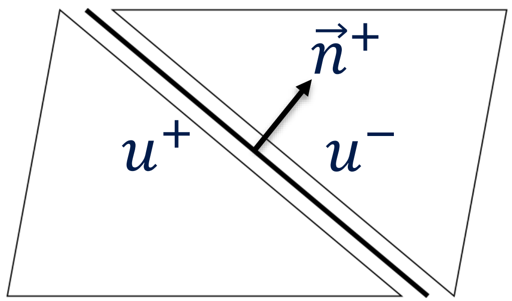



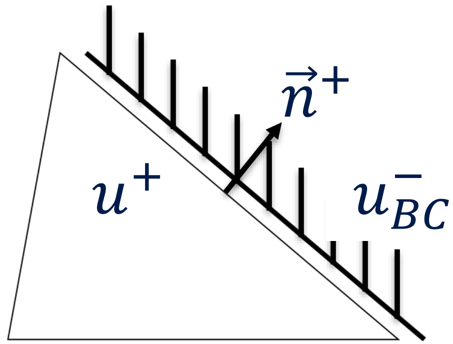
# Interior penalty method

How to construct appropriate **central** scheme?

$$\gamma^e = [\![\phi_i^e]\!] \cdot \{\vec{d}\} + \sigma_f [\![\phi_i^e]\!] \cdot [\![u]\!]$$

$\sigma_f = f(h, p)$, impact **stability** & **conditioning**



# Interior penalty method

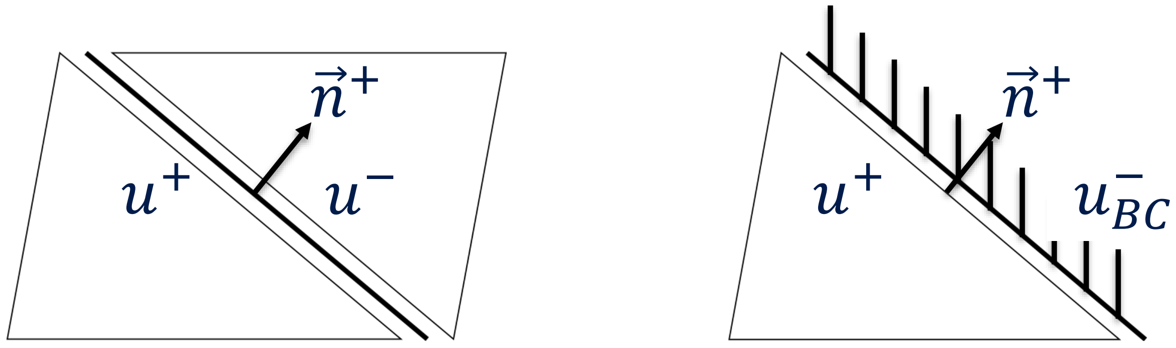
How to construct appropriate **central** scheme?

$$\gamma^e = [\![\phi_i^e]\!] \cdot \{\vec{d}\} + \sigma_f [\![\phi_i^e]\!] \cdot [\![u]\!] + \beta [\![u]\!] \cdot \{\mu \nabla \phi_i^e\}$$

Linearised diffusivity

$\sigma_f = f(h, p)$, impact **stability** & **conditioning**

$\beta$ to get a (anti-)symmetric or incomplete formulation



# Interior penalty method

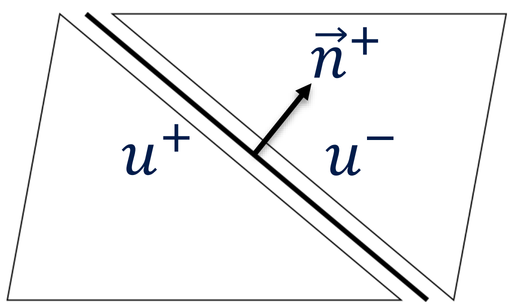
How to construct appropriate **central** scheme?

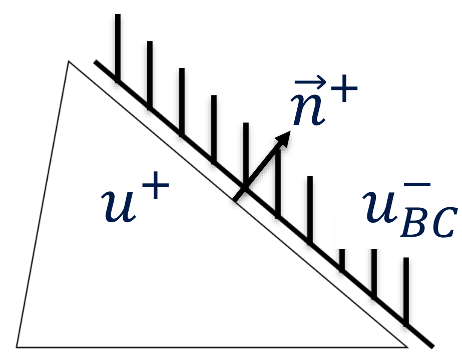
$$\gamma^e = [\![\phi_i^e]\!] \cdot \{\vec{d}\} + \sigma_f [\![\phi_i^e]\!] \cdot [\![u]\!] + \beta [\![u]\!] \cdot \{\mu \nabla \phi_i^e\}$$

At convergence

$\sigma_f = f(h, p)$, impact **stability** & **conditioning**

$\beta$ to get a (anti-)symmetric or incomplete formulation



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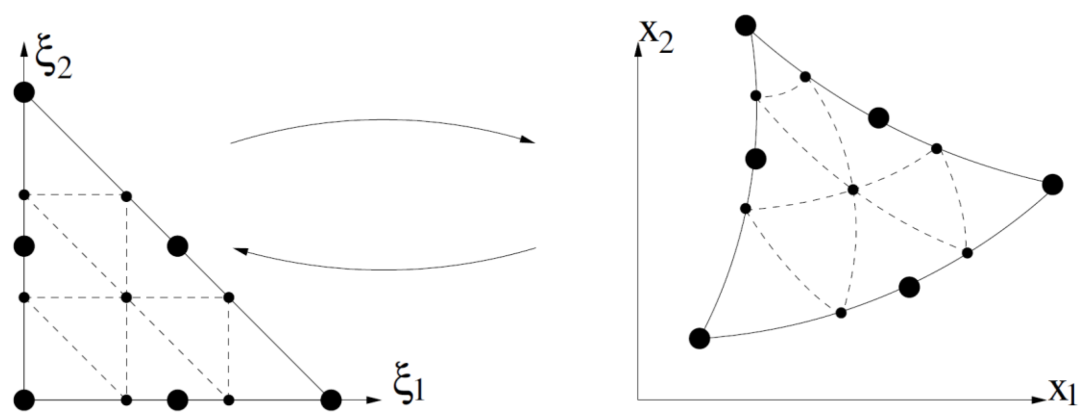
# Quadrature rules

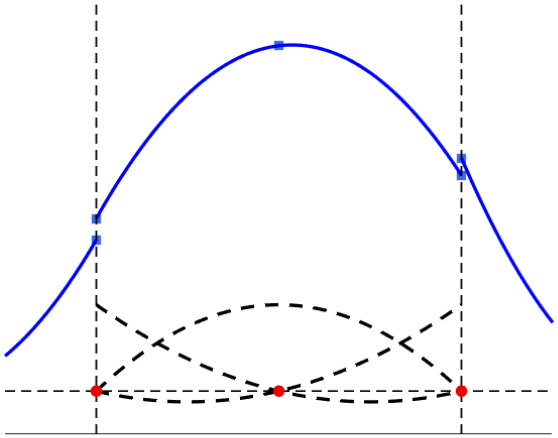
Accuracy of DGM ▶ Precision of integrals computation

$$\int_{-1}^{1} f(\xi) d\xi \approx \sum_{q=0}^{n} w_q f(\xi_q)$$

- Gaussian QR are exact for polynomials $f(\xi)$ of degree 2 $n$ -1
- Integration is performed on parametric reference elements

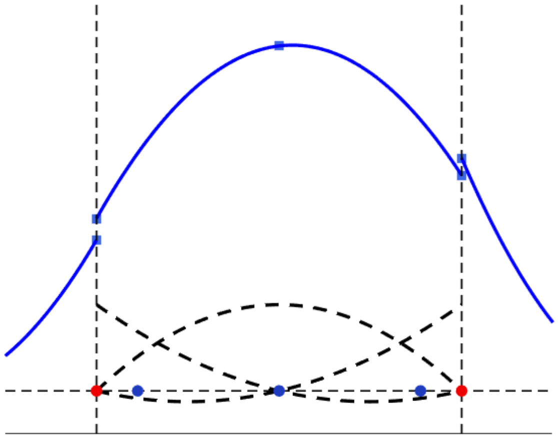


# Expansion to quadrature points



$$u^e\left(x_q\right) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$

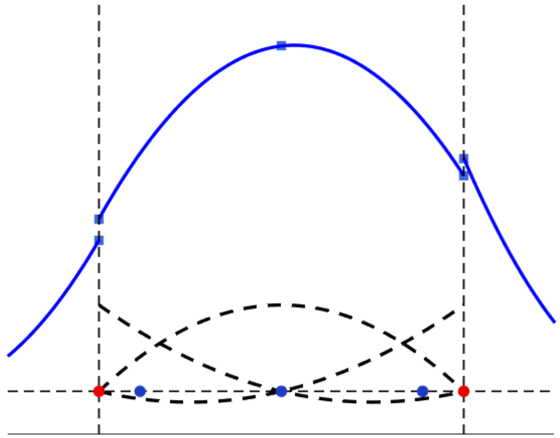
# Expansion to quadrature points



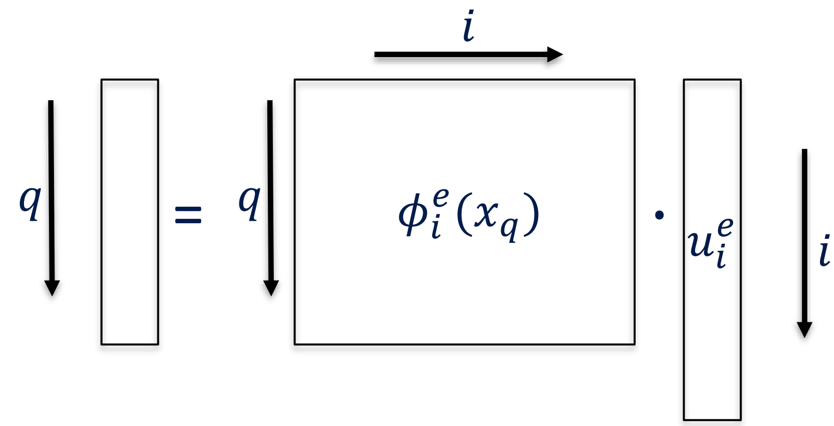
$$u^e(x_q) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$

Quadrature point

# Expansion to quadrature points



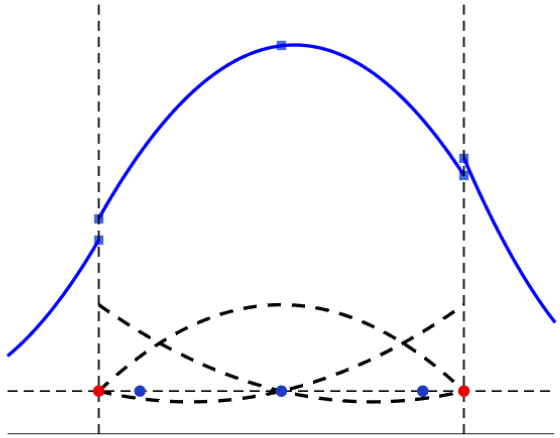
$$u^e(x_q) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$



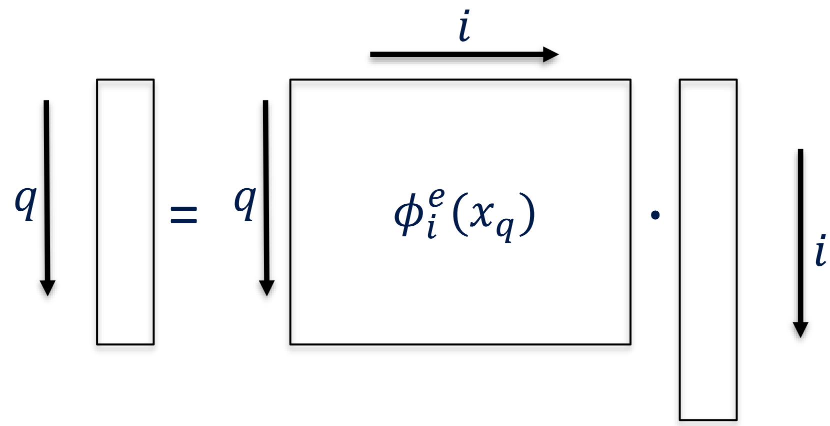

$$\boldsymbol{U_q} = \boldsymbol{C} \cdot \boldsymbol{U_i}$$

$$\sum_e \left( \int_e \phi_i^e \frac{\partial u^e}{\partial t} dV - \int_e \nabla \phi_i^e \cdot \vec{g}^e dV + \sum_f \int_f [\![ \phi_i^e ]\!] \cdot H^e dS \right) = 0, \forall\, \phi_i^e$$

# Inertia term



$$u^e(x_q) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$

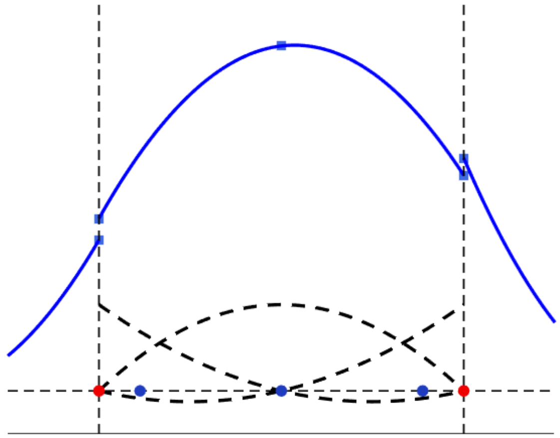



$$\dot{\boldsymbol{U}}_{\boldsymbol{q}} = \boldsymbol{C} \cdot \frac{\partial u_i^e}{\partial t}$$

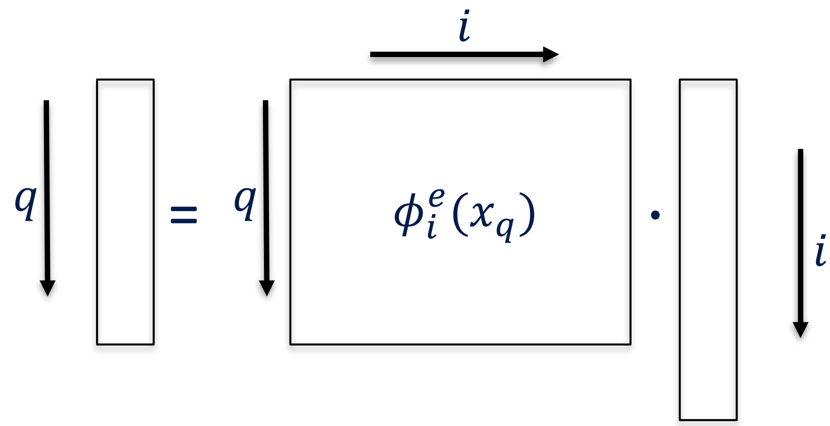
$$\sum_e \left( \int_e \phi_i^e \frac{\partial u^e}{\partial t} dV - \int_e \nabla\phi_i^e \cdot \vec{g}^e dV + \sum_f \int_f [\![\phi_i^e]\!] \cdot H^e dS \right) = 0, \forall\, \phi_i^e$$

$$\approx \sum_{q=0}^{n} w_q \phi_i^e(x_q) \cdot \frac{\partial u^e}{\partial t}(x_q) = \boldsymbol{M} \frac{\partial u_i^e}{\partial t}$$

# Volume term



$$u^e(x_q) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$

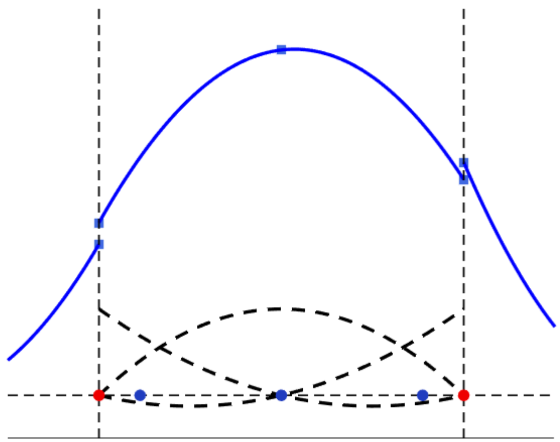



$$\dot{\boldsymbol{U}}_{\boldsymbol{q}} = \boldsymbol{C} \cdot \frac{\partial u_i^e}{\partial t}$$

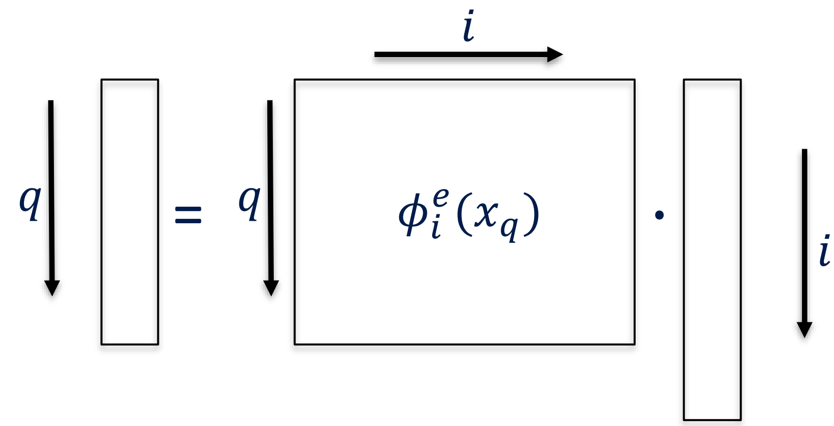
$$\sum_e \left( \int_e \phi_i^e \frac{\partial u^e}{\partial t} dV - \int_e \nabla \phi_i^e \cdot \vec{g}^e dV + \sum_f \int_f [\![\phi_i^e]\!] \cdot H^e dS \right) = 0, \forall\, \phi_i^e$$

$$\approx \sum_{q=0}^{n} w_q \nabla \phi_i^e(x_q) \cdot \vec{g}^e \Big|_{x_q} = \boldsymbol{R_v G}(\boldsymbol{U_q})$$

# Interface term



$$u^e(x_q) = \sum_{i=0}^{p} u_i^e \phi_i^e(x_q)$$



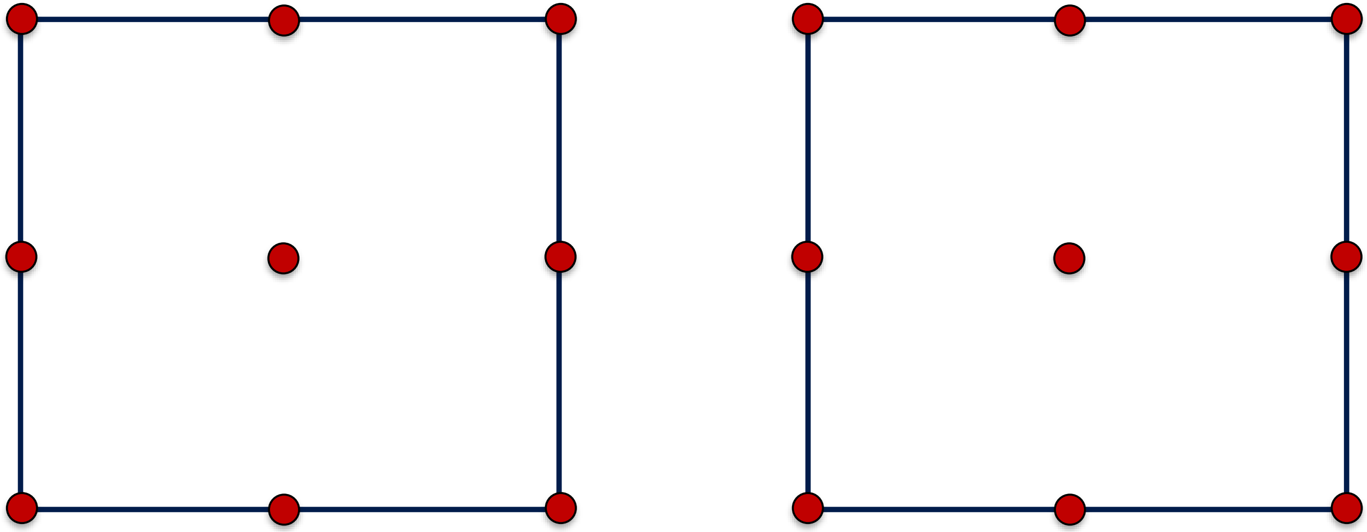

$$\dot{\boldsymbol{U}}_{\boldsymbol{q}} = \boldsymbol{C} \cdot \frac{\partial u_i^e}{\partial t}$$

$$\sum_e \left( \int_e \phi_i^e \frac{\partial u^e}{\partial t} dV - \int_e \nabla \phi_i^e \cdot \vec{g}^e dV + \sum_f \int_f [\![\phi_i^e]\!] \cdot H^e dS \right) = 0, \forall\, \phi_i^e$$

$$\approx \sum_{q=0}^{n} w_q \phi_i^e(x_q)\, H^e \Big|_{x_q} = \boldsymbol{R_f H(U_q)}$$

# Practical Implementation

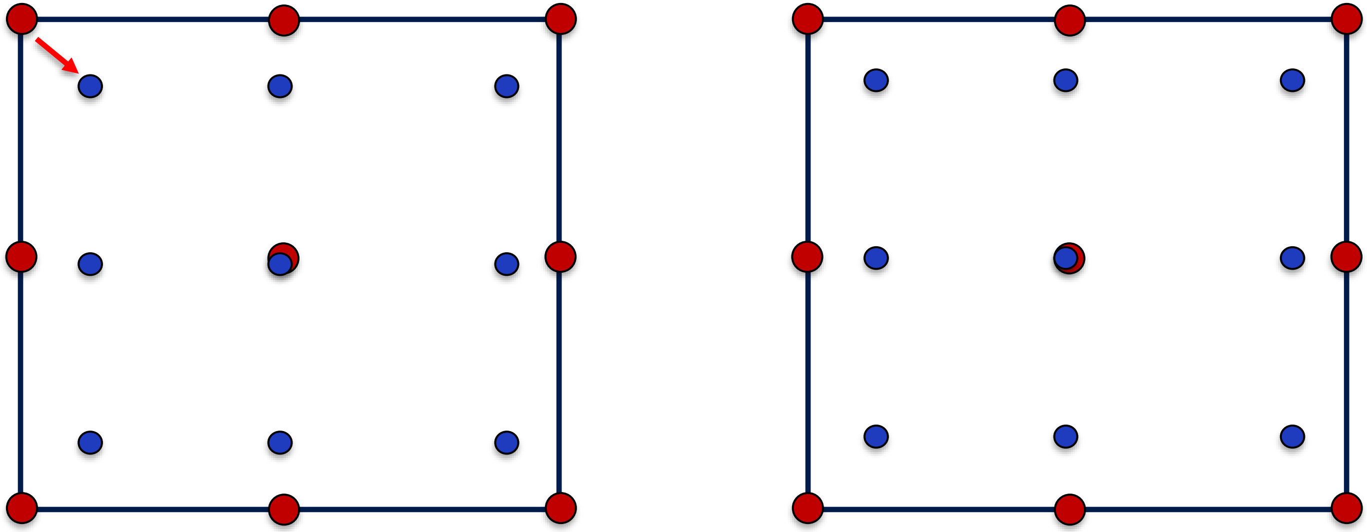
$$M\frac{\partial u_i^e}{\partial t} + R_v G(U_q) + \sum_f R_f H(U_{qf}^+, U_{qf}^-) = \mathbf{0}$$



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + R_v G(\boldsymbol{U_q}) + \sum_f R_f H(U_{qf}^+, U_{qf}^-) = \mathbf{0}$$

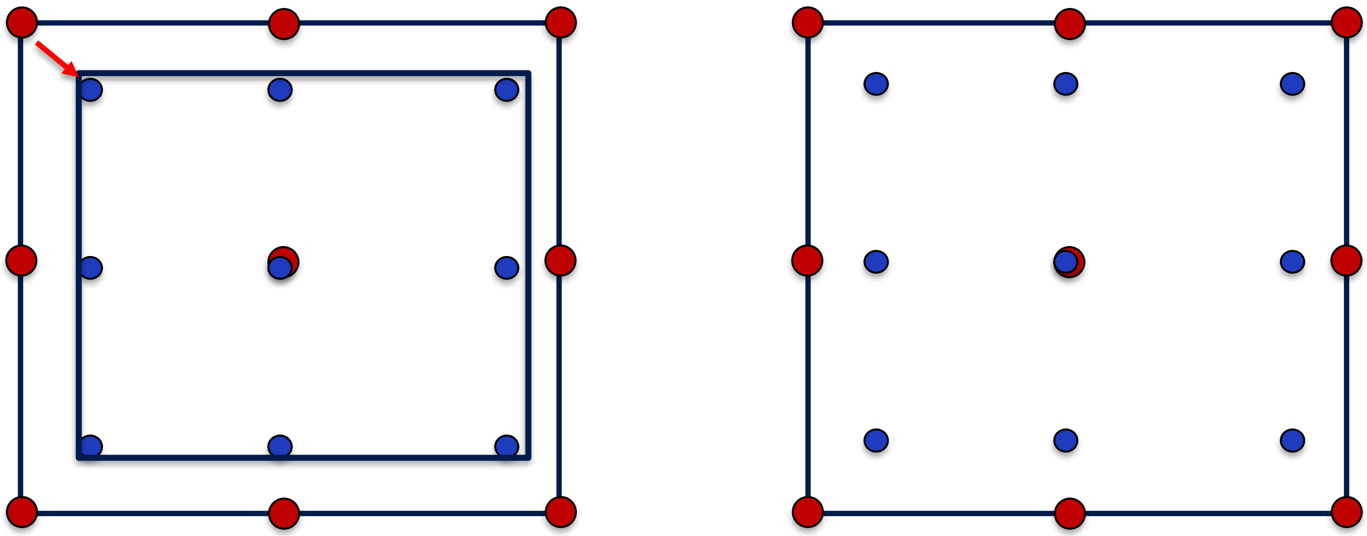
1. Collocation



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + R_v \boldsymbol{G}(\boldsymbol{U_q}) + \sum_f R_f H(U_{qf}^+, U_{qf}^-) = \mathbf{0}$$

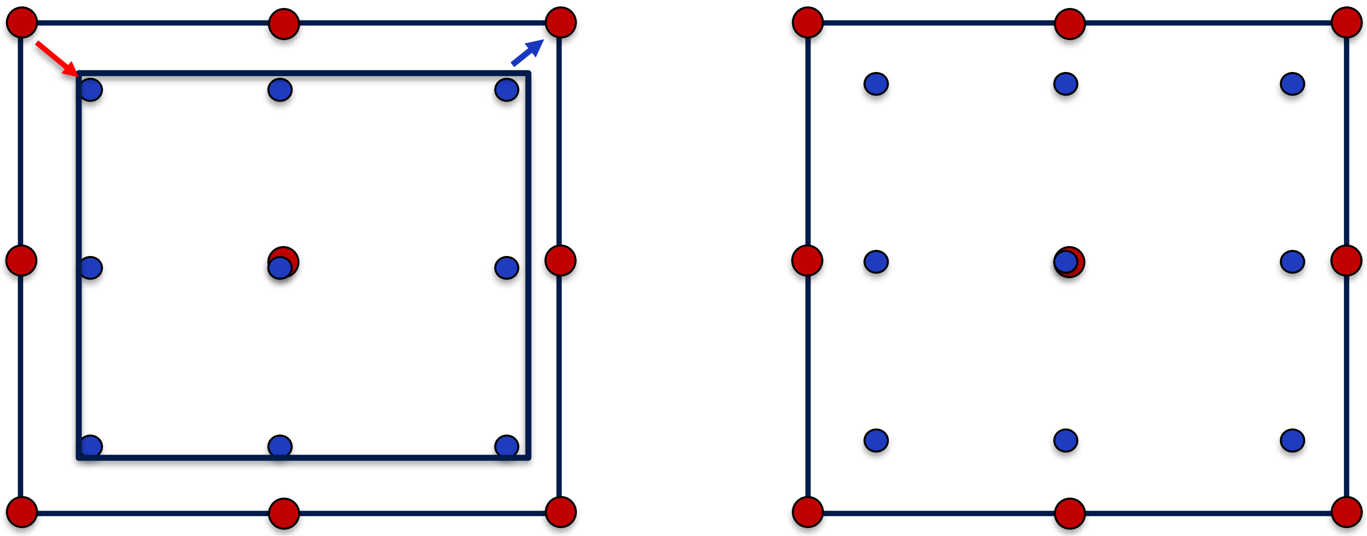
1. Collocation
2. Evaluation of volume fluxes



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R}_v \boldsymbol{G}(\boldsymbol{U}_q) + \sum_f \boldsymbol{R}_f \boldsymbol{H}(\boldsymbol{U}_{qf}^+, \boldsymbol{U}_{qf}^-) = \boldsymbol{0}$$

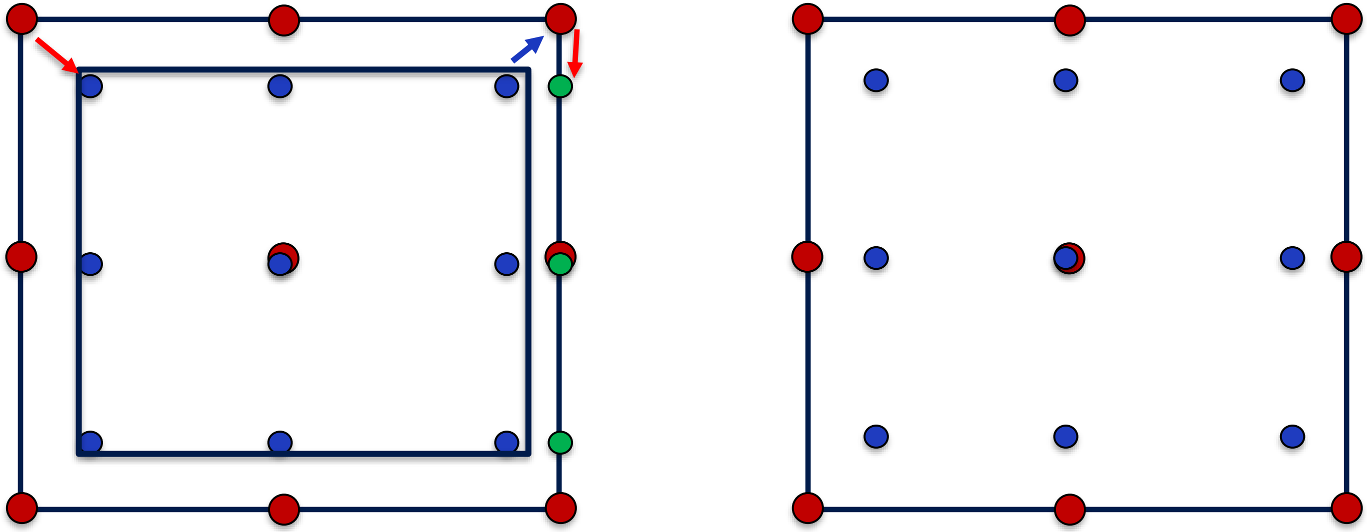
1. Collocation
2. Evaluation of volume fluxes
3. Integration & redistribution



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R_v G}(\boldsymbol{U_q}) + \sum_f R_f H\left(\boldsymbol{U_{qf}^+}, U_{qf}^-\right) = \boldsymbol{0}$$

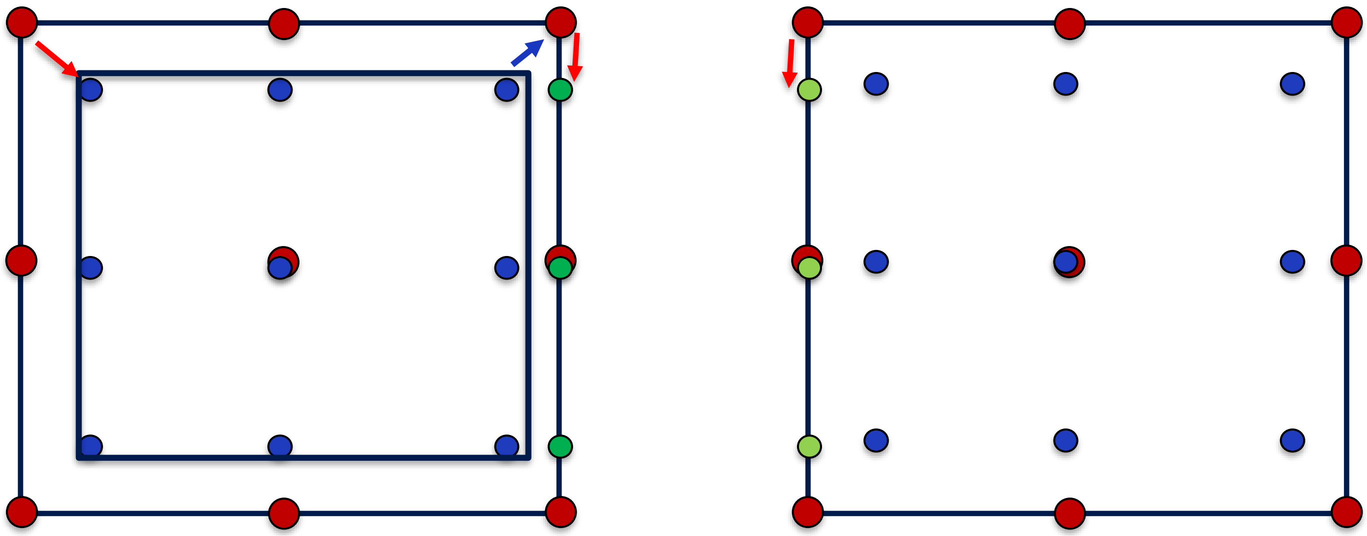
1. Collocation



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R_v G(U_q)} + \sum_f \boldsymbol{R_f H(U_{qf}^+, U_{qf}^-)} = \boldsymbol{0}$$

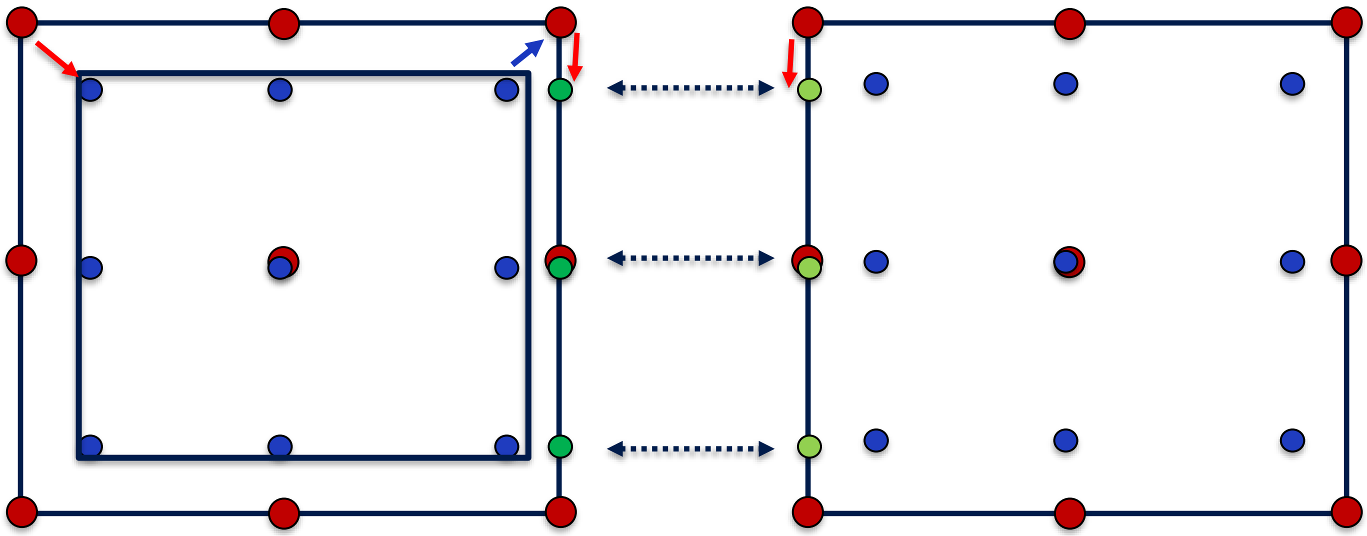
1. Collocation



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R}_v \boldsymbol{G}(\boldsymbol{U}_q) + \sum_f R_f \boldsymbol{H}(\boldsymbol{U}_{qf}^+, \boldsymbol{U}_{qf}^-) = \boldsymbol{0}$$

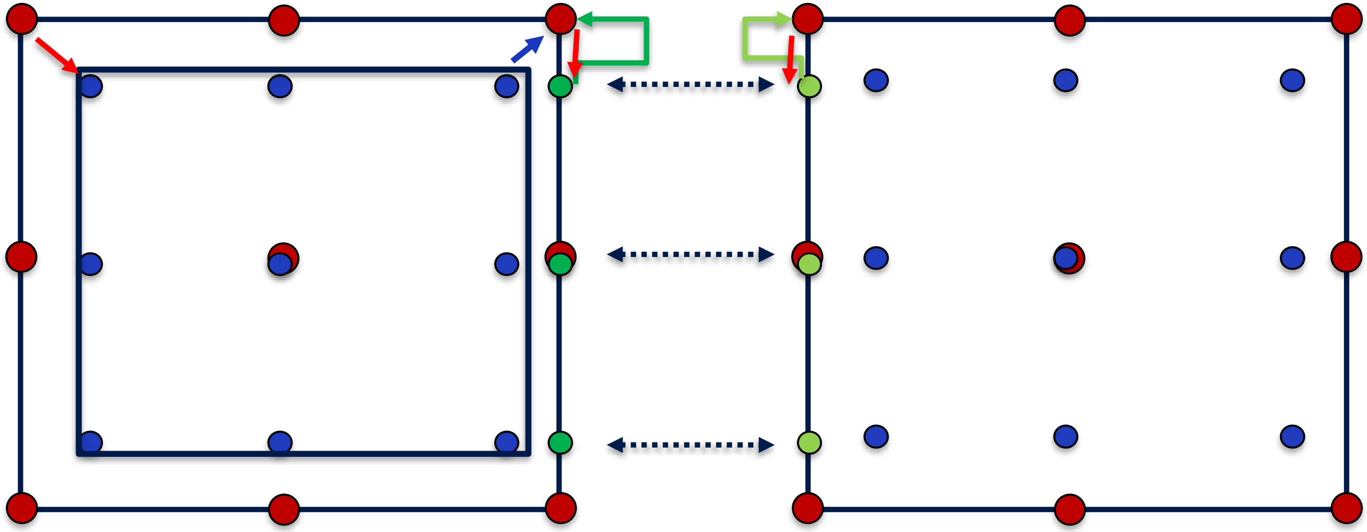
1. Collocation
2. Evaluation of interface fluxes



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R}_v \boldsymbol{G}(\boldsymbol{U}_q) + \sum_f \boldsymbol{R}_f \boldsymbol{H}(\boldsymbol{U}_{qf}^+, \boldsymbol{U}_{qf}^-) = \boldsymbol{0}$$

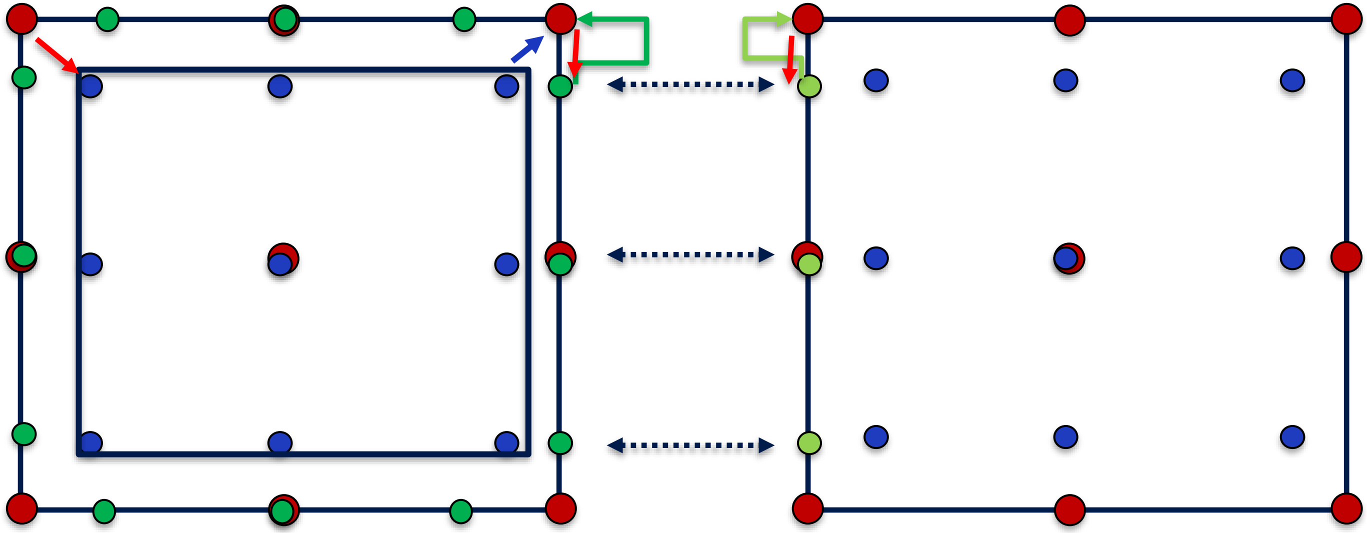
1. Collocation
2. Evaluation of interface fluxes
3. Integration & redistribution



# Practical Implementation

$$M\frac{\partial u_i^e}{\partial t} + \boldsymbol{R_v G(U_q)} + \sum_f \boldsymbol{R_f H(U_{qf}^+, U_{qf}^-)} = \mathbf{0}$$

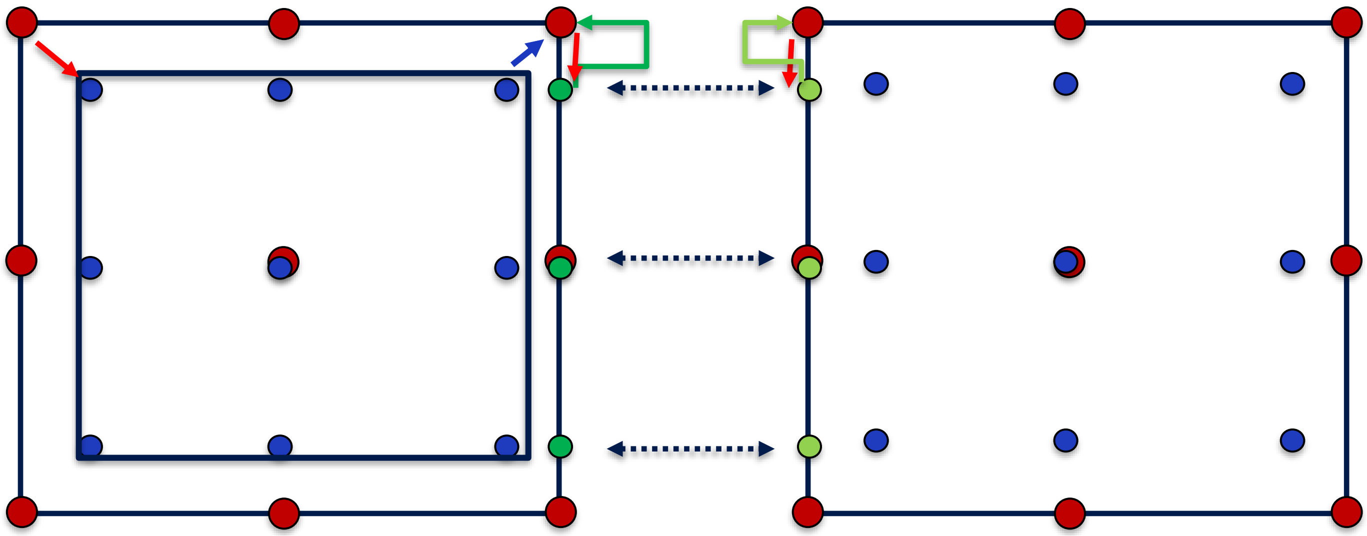
1. Collocation
2. Evaluation of interface fluxes
3. Integration & redistribution



# Practical Implementation

Residual, $\boldsymbol{L}$

$$M\frac{\partial u_i^e}{\partial t} + \overbrace{\boldsymbol{R}_v\boldsymbol{G}(\boldsymbol{U}_q) + \sum_f \boldsymbol{R}_f\boldsymbol{H}(\boldsymbol{U}_{qf}^+, \boldsymbol{U}_{qf}^-)} = \boldsymbol{0}$$



Semi-discrete eqn.

$$\frac{\partial u_i^e}{\partial t} = \boldsymbol{M}^{-1}\boldsymbol{L}$$

# Outline

What is Discontinuous Galerkin method?

Variational formulation

Interface fluxes

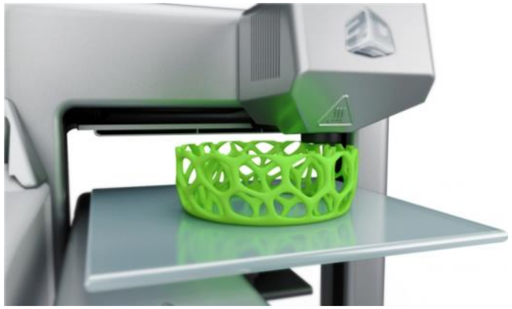
In practice

Our PhD theses

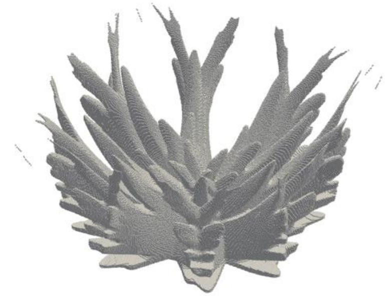
# ForDGe, a generic DG solver

Complex / evolving geometry

3D printing



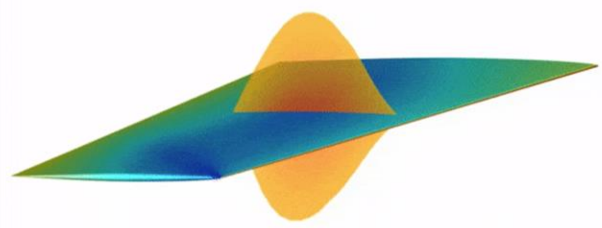
Topology optimization



[Alexandersen, 2016]

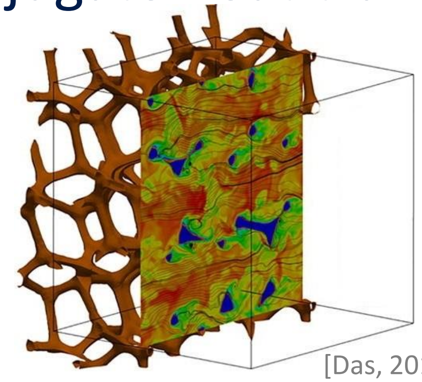
Multiphysics

Fluid-solid interactions



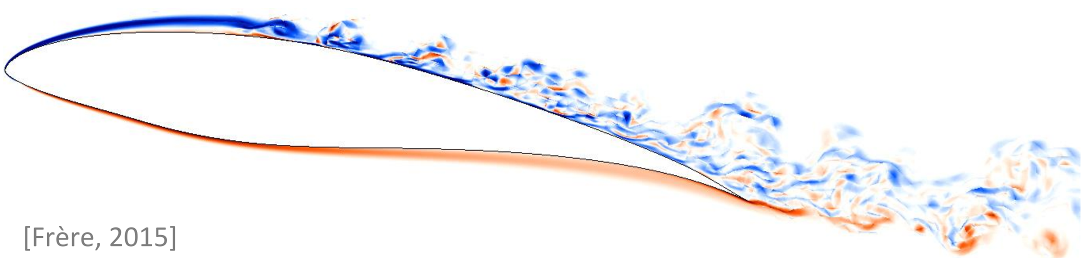
[Thomas, 2019]

Conjugate heat transfer



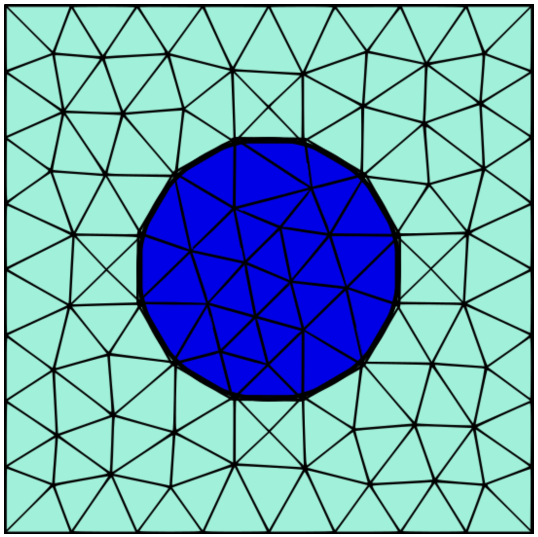
[Das, 2018]

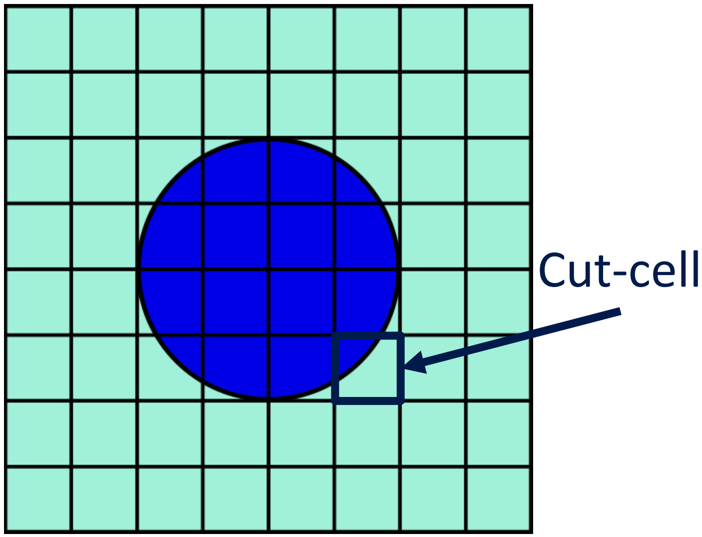
High resolution



[Frère, 2015]

# Geometry discretization






Body-fitted approach

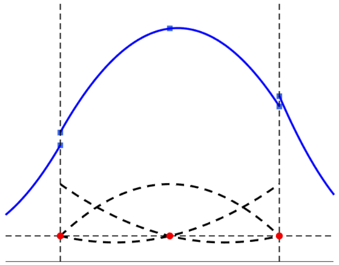
Immersed approach

- ✓ Simple interface conditions
- ✗ Challenging mesh generation
- ✗ Remeshing if evolving geometry

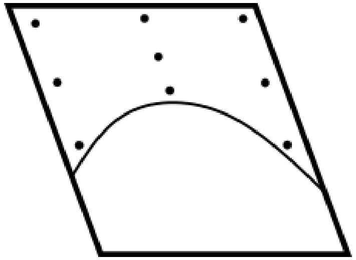
- ✓ Simple mesh generation
- ✗ Numerical errors at **interface**
- ✗ Lack of resolution in vicinity of interfaces

# Towards high accuracy immersed

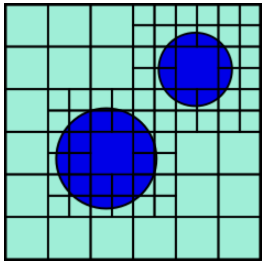
How to tackle lack of accuracy?



High-order DG
on cartesian grid



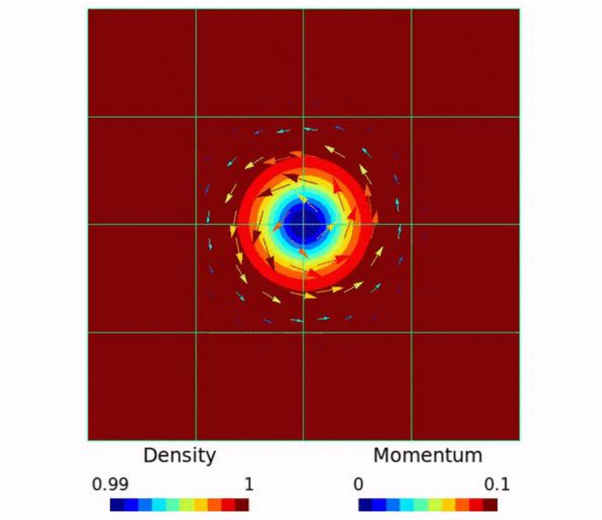
High-order
cut-cells



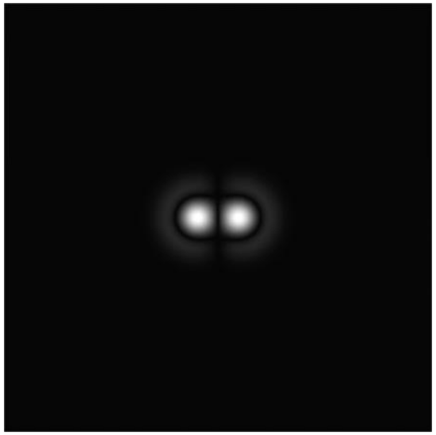
Adaptive mesh refinement

# Validation of DG solver

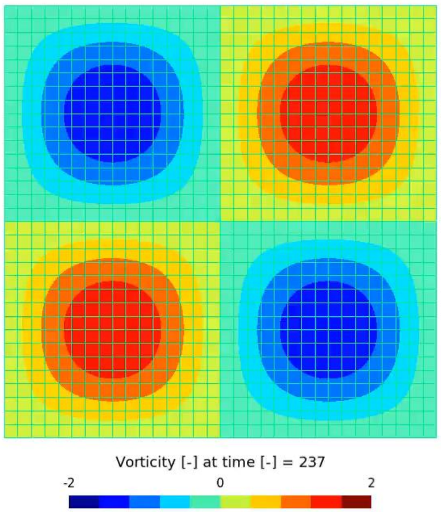
## Convection



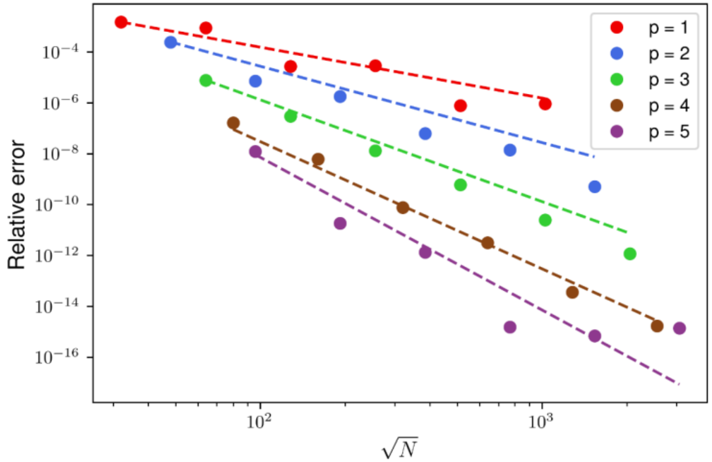

## Boundary conditions



## Convection-Diffusion

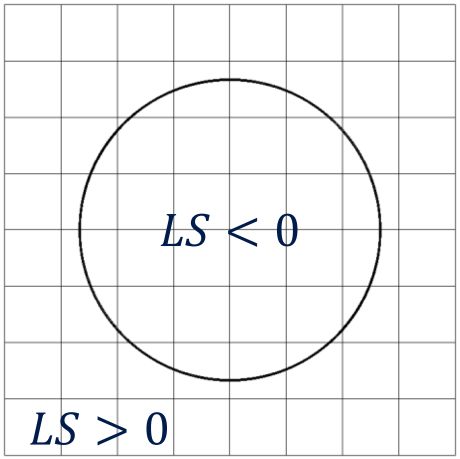



## Convergence



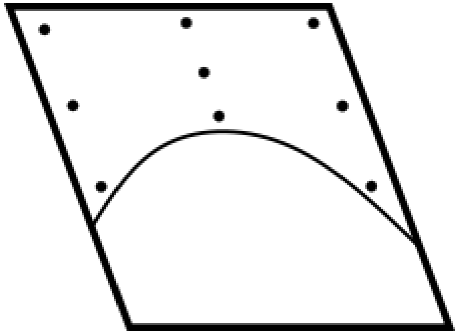
# Immersed approach

## Geometry definition



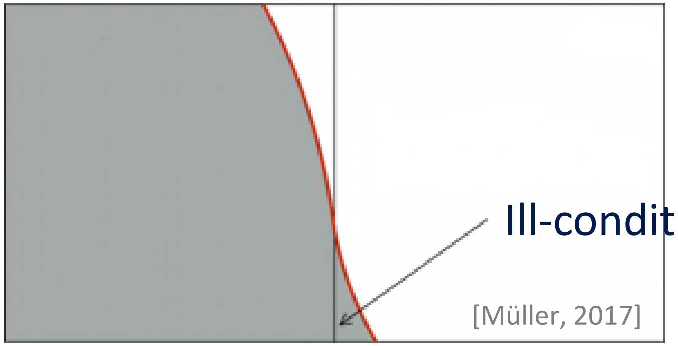

- Implicit sharp definition
- Cut elements identification

## Quadrature rules



- Accuracy
- Computational cost

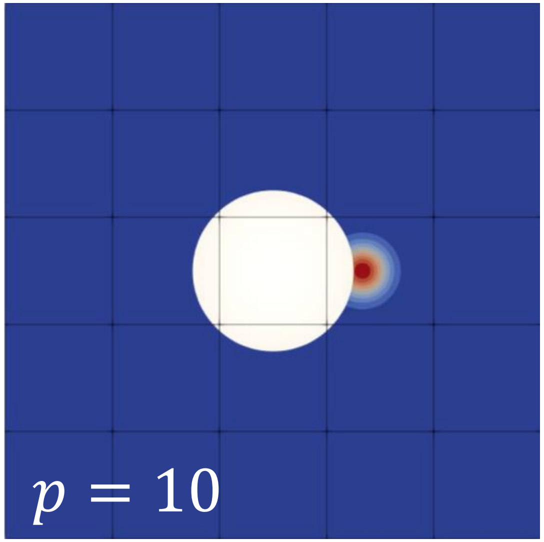
## Stability & conditionning



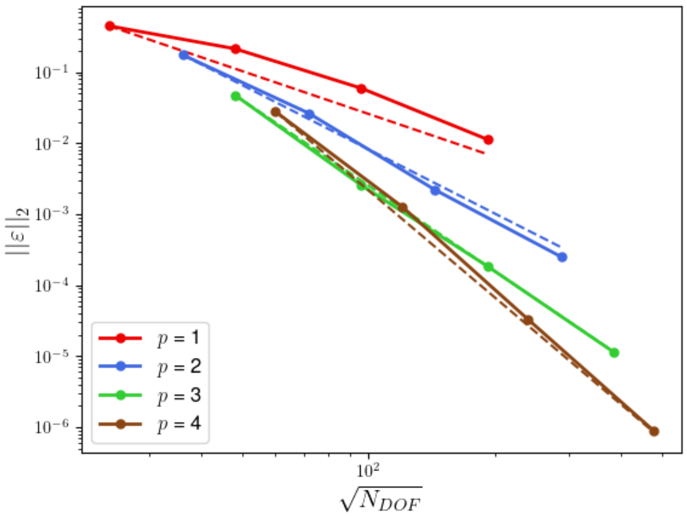

- Cell-agglomeration
- Interior penalty

# Validation of Immersed approach

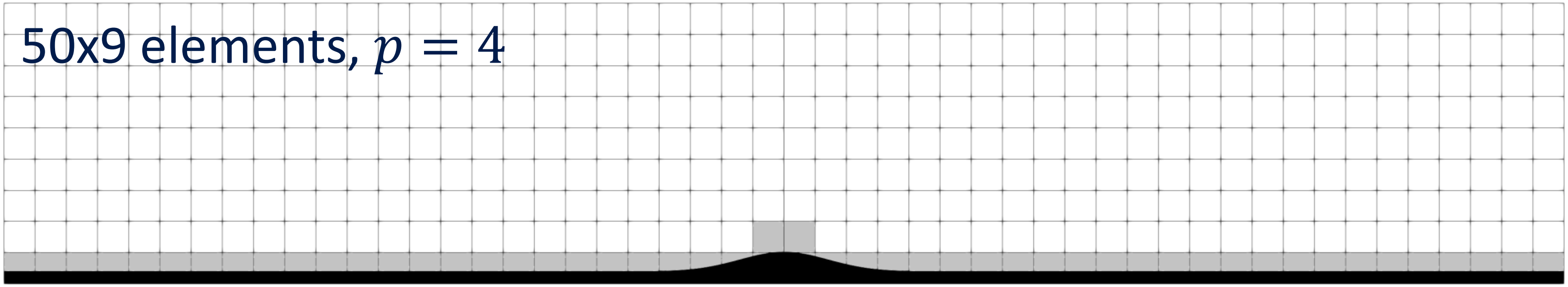
Convection




Convergence $p + 1$

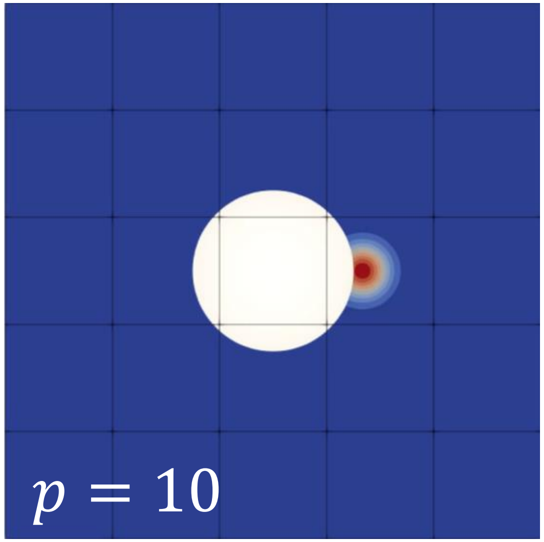



Euler, $M = 0.5$

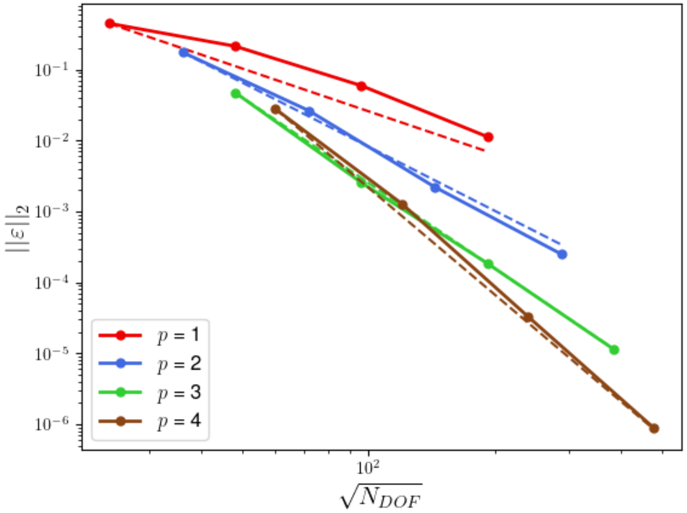


# Validation of Immersed approach

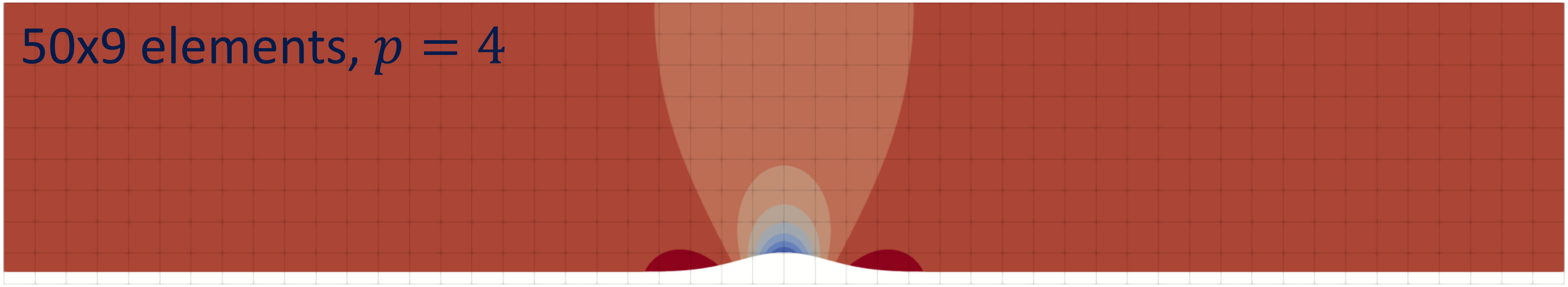
Convection




Convergence $p + 1$

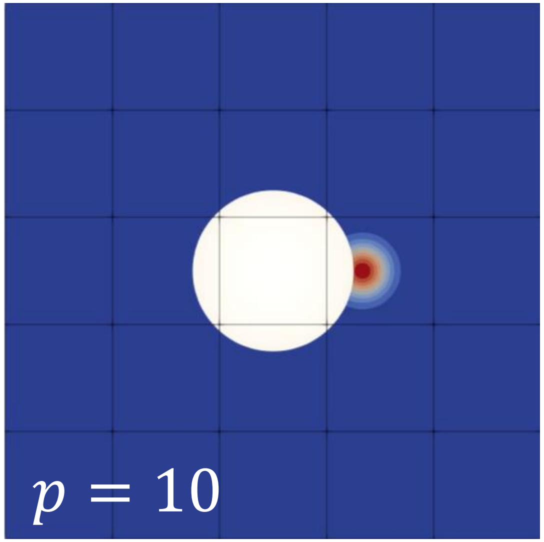



Euler, $M = 0.5$

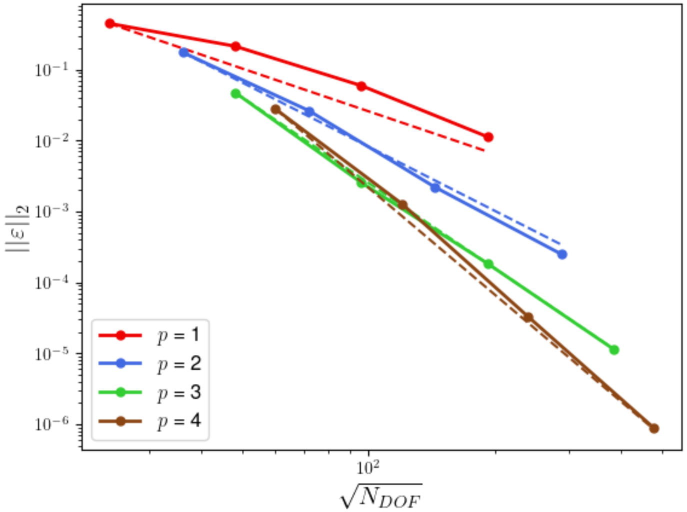


# Validation of Immersed approach

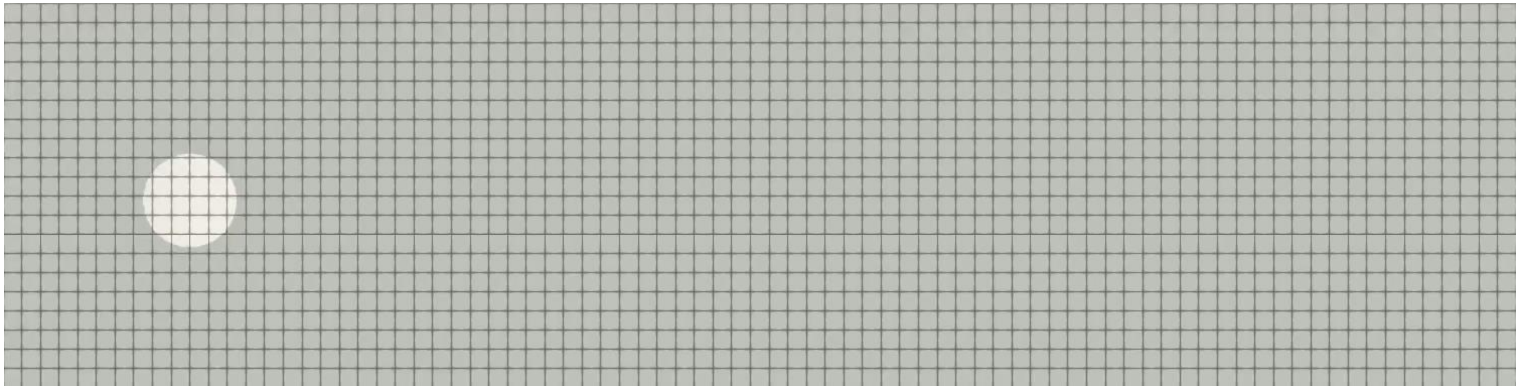
Convection



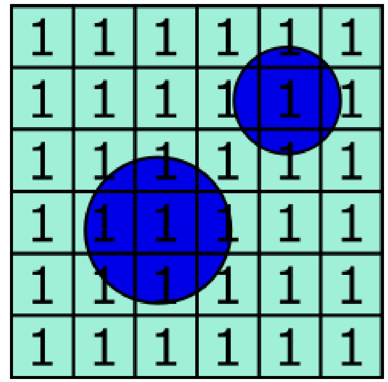

Convergence $p + 1$



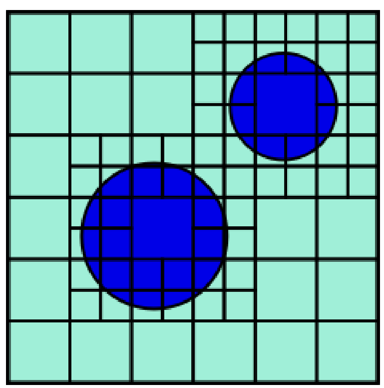

Navier-Stokes, $M = 0.1$, $Re = 100$



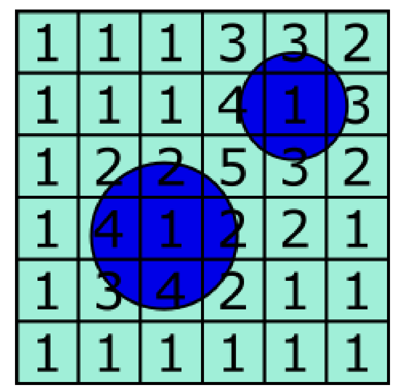
# Adaptive mesh / order refinement




Local dynamic refinement



Mesh

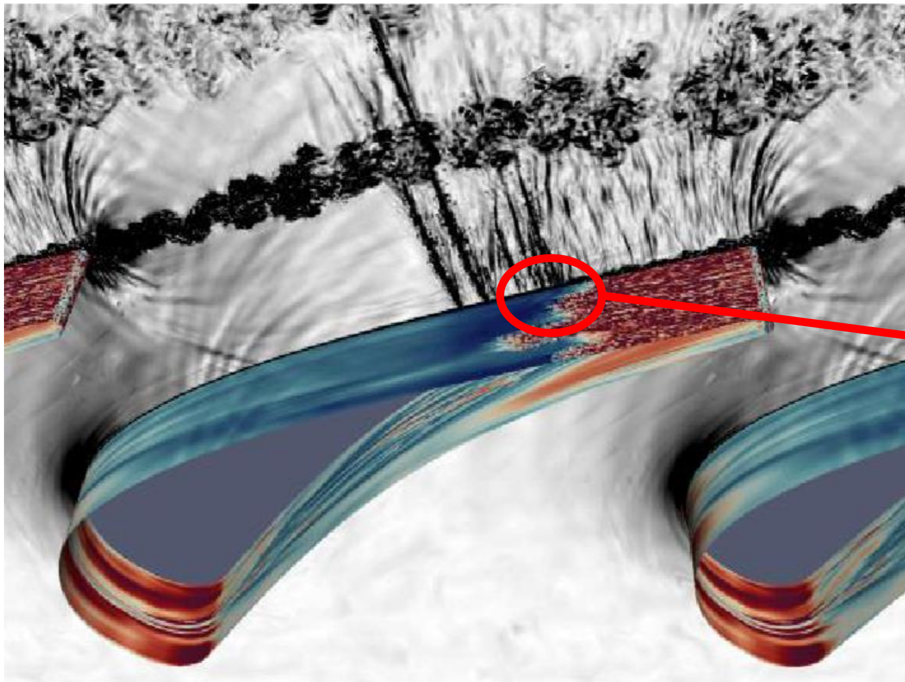



Order

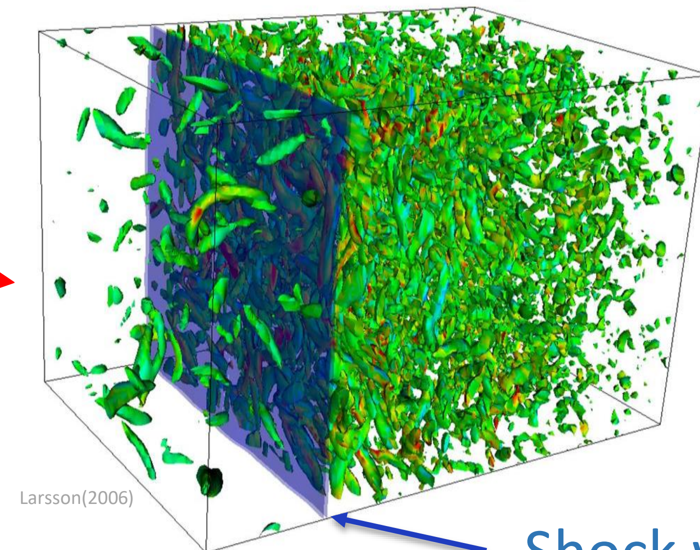
✔ Reduce computational cost

Interested? Reach us for a master thesis

# Transonic turbulent flows



Turbulence




Shock wave

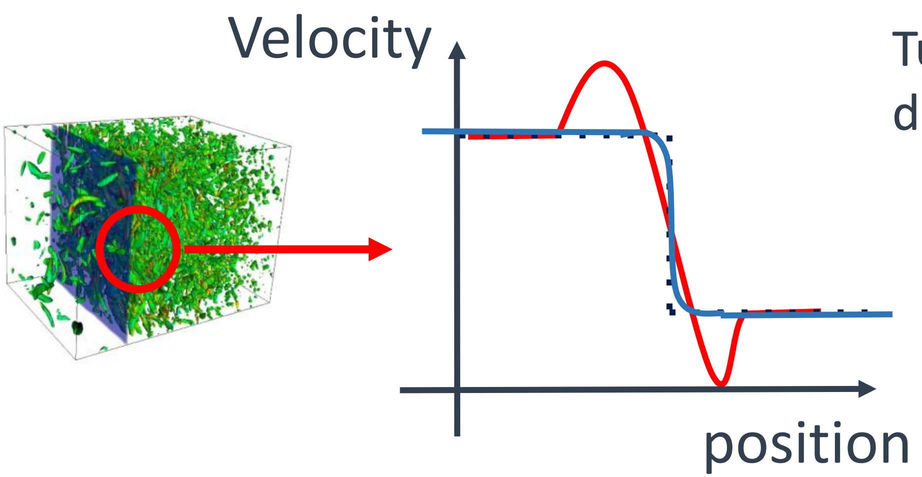
Turbulence ⟷ Shock wave

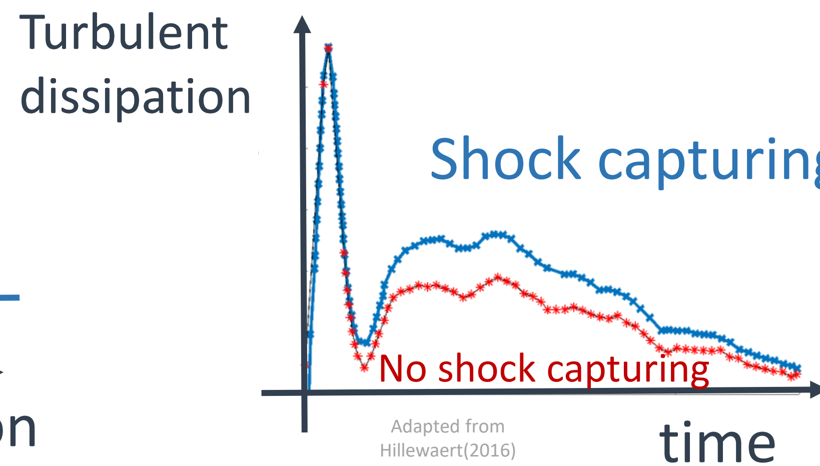
- Increase heat flux
- Aerodynamic losses
- Structure fatigue

Turbulence amplification

# Challenges

Shock waves induce **oscillations** → **Numerical dissipation** is added → The physics is **modified** hence turbulence is **damped**



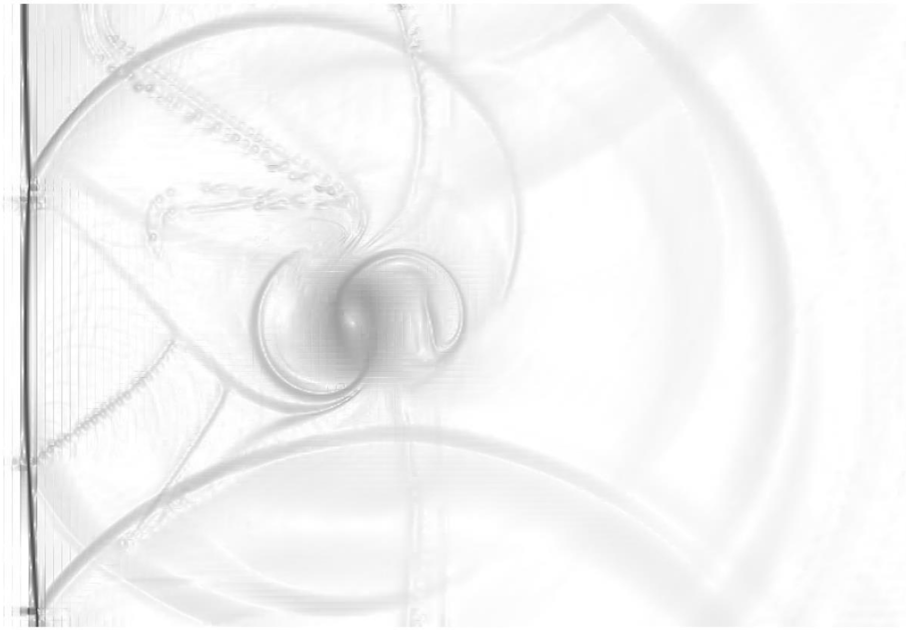




How can we **minimise** the impact of the **shock capturing method** on the **turbulence**?

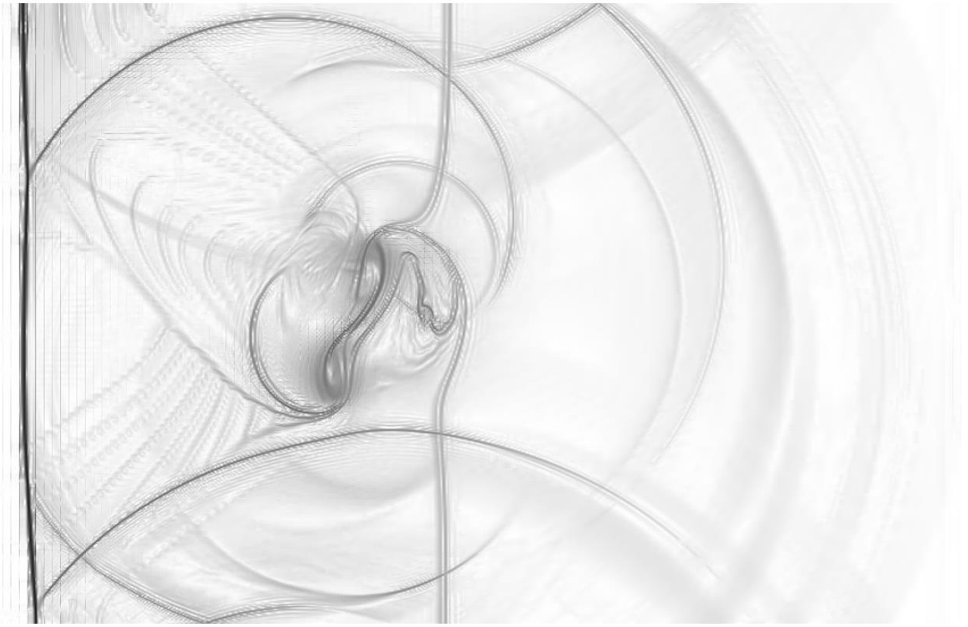
# Strong vortex – shock interaction

# Strong vortex – shock interaction

Sensor A



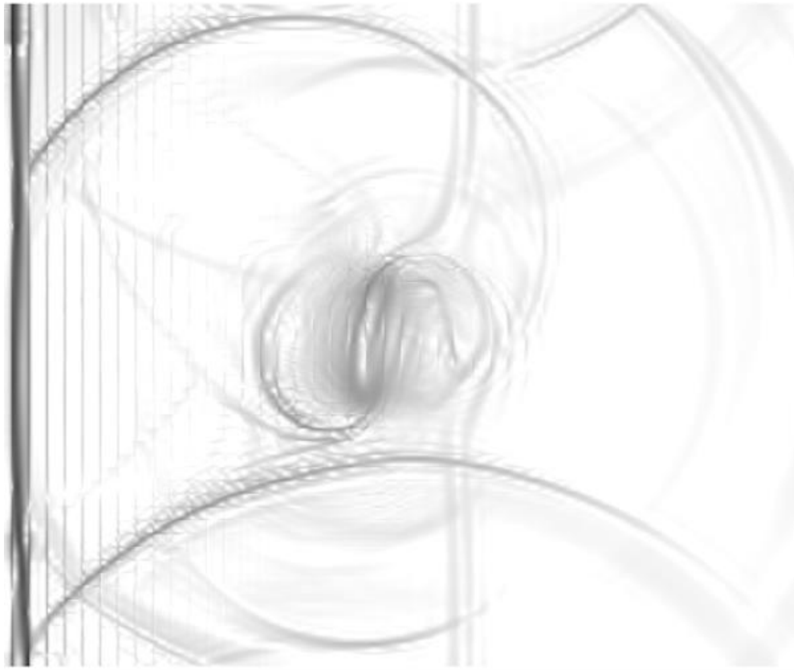
Sensor B



# Strong vortex – shock interaction

Impact of the stabilization method

Method A



Method B