Analogues of Cobham's theorem in three different areas of mathematics

Émilie Charlier

Département de Mathématique, Université de Liège

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This talk is based on

- An analogue of Cobham's theorem for fractals [Adamczewski-Bell 2011]
- On the sets of real numbers recognized by finite automata in multiple bases [Boigelot-Brusten-Bruyère 2010]
- First-order logic and Numeration Systems [Charlier 2017]
- An analogue of Cobham's theorem for graph-directed iterated function systems [Charlier-Leroy-Rigo 2015]
- On the structures of generating iterated function systems of Cantor sets [Feng-Wang 2009]

Part 1

Three Cobham-like theorems: links between them and generalizations

IFS and their attractors

An iterated function system (IFS) is a family of contraction maps $\Phi = (\phi_1, \dots, \phi_k)$ of $\mathbb{R}^d$.

Theorem (Hutchinson 1981)

There is a unique nonempty compact subset K of $\mathbb{R}^d$ with the property $K = \cup_{i=1}^{k} \phi_i(K)$.

This set K is called the attractor of the IFS Φ.

Fractals and self-similarity [Hutchinson 1981]

The Cantor set

Example

The Cantor set C is the attractor of the IFS (ϕ_1, ϕ_2) where $\phi_1 : x \mapsto \frac{1}{3}x$ and $\phi_2 : x \mapsto \frac{1}{3}x + \frac{2}{3}$.

As is the case for the Cantor set, we will restrict to IFS consisting of contracting affine maps.

- Can C be the attractor of another IFS? If yes, which ones?

On the structures of generating iterated function systems of Cantor sets [Feng-Wang 2009]

A Cobham theorem for IFS

A homogeneous IFS is an IFS Φ whose contracting affine maps all share the same contraction ratio r_Φ.

An IFS $\Phi = (\phi_1, \ldots, \phi_k)$ satisfies the open set condition (OSC) if there exists a nonempty open set V s.t. $\phi_1(V), \ldots, \phi_k(V)$ are pairwise disjoint subsets of V.

Theorem (Feng-Wang 2009)

Let Φ be a homogeneous IFS of $\mathbb{R}$ satisfying the OSC, let $\Psi = (r_1 x + t_1, \ldots, r_k x + t_k)$ and suppose that K is the attractor of both Φ and Ψ.

- *If $\dim_H(K) < 1$ then $\frac{\log |r_i|}{\log |r_\Phi|} \in \mathbb{Q}$ for each $1 \leq i \leq k$.*
- *If $\dim_H(K) = 1$, Ψ is homogeneous, and K is not a finite union of intervals, then $\frac{\log |r_\Psi|}{\log |r_\Phi|} \in \mathbb{Q}$.*

A Cobham theorem for real numbers in integer bases

Theorem (Boigelot-Brusten-Bruyère 2010)

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A subset of $\mathbb{R}^d$ is simultaneously weakly b-recognizable and b′-recognizable iff it is definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$.

Subsets of $\mathbb{R}^d$ that are definable (by a first order formula) in the structure $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$ are the finite unions of periodic repetitions of polyhedra with rational vertices.

On the sets of real numbers recognized by finite automata in multiple bases

[Boigelot-Brusten-Bruyere 2010]

A Cobham theorem for self-similar subsets

Theorem (Adamczewski-Bell 2011)

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A compact subset of $[0, 1]$ is simultaneously b-self-similar and b'-self-similar iff it is a finite union of intervals with rational endpoints.

They conjectured an equivalent result in higher dimension:

Conjecture

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A compact subset of $[0, 1]^d$ is simultaneously b-self-similar and b'-self-similar iff it is a finite union of polyhedra with rational vertices.

The Cobham theorem

Theorem (Cobham 1969)

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A subset of $\mathbb{N}$ is simultaneously b-recognizable and b'-recognizable iff it is a finite union of arithmetic progressions.

A subset X of $\mathbb{N}$ is b-recognizable if the set of all b-representations $\mathrm{val}_b^{-1}(X)$ is accepted by a finite automaton, where

$$\mathrm{val}_b \colon \{0, 1, \ldots, b-1\}^* \to \mathbb{N}, \; u_{\ell-1} \cdots u_0 \mapsto \sum_{i=0}^{\ell-1} u_i \, b^i .$$

On the base-dependance of sets of numbers recognizable by finite automata [Cobham 1969]

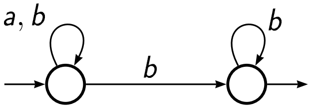
Recognizing real numbers

In general real numbers are represented by infinite words.

In this context, we consider Büchi automata. An infinite word is accepted when the corresponding path goes infinitely many times through an accepting state.

We talk about ω-languages and ω-regular languages.

Example (A Büchi automaton)



Regular languages vs ω-regular languages

Regular and ω-regular languages share some important properties: they both are stable under

- complementation
- finite union
- finite intersection
- morphic image
- inverse image under a morphism.

Nevertheless, they differ by some other aspects. One of them is determinism.

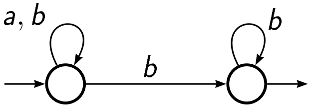
Deterministic Büchi automata

As for DFAs, we can define deterministic Büchi automata.

But one has to be careful as the family of ω-languages that are accepted by deterministic Büchi automata is strictly included in that of ω-regular languages.

Example

No deterministic Büchi automaton accepts the ω-language accepted by



Weak Büchi automata

- A Büchi automaton is weak if each of its strongly connected components contains either only final states or only non-final states.
- Deterministic weak Büchi automata admit a canonical form.
- Therefore, such automata can be viewed as the analogues of DFAs for infinite words.

β-representation of real numbers

Let $\beta > 1$ be a real number. For a real number x, any infinite word $u = u_k \cdots u_1 u_0 \star u_{-1} u_{-2} \cdots$ over $\mathbb{Z}$ s.t.

$$\mathrm{val}_\beta(u) := \sum_{-\infty < i \leq k} u_i \, \beta^i = x$$

is a β-representation of x.

In general, this is not unique.

Example ($\beta = \frac{1+\sqrt{5}}{2}$, the golden ratio)

Consider $x = \sum_{i \geq 1} \beta^{-2i}$.

As we also have $x = \sum_{i \geq 3} \beta^{-i} = \beta^{-1}$, the words

$$u = 0 \star 001111 \cdots, \quad v = 0 \star 0101010 \cdots \quad \text{and} \quad w = 0 \star 10000 \cdots$$

are all β-representations of x.

β-expansions of real numbers

For $x \geq 0$, among all such β-representations of x, we distinguish the β-expansion

$$d_\beta(x) = x_k \cdots x_1 x_0 \star x_{-1} x_{-2} \cdots$$

which is the infinite word over $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ obtained by the greedy algorithm.

Reals in $[0, 1)$ have a β-expansion of the form $0 \star u$ with $u \in A_\beta^\omega$.

In particular $d_\beta(0) = 0 \star 0^\omega$.

Parry's criterion

- We let $D_\beta = 0^* d_\beta(\mathbb{R}^{\geq 0})$.
- Then we let S_β denote the topological closure of D_β.
- Finally, $d_\beta^*(1)$ denotes the lexicographically greatest $w \in A_\beta^\omega$ not ending in 0^ω s.t. $\mathrm{val}_\beta(0 \star w) = 1$.

Theorem (Parry 1960)

Let $u = u_\ell \cdots u_1 u_0 \star u_{-1} u_{-2} \cdots$ with $u_i \in \mathbb{N}$ for all $i \leq \ell$. Then

$$u \in D_\beta \iff \forall k \leq \ell,\ u_k u_{k-1} \cdots < d_\beta^*(1), \text{ and}$$
$$u \in S_\beta \iff \forall k \leq \ell,\ u_k u_{k-1} \cdots \leq d_\beta^*(1).$$

Example ($\beta = \frac{1+\sqrt{5}}{2}$, the Golden ratio)

We have seen that the words

$$u = 0 \star 001111\cdots,\ v = 0 \star 0101010\cdots,\ w = 0 \star 1000\cdots$$

are all β-representations of $x = \sum_{i \geq 1} \beta^{-2i}$.

We have $d_\beta^*(1) = 101010\cdots$.

Thanks to Parry's theorem, the β-expansions of real numbers in $[0, 1)$ are of the form $0 \star u$, where $u \in \{0, 1\}^\omega$ does not contain 11 as a factor and not ending in $(10)^\omega$.

So w is *the* β-expansion of x, and both v and w belongs to S_β.

Representing negative numbers

In order to deal with negative numbers, we let $\overline{a}$ denote the integer $-a$ for all $a \in \mathbb{Z}$. Moreover we write

$$\overline{u\,v} = \overline{u}\,\overline{v}, \quad \overline{u \star v} = \overline{u} \star \overline{v} \quad \text{and } \overline{\overline{u}} = u.$$

For $x < 0$, the β-expansion of x is defined as

$$d_\beta(x) = \overline{d_\beta(-x)}.$$

We let $\overline{A_\beta} = \{\overline{0}, \overline{1}, \ldots, \overline{\lceil\beta\rceil - 1}\}$ and $\tilde{A_\beta} = A_\beta \cup \overline{A_\beta}$ (with $\overline{0} = 0$).

Multidimensional framework

Let $\beta = \frac{1+\sqrt{5}}{2}$.

Consider $\mathbf{x} = (x_1, x_2) = (\frac{1+\sqrt{5}}{4}, 2+\sqrt{5})$. We have

$$d_\beta(\mathbf{x}) = \begin{matrix} 0 & 0 & 0 & \star & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & \star & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \end{matrix}$$

where the first β-expansion is padded with some leading zeroes.

With $\mathbf{y} = (x_1, x_2) = (\frac{1+\sqrt{5}}{4}, -\frac{1}{2})$, we get

$$d_\beta(\mathbf{y}) = \begin{matrix} 0 & \star & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & \star & 0 & \overline{1} & 0 & 0 & \overline{1} & 0 & \cdots \end{matrix}$$

Quasi-greedy representations

- We let $S_\beta(\mathbb{R}^d)$ be the topological closure of $0^* d_\beta(\mathbb{R}^d)$.
- In particular, for $d = 1$, we have $S_\beta(\mathbb{R}) = S_\beta \cup \overline{S_\beta}$.
- Let $\mathrm{val}_\beta(u \bigstar v)$ to be the vector in $\mathbb{R}^d$ obtained by evaluating each component of $u \bigstar v$.
- For $X \subseteq \mathbb{R}^d$, we define $S_\beta(X) = S_\beta(\mathbb{R}^d) \cap \mathrm{val}_\beta^{-1}(X)$.
- The quasi-greedy β-representations of $\mathbf{x} \in \mathbb{R}^d$ are the elements in $S_\beta(\mathbf{x})$.

Closed is closed

A subset X of $\mathbb{R}^d$ is closed iff $S_\beta(X)$ is closed.

β-recognizable subsets of $\mathbb{R}^d$

A subset X of $\mathbb{R}^d$ is β-recognizable if $S_\beta(X)$ is ω-regular.

Remarks and properties

- Two β-recognizable subsets of $\mathbb{R}^d$ coincide iff they have the same ultimately periodic quasi-greedy β-representations.
- A β-recognizable subset X of $\mathbb{R}^d$ is closed iff $S_\beta(X)$ is accepted by a deterministic Büchi automaton all of whose states are final.
- But how to understand those sets in another way than the regularity of their β-representations itself ? How to prove that a subset we are interested in is or is not β-recognizable?

Parry numbers

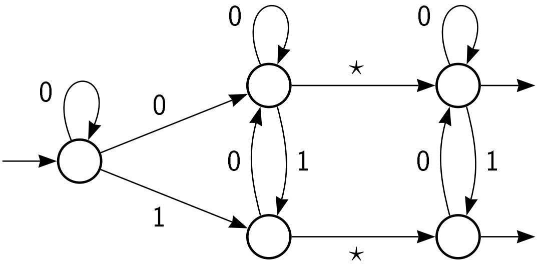
A Parry number is a real number $\beta > 1$ for which $d^*_\beta(1)$ is ultimately periodic.

Remarks and properties for Parry bases β

- Corollary of Parry's theorem: S_β is accepted by a weak deterministic Büchi automaton, and hence so is $S_\beta(\mathbb{R}^d)$.
- $S_\beta(X)$ is ω-regular iff so is $d_\beta(X)$.
- As a consequence, it is easy to provide examples of β-recognizable sets.

Example ($\beta = \frac{1+\sqrt{5}}{2}$, the Golden ratio)

The following Büchi automaton accepts the ω-language S_β.



To handle negative numbers, we make the union of two such automata. For $d > 1$, we handle the sign of each components separately: we get a union of 2^d of such automata.

Weak β-recognizability

A subset X of $\mathbb{R}^d$ is weakly β-recognizable if $S_\beta(X)$ is accepted by a weak deterministic Büchi automaton.

About closed sets

A closed subset of $\mathbb{R}^d$ is β-recognizable iff it is weakly β-recognizable.

Back to the Cobham-like theorems

Let $b, b' \geq 2$ be integers s.t. $\frac{\log(b)}{\log(b')} \notin \mathbb{Q}$.

Theorem (Boigelot-Brusten-Bruyère 2010)

A subset of $\mathbb{R}^d$ is simultaneously weakly b-recognizable and b'-recognizable iff it is definable in $\langle \mathbb{R}, +, \leq, \mathbb{Z} \rangle$.

Theorem (Adamczewski-Bell 2011)

A compact subset of $[0, 1]$ is simultaneously b-self-similar and b'-self-similar iff it is a finite union of closed intervals with rational endpoints.

b-self-similarity

Let $b \geq 2$ be an integer.

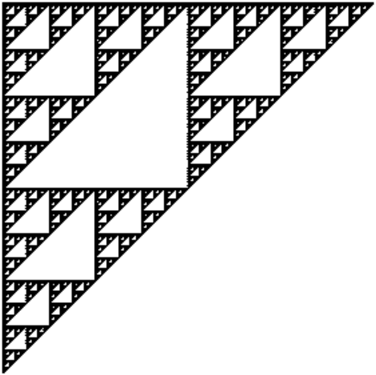
A compact set $X \subset [0,1]^d$ is b-self-similar if its b-kernel

$$\left\{(b^a X - \mathbf{t}) \cap [0,1]^d : a \in \mathbb{N},\ \mathbf{t} \in ([0, b^a) \cap \mathbb{Z})^d\right\}$$

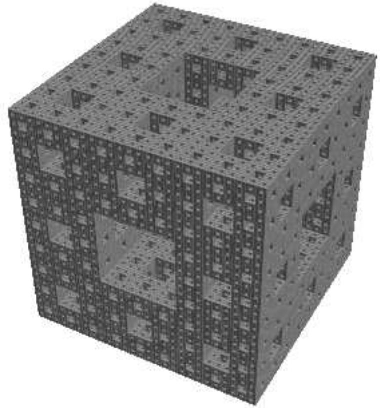
is finite.

An analogue of Cobham's theorem for fractals [Adamczewski-Bell 2011]

Example (Pascal's triangle modulo 2 is 2-self-similar)



Example (The Menger sponge is 3-self-similar)



β-self-similarity

Instead of working in $[0,1]$, we work in $I_\beta = \left[0, \frac{\lceil\beta\rceil - 1}{\beta - 1}\right]$.
We also let $J_\beta = \left[0, \frac{\lceil\beta\rceil - 1}{\beta - 1}\right)$

A compact subset X of ${I_\beta}^d$ is β-self-similar if its β-kernel

$$\left\{(\beta^a X - \mathbf{t}) \cap {I_\beta}^d : a \in \mathbb{N},\ \mathbf{t} \in (\beta^a J_\beta \cap \mathbb{Z}[\beta])^d.\right\}$$

is finite.

Graph-directed iterated function systems (GDIFS)

A GDIFS is a 4-tuple $(V, E, (X_v, v \in V), (\phi_e, e \in E))$ where

- (V, E) is a connected digraph s.t. each vertex has at least one outgoing edge,
- for each $v \in V$, X_v is a complete metric space,
- for each edge in E from u to v, $\phi_e \colon X_v \to X_u$ is a contraction map.

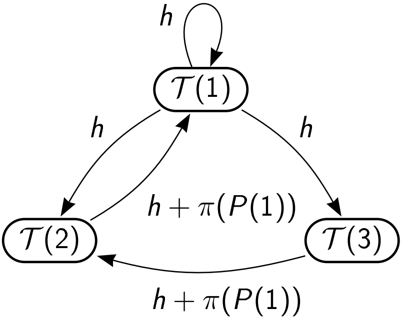
Theorem

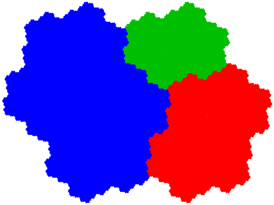
There is a unique list of non-empty compact subsets $(K_u, u \in V)$ s.t., for all $u \in V$, $K_u \subseteq X_u$ and

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} \phi_e(K_v).$$

The list $(K_u, u \in V)$ is called the attractor of the GDIFS.

Example (The Rauzy fractal is the attractor of a GDIFS)





Linking Büchi automata, β-self-similarity and GDIFS

Theorem (C-Leroy-Rigo 2015)

If β is Pisot then, for any compact $X \subseteq \left[0, \frac{\lceil\beta\rceil-1}{\beta-1}\right]^d$, the f.a.a.e:

1. *There is a Büchi automaton $\mathcal{A}$ over the alphabet A_β^d s.t.* $\mathrm{val}_\beta(\mathbf{0} \star L(\mathcal{A})) = X$.
2. *X belongs to the attractor of a GDIFS on $\mathbb{R}^d$ whose contraction maps are of the form $\mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{t}}{\beta}$ with $\mathbf{t} \in A_\beta^d$.*
3. *X is β-self-similar.*

An analogue of Cobham's theorem for graph-directed iterated function systems [Charlier-Leroy-Rigo 2015]

A Cobham-like theorem for multidimensional b-self-similar sets

Corollary

Any b-self-similar subset of $[0,1]^d$ is weakly b-recognizable.

Corollary (simultaneously obtained by Chan-Hare 2014)

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$. A compact subset of $[0,1]^d$ is simultaneously b-self-similar and b'-self-similar iff it is a finite union of rational polyhedra.

A Cobham-like theorem for GDIFS

Corollary

Let $b, b' \geq 2$ be integers s.t. $\frac{\log b}{\log b'} \notin \mathbb{Q}$.
A compact subset of $\mathbb{R}^d$ is the attractor of two GDIFS,
one with contraction maps of the form $\mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{t}}{b}$ with $\mathbf{t} \in {A_b}^d$ and
the other with contraction maps of the form $\mathbf{x} \mapsto \frac{\mathbf{x}+\mathbf{t}}{b'}$ with $\mathbf{t} \in {A_{b'}}^d$,
iff it is a finite union of rational polyhedra.

References

- An analogue of Cobham's theorem for fractals [Adamczewski-Bell 2011]
- On the sets of real numbers recognized by finite automata in multiple bases [Boigelot-Brusten-Bruyère 2010]
- First-order logic and Numeration Systems [Charlier 2017]
- An analogue of Cobham's theorem for graph-directed iterated function systems [Charlier-Leroy-Rigo 2015]
- On the base-dependance of sets of numbers recognizable by finite automata [Cobham 1969]
- On the structures of generating iterated function systems of Cantor sets [Feng-Wang 2009]
- Fractals and self-similarity [Hutchinson 1981]