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COSHEAVES HOMOLOGY

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§ 0 Introduction

The theory of cosheaves is naively the theory of sheaves with values in the opposite category of the category of abelian groups. However, since colimits are not exact in Ab^{op} it is well known that such cosheaves do not possess the good properties of classical sheaves. In particular, one cannot associate a cosheaf to each precosheaf, there is no inverse image functor and one cannot derive the direct image functor. The aim of this paper is to correct all these bad things by using a more general kind of cosheaves. In fact, we choose a pair of universes U,V such that $U \in V$ and we study the categories of cosheaves with values in $Pro_V\text{-}Ab_U$. We show that these categories have most of the classical properties of the categories of abelian sheaves. Let us now describe briefly the contents of the different paragraphs.

In paragraph 1, we first review in details the elementary categories (see DEFINITION 1.1). Then we prove that Ind-$\mathcal{A}$ is elementary when $\mathcal{A}$ is a U-small abelian category with enough projective objects. Finally we consider sheaves with values in an elementary category.

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In the second paragraph, we study the category Pro_V-Ab_U and construct intersting functors connecting this category and the category Ab_V.

Paragraph 3 is devoted to the homology of cosheaves. We define the homology and hyperhomology functors, we prove a kind of duality theorem between homology of cosheaves and cohomology of sheaves and we extend the basic property of Φ-taut subspaces to cosheaves. We construct also a kind of universal KRONECKER product and a kind of universal cap-product. The left and right adjoints of these functors are obtained. We conclude this paragraph by creating the functor $\overline{P}^{\bullet}$. We shall investigate the relation of this functor with POINCARE and VERDIER dualities in a subsequent paper.

The fourth paragraph is rather brief. We just relate our homology of the constant cosheaf $\mathbb{Z}^X$ with the singular homology with integer coefficients and with the Borel-Moore homology with compact supports of the constant sheaf $\mathbb{Z}_X$.

In this paper, unless they clearly refer to open subsets, the letters U,V denote two universes such that $U \in V$.

§1 Sheaves witch values in an elementary category.

Let us first recall two definitions

A *small* object of an abelian U-category $\mathcal{A}$ is an object P such that the functor h_P is coproduct preserving. Explicitly this means that we have $\mathrm{Hom}_{\mathcal{A}}(P, \bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P, A_i)$ for every family $(A_i)_{i\in I}$ of objects of $\mathcal{A}$ such that I belongs to U.

A *generating family* of an abelian U-category $\mathcal{A}$ is a family $(A_i)_{i\in I}$ of objects of $\mathcal{A}$ such that for every non zero arrow $f: A \to B$ of $\mathcal{A}$ there exist an element i of I and an arrow $g: A_i \to A$ such that $f \circ g \neq 0$.

Now we are able to give the following definition

DEFINITION 1.1 An *elementary* U-category is an abelian U-category wich is complete, cocomplete and wich has a U-small generating family of U-small projective objects.

It is very easy to work with an elementary category because for each object A we can use arrows from small projective objects to A essentially in the same way we use elements in the category of abelian U-groups.

In fact we have the following propositions

PROPOSITION 1.2

a) An elementary category has a projective generator and thus enough projective objects.

b) If $\mathcal{A}$ is elementary then the sequence:

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

is exact if and only if the sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(P, A) \xrightarrow{\mathrm{Hom}(P,u)} \mathrm{Hom}_{\mathcal{A}}(P, B) \xrightarrow{\mathrm{Hom}(P,v)} \mathrm{Hom}_{\mathcal{A}}(P, C) \longrightarrow 0$$

is exact for each P belonging to a U-small generating family of

U-small projective objects of A.

c) If P is a small projective of the abelian category A then $\mathrm{Hom}_A(P,?)$ is colimit preserving.

Proof:

a) Let A be an elementary category and $(P_i)_{i\in I}$ be a U-small generating family of U-small projective objects. Since A is cocomplete the object $P=\bigoplus_{i\in I} P_i$ exists and since

$$\mathrm{Hom}\ (\bigoplus_{i\in I} P_i,A)=\prod_{i\in I}\mathrm{Hom}\ (P_i,A)$$

this object is a projective generator . To conclude,we just have to note that the canonical map $\bigoplus_{f\in\mathrm{Hom}(P,A)} P\to A$ expresses A as a quotient of the projective object $\bigoplus_{f\in\mathrm{Hom}(P,A)} P$

b) Let P denote a U-small generating family of U-small projective objects of A. Since each object of P is projective,it is clear that the sequence

$$0\longrightarrow \mathrm{Hom}(P,A)\xrightarrow{\mathrm{Hom}(P,u)}\mathrm{Hom}(P,B)\xrightarrow{\mathrm{Hom}(P,v)}\mathrm{Hom}(P,C)\longrightarrow 0$$

is exact when $P\in P$. So the "only if" part of the proposition is trivial.

To prove the "if" part, consider a sequence:

$$0\longrightarrow A\xrightarrow{u} B\xrightarrow{v} C\longrightarrow 0$$

and suppose that the sequence:

$$0\longrightarrow \mathrm{Hom}(P,A)\xrightarrow{\mathrm{Hom}(P,u)}\mathrm{Hom}(P,B)\xrightarrow{\mathrm{Hom}(P,v)}\mathrm{Hom}(P,C)\longrightarrow 0$$

is exact for each P belonging to P.

Let D be an arbitrary object of A and P.(D) a resolution of D by direct sums of objects of P (cf. Proposition 1.2).

Clearly we have the following exact sequence of complexes:

$$0\to \mathrm{Hom}(P.(D),A)\to \mathrm{Hom}(P.(D),B)\to \mathrm{Hom}(P.(D),C)\to 0$$

From this sequence we deduce the following long exact sequence:

$$0 \to \mathrm{Hom}(D,A) \to \mathrm{Hom}(D,B) \to \mathrm{Hom}(D,C) \to \mathrm{Ext}^1(D,A) \to \mathrm{Ext}^1(D,B) \to \mathrm{Ext}^1(D,C) \ldots$$

This establishes the exactness of the sequence

$$0 \longrightarrow \mathrm{Hom}(D,A) \longrightarrow \mathrm{Hom}(D,B) \longrightarrow \mathrm{Hom}(D,C)$$

for each D in $Ob(\mathcal{A})$ and thus the exactness of the sequence

$$0 \to A \xrightarrow{u} B \xrightarrow{v} C$$

To show that v is an epimorphism consider an arrow $w:C \to E$ such that $w \circ v = 0$. For each P belonging to $\mathcal{P}$ and each arrow $f:P \to C$ there exists an arrow $g:P \to B$ such that $v \circ g = f$. It follows easily that $w \circ f = 0$. Using the fact that $\mathcal{P}$ is a generating family we deduce that $w=0$ and the proof is complete.

c) Let $F: \mathcal{B} \longrightarrow \mathcal{A}$ be a functor from a small category to the abelian category $\mathcal{A}$. By a classical result of the theory of colimits, we know that the following sequence is exact:

$$\bigoplus_{f \in Ar(\mathcal{B})} F(\mathrm{dom}(f)) \to \bigoplus_{A \in Ob(\mathcal{B})} F(A) \to \mathrm{Colim}\, F \longrightarrow 0 \qquad (1)$$

Thus if P is a small projective object of $\mathcal{A}$ it follows that the following sequence is exact:

$$\bigoplus_{f \in Ar(\mathcal{B})} \mathrm{Hom}(P,F(\mathrm{dom}(f))) \to \bigoplus_{A \in Ob(\mathcal{B})} \mathrm{Hom}(P,F(A)) \to \mathrm{Hom}(P,\mathrm{Colim}\, F) \to 0$$

Using (1) for the functor Hom(P,F(?)) shows that:

$$\mathrm{Colim}\, \mathrm{Hom}(P,F(?)) = \mathrm{Hom}(P,\mathrm{Colim}\, F). ///$$

PROPOSITION 1.3

a) In an elementary category the small filtering colimits are exact.

b) An elementary category has enough injective objects.

Proof:

a) Let $0 \to F \to G \to H \to 0$ be an exact sequence of functors from the U-small filtering category $\mathcal{B}$ to the elementary category $\mathcal{A}$. It follows from Proposition 1.2.b that:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P,F(?)) \to \mathrm{Hom}_{\mathcal{A}}(P,G(?)) \to \mathrm{Hom}_{\mathcal{A}}(P,H(?)) \to 0$$

is an exact sequence of functors from $\mathcal{B}$ to Ab_U for every small projective object P of $\mathcal{A}$. This implies that the sequence:

$$0 \to \mathrm{Colim}\ \mathrm{Hom}_{\mathcal{A}}(P,F(?)) \to \mathrm{Colim}\ \mathrm{Hom}_{\mathcal{A}}(P,G(?)) \to \mathrm{Colim}\ \mathrm{Hom}_{\mathcal{A}}(P,H(?)) \to 0$$

is exact for all P belonging to a U-small generating family of U-small projective objects of $\mathcal{A}$. To conclude we just have to use the parts b and c of Proposition 1.2.

b) We know that an elementary category $\mathcal{A}$ has a generator and exact U-small filtering colimits, so that we may use theorem 1.10.1 proved by GROTHENDIECK in (5,p.13) and this proves that $\mathcal{A}$ has enough injective objects.///

The structure of elementary category is given by the following theorem due to FREYD.

THEOREM 1.4.
An abelian category is elementary if and only if it is equivalent to the category of additive functors from a U-small additive category to Ab_U.

Proof:

See (9,p.106) for details.///

Of course, the category $A\text{-}Mod_U$ of left A-modules belonging to U is an example of elementary U-category (A is a projective generator) but the following result is more interesting.

PROPOSITION 1.5.

If $\mathcal{A}$ is a U-small abelian category with enough projective objects then Ind-$\mathcal{A}$ is an elementary U-category.
A U-small generating family of U-small projective objects of Ind-$\mathcal{A}$ is given by the family $(\mathrm{Hom}_{\mathcal{A}}(?,P))_{P\in\mathcal{P}}$ where $\mathcal{P}$ is the U-set of projective objets of $\mathcal{A}$.

Proof:

Let us denote as usual the functor Hom (?,A) by h^A. Clearly Ind-$\mathcal{A}$ is abelian, complete and cocomplete. Let us prove that h^P is a projective object of Ind-$\mathcal{A}$ when $P\in\mathcal{P}$. Let

$$0 \longrightarrow F \xrightarrow{u} G \xrightarrow{v} H \longrightarrow 0$$

be an exact sequence of Ind-$\mathcal{A}$. By a classical result ([11], [1]) we know that there exist a directed U-set I, direct systems

$$(A_i, f_{ji})_{i\in I}, (B_i, g_{ji})_{i\in I}, (C_i, h_{ji})_{i\in I}$$

of objects of $\mathcal{A}$ and morphims:

$$(u_i)_{i\in I} : (A_i, f_{ji})_{i\in I} \longrightarrow (B_i, g_{ji})_{i\in I}$$

$$(v_i)_{i\in I} : (B_i, g_{ji})_{i\in I} \longrightarrow (C_i, h_{ji})_{i\in I}$$

such that we have the following commutative diagram in Ind-$\mathcal{A}$:

$$\begin{array}{ccccc}
F & \xrightarrow{u} & G & \xrightarrow{v} & H \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr \\
\operatorname{colim}_{i\in I} h^{A_i} & \xrightarrow{\operatorname{colim}_{i\in I} h^{u_i}} & \operatorname{colim}_{i\in I} h^{B_i} & \xrightarrow{\operatorname{colim}_{i\in I} h^{v_i}} & \operatorname{colim}_{i\in I} h^{C_i}
\end{array}$$

and such that the sequence:

$$0 \longrightarrow A_i \xrightarrow{u_i} B_i \xrightarrow{v_i} C_i \longrightarrow 0 \qquad (2)$$

is exact for every i belonging to I. We deduce from (2) that the sequences:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P,A_i) \xrightarrow{\mathrm{Hom}(P,u_i)} \mathrm{Hom}_{\mathcal{A}}(P,B_i) \xrightarrow{\mathrm{Hom}(P,v_i)} \mathrm{Hom}_{\mathcal{A}}(P,C_i) \longrightarrow 0$$

are exact. By consequence, we have the exact sequence:

$$0 \to \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P,A_i) \to \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P,B_i) \to \operatorname*{colim}_{i\in I} \mathrm{Hom}_{\mathcal{A}}(P,C_i) \to 0 \quad (3)$$

Using (3) and a well known property of Ind-$\mathcal{A}$ we get the following exact sequence:

$$0 \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P, \operatorname*{colim}_{i\in I} h^{A_i}) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P, \operatorname*{colim}_{i\in I} h^{B_i}) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P, \operatorname*{colim}_{i\in I} h^{C_i}) \to 0$$

Using (1) we see that the sequence:

$$0 \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P,F) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P,G) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P,H) \to 0$$

is exact and this shows that h^P is a projective object of Ind-$\mathcal{A}$. Now we prove that h^P is small. Let I be a U-set and $(F_i)_{i\in I}$ a family of Ind-$\mathcal{A}$ indexed by I.

Consider the U-set $\mathcal{F} = \{J : J\subset I, J \text{ finite}\}$ ordered by inclusion. Clearly, the direct system $((\bigoplus_{i\in J} F_i), s_{KJ})_{J\in\mathcal{F}}$ of objects of Ind-$\mathcal{A}$ has a colimit isomorphic to $\bigoplus_{i\in I} F_i$. By a well known property of Ind-$\mathcal{A}$ we see that

$$\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P, \bigoplus_{i\in I} F_i) \simeq \operatorname*{Colim}_{J\in\mathcal{F}} \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{A}}(h^P, \bigoplus_{i\in J} F_i)$$

In an abelian category such as Ind-$\mathcal{A}$, finite direct sums and finite direct products coincide so we have:

$$\operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^P, \bigoplus_{i\in I} F_i) \simeq \operatorname*{Colim}_{J\in\mathcal{F}} \bigoplus_{i\in J} \operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^P, F_i)$$

$$\simeq \bigoplus_{i\ I} \operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^P, F_i)$$

This last isomorphisme shows that h^P is a small object of Ind-$\mathcal{A}$. Finally we establish that $(h^P)_{P\in\mathcal{P}}$ is a generating family of Ind-$\mathcal{A}$. Let us denote by I the canonical functor from Ind-$\mathcal{A}$ to $Ab^{\mathcal{A}^{op}}$. We know that

$$\operatorname{Hom}_{\text{Ind-}\mathcal{A}}(F,G) = \operatorname{Hom}_{Ab^{\mathcal{A}^{op}}}(I(F), I(G))$$

thus two morphism u,v from F to G are equal if and only if

$$u_A : F(A) \longrightarrow G(A) \quad \text{and} \quad v_A : F(A) \longrightarrow G(A)$$

are equal for every A belonging to $\mathcal{O}b(\mathcal{A})$. This proves that u,v are equal if and only if $A\in\mathcal{O}b(\mathcal{A})$ implies that the morphisms

$$\operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^A, u) \quad \text{and} \quad \operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^A, v)$$

are equal. Now, let A be an object of $\mathcal{A}$. Choose an epimorphism $P \twoheadrightarrow A$ with P in $\mathcal{P}$; this gives us an epimorphism $h^P \twoheadrightarrow h^A$ and the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^A, F) & \longrightarrow & \operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^P, F) \\
\operatorname{Hom}(h^A,u)\downarrow \ \downarrow\operatorname{Hom}(h^A,v) & & \operatorname{Hom}(h^P,u)\downarrow \ \downarrow\operatorname{Hom}(h^P,v) \\
\operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^A, G) & \longrightarrow & \operatorname{Hom}_{\text{Ind-}\mathcal{A}}(h^P, G)
\end{array}$$

where u,v are two morphism from F to G.

On the preceding diagram we read that

$$\mathrm{Hom}(h^A,u) \quad \text{and} \quad \mathrm{Hom}(h^A,v)$$

are equal if

$$\mathrm{Hom}(h^P,u) \quad \text{and} \quad \mathrm{Hom}(h^P,v)$$

are equal and combining this with the preceding remarks, we see that u and v are equal if and only if

$$\mathrm{Hom}(h^P,u) \quad \text{and} \quad \mathrm{Hom}(h^P,v)$$

for every P belonging to $\mathcal{P}$.
This proves that $(h^P)_{P\in\mathcal{P}}$ is a generating family of Ind-$\mathcal{A}$.///

The main reason which justifies our interest in elementary categories besides the fact that they support a good homological algebra is that the sheaves with values in such a category are as easy to handle as the abelian sheaves. One can easily extend the results of the theory of abelian sheaves to the theory of sheaves with values in an elementary category by using the same procedure as at the beginning of this paragraphe or by using THEOREM 1.4 to reduce everything to the classical case. We shall just show how the process works in the following example:

PROPOSITION 1.6

If $\underline{F},\underline{G},\underline{H}$ *are sheaves on* X *with values in an elementary* U-*category* $\mathcal{A}$ *if* $\underline{F}$ *is flabby and if the sequence*

$$0 \longrightarrow \underline{F} \longrightarrow \underline{G} \rightarrow \underline{H} \rightarrow 0$$

is exact then the sequence

$$0 \longrightarrow \underline{F}(X) \longrightarrow \underline{G}(X) \longrightarrow \underline{H}(X) \longrightarrow 0$$

is exact.

Proof:

Let $\mathcal{P}$ be a U-small generating family of small projective objects of $\mathcal{A}$. For every P belonging to $\mathcal{P}$ the functors

$$\text{Hom}(P,\underline{F}(?)),\ \text{Hom}(P,\underline{G}(?)),\ \text{Hom}(P\underline{H}(?))$$

are abelian sheaves on X since Hom(P,?) is limit preserving. The sheaf $\text{Hom}(P,\underline{F}(?))$ is flabby since P is projective.

The sequence

$$0 \rightarrow \text{Hom}(P,\underline{F}(?)) \rightarrow \text{Hom}(P,\underline{G}(?)) \rightarrow \text{Hom}(P,\underline{H}(?)) \rightarrow 0$$

is an exact sequence of abelian sheaves since Hom(P,?) is colimit preserving. Using a result of classical sheaf theory we deduce that the sequence:

$$0 \rightarrow \text{Hom}(P,\underline{F}(X)) \rightarrow \text{Hom}(P,\underline{G}(X)) \rightarrow \text{Hom}(P,\underline{H}(X)) \rightarrow 0$$

is exact. To conclude the proof,we just have to use part b of Proposition 1.2.///

§ 2 Canonical functors associated with Pro_V-Ab_U.

In this paragraph we shall use freely some results on Ind and Pro categories proved for example in (1), (6), (7), (11) and (13).
First we construct the mixed tensor product.

PROPOSITION 2.1

There exists a functor $\otimes: Ab_V \times Pro_V\text{-}Ab_U \to Pro_V\text{-}Ab_U$ *such that the functor* $?\otimes B$ *is a canonical left adjoint of* $Hom_{Pro_V\text{-}Ab_U}(B,?)$ *for every object* B *of* Pro_V-Ab_U . *If* $B = \lim\limits_{i\in I} B_i$ *with* $B_i \in Ob(Ab_U)$, I *directed* V-set *and* $A = \operatorname{colim}\limits_{j\in J} A_j$ *with* $A_j \in Ob(FAb_U)$ J *directed* V-set *then one has* $A\otimes B = \operatorname{colim}\limits_{j\in J} \lim\limits_{i\in I} I(A_j\otimes B_i)$ *where* I *denotes the canonical functor from* Ab_V *to* Pro_V-Ab_U.

Proof:

Let A,B,C be a finitely generated abelian U-groups.
It is clear that the functor $I(A\otimes ?): Ab_U \to Pro_V\text{-}Ab_U$ has a limit preserving extension to Pro_V-Ab_U which is unique up to a natural isomorphism . Denote temporarily this extention by F_A, every arrow $f: A \to B$ in Ab_U gives rise to a natural transformation from $I(A\otimes ?)$ to $I(B\otimes ?)$ and thus to a natural transformation F_f from F_A to F_B. Moreover, it is easily seen that $F_g \circ F_f = F_{g\circ f}$ if $G: B \to C$ is an arrow in Ab_U. Now, if we choose an object D of Pro_V-Ab_U, we get a functor:

$$F_? (D) : FAb_U \to Pro_V - Ab_U$$

where FAb_U stands for the category of all finitely generated abelian U-groups.
This functor extends to a "unique" colimit preserving functor:

$$G_D : Ind_V - FAb_U \to Pro_V - Ab_U$$

But $Ind_V\text{-}FAb_U$ is canonically equivalent to Ab_V thus we have constructed a functor $G'_D : Ab_V \longrightarrow Pro_V\text{-}Ab_U$. We can also associate canonically to each arrow $f : D \longrightarrow E$ in $Pro_V\text{-}Ab_U$ an arrow $G'_f : G'_D \longrightarrow G'_E$.

Moreover if $g : E \longrightarrow H$ is a second arrow $Pro_V\text{-}Ab_U$ we have $G'_g \circ G'_f = G'_{g \circ f}$. The preceding construction gives rise to a functor $\otimes : Ab_U \times Pro_V\text{-}Ab_U \longrightarrow Pro_V\text{-}Ab_U$ such that $A \otimes D = G'_D(A)$ if $A \in \mathcal{O}b(Ab_V)$, $D \in \mathcal{O}b(Pro_V\text{-}Ab_U)$. By construction, we have

$$(\operatorname*{colim}_{j\in J} A_j) \otimes (\lim_{i\in I} D_i) = \operatorname*{colim}_{j\in J} \lim_{i\in I} I(A_j \otimes D_i)$$

if $A_j \in \mathcal{O}b(FAb_U)$, $D_i \in \mathcal{O}b(Ab_U)$. This shows that

$$A \otimes I(D) = I(A \otimes D)$$

if $A, D \in \mathcal{O}b(Ab_U)$. So it is not dangerous to use the same notation for our functor as for the classical tensor product.

Now let A be an abelian V-group and B, C objects of $Pro_V\text{-}Ab_U$. We can find one directed V-direct system $(A_i, f_{ji})_{i\in I}$ of FAb_U and two directed V-inverse system $(B_k, g_{lk})_{k\in K}$, $(C_m, h_{nm})_{m\in M}$ of Ab_U such that $A = \operatorname*{colim}_{i\in I} A_i$, $B = \lim_{k\in K} I(B_k)$, $C = \lim_{m\in M} I(C_m)$.

We have successively

$$\begin{aligned}
&Hom_{Ab_V}(A, Hom_{Pro_V\text{-}Ab_U}(B,C)) \\
&\simeq Hom_{Ab_V}(\operatorname*{colim}_{i\in I} A_i, Hom_{Pro_V\text{-}Ab_U}(\lim_{k\in K} I(B_k), \lim_{m\in M} I(C_m))) \\
&\simeq \lim_{i\in I} \lim_{m\in M} Hom_{Ab_V}(A_i, \operatorname*{colim}_{k\in K} Hom_{Ab_U}(B_k, C_m)) \\
&\simeq \lim_{i\in I} \lim_{m\in M} \operatorname*{colim}_{k\in K} Hom_{Ab_U}(A_i \otimes B_k, C_m) \\
&\simeq \lim_{i\in I} \lim_{m\in M} \operatorname*{colim}_{k\in K} Hom_{Pro_V\text{-}Ab_U}(I(A_i \otimes B_k), I(C_m)) \\
&\simeq Hom_{Pro_V\text{-}Ab_U}(\operatorname*{colim}_{i\in I} \lim_{k\in K} I(A_i \otimes B_k), \lim_{m\in M} I(C_m)) \\
&\simeq Hom_{Pro_V\text{-}Ab_U}(A \otimes B, C).
\end{aligned}$$

One sees easily that the isomorphism resulting from the preceding chain of isomorphisms is natural in A,B,C and this conclude the proof.///

DEFINITION 2.2

The *mixed tensor product* is the functor $\otimes : Ab_V \times Pro_V\text{-}Ab_U \longrightarrow Pro_V\text{-}Ab_U$ given by the preceding proposition. It results from the preceding construction that $A\otimes ?$ and $?\otimes B$ are additive functors.
Now, we construct a right adjoint to $A\otimes$?

PROPOSITION 2.3.

There exist a functor $D : Ab_V^{op} \times Pro_V\text{-}Ab_U \longrightarrow Pro_V\text{-}Ab_U$ *such that* $D(A,?)$ *is a canonical right adjoint of* $A\otimes ?$ *for every object* A *of* Ab_V. *Moreover, one has*

$$D(A,B) = \lim_{i\in I} \lim_{j\in J} I(\mathrm{Hom}_{Ab_U}(A_i, B_j))$$

if $A = \operatorname{colim}_{i\in I} A_i$ *with* $A_i \in Ob(Fab_U)$ *and* $B = \lim_{j\in J} I(B_j)$ *with* $B_j \in Ob(Ab_U)$ I, J *being* V-*sets*.

Proof:

One just has to work as in the preceding proof.///

§ 3 Homology of sheaves

The cosheaves we study in this paragraph are in fact cosheaves with values in $Pro_V\text{-}Ab_U$ which are defined on a V-topological space. Associated to each V-topological space there is a category of cosheaves, denoted by Cosh(X), and which is precisely defined by

$$Cosh(X) = Shv^{op}(X,(Pro_V\text{-}Ab_U)^{op}) = Shv^{op}(X,Ind\text{-}Ab_U^{op})$$

Since Ab_U has enough injective objects it is clear that Ab_U^{op} is a V-small category with enough projective objects thus $Ind_V\text{-}Ab_U^{op}$ is an elementary category and the categories $Cosh^{op}(X)$ have most of the usual properties of the categories of abelian sheaves. In particular:

a) For every continuous map $f:X \to Y$ we have the two functors f_*, f^* which satisfy the following relations:

$$f_*(\overline{F})(U) = \overline{F}(f^{-1}(U))$$

for every $\overline{F} \in Ob(Cosh(X))$, U open in Y and

$$Hom_{Cosh(Y)}(f_*\overline{F},\overline{G}) = Hom_{Cosh(X)}(\overline{F},f^*\overline{g})$$

for every $\overline{F} \in Ob(Cosh(X)), \overline{G} \in Ob(Cosh(Y))$.

b) We note Γ^{Φ} the functor which corresponds to the classical Γ_{Φ} when Φ is a family of supports. We have:

$$\Gamma^F(X,\overline{F}) = \overline{F}(X)/\mathrm{im}\ i^{\overline{F}}_{X,X\setminus F}$$

where $\overline{F}$ is a cosheaf on X, F is a closed subset of X and $i^{\overline{F}}_{X,X\setminus F}$ is the canonical map from $\overline{F}(X\setminus F)$ to $\overline{F}(X)$ corresponding to the classical restriction map. Moreover, one has:

$$\Gamma^{\Phi}(X,\overline{F}) = \lim_{F\in\Phi} \Gamma^F(X;\overline{F})$$

for every cosheaf $\overline{F}$ on X and every family of supports Φ on X.

c) Since the category of cosheaves on X has enough projective objects one can construct the left derived functors:

$$\mathbb{L}f_* : D^-(Cosh(X)) \to D^-(Cosh(Y))$$
$$\mathbb{L}\Gamma^{\Phi} : D^-(Cosh(X)) \to D^-(Cosh(X))$$

One sets

$$\mathbb{L}_i f_* = H^{-i}\mathbb{L}f_*$$

and

$$\mathbb{H}_i^{\Phi}(X;?) = H^{-i}\mathbb{L}\Gamma^{\Phi}$$

these are the hyperhomology functors.

Moreover if $\overline{F}$ is a cosheaf on X one sets

$$H_i^{\Phi}(X;\overline{F}) = \mathbb{H}_i^{\Phi}(X;Q\overline{F}));$$

H_i^{Φ} (X;?) is the ordinary homology functor.

The following construction is very useful :

DEFINITION 3.1.

Let $\overline{F}$ be a cosheaf on X and G be an object of $Pro_V\text{-}Ab_U$. Since the functor $Hom_{Pro_V\text{-}Ab_U}(?,G):Pro_V\text{-}Ab_U^{op} \longrightarrow Ab_V$ is limit preserving we know that the functor:

$$Hom_{Pro_V\text{-}Ab_U}(\overline{F}(?),G):Open(X)^{op} \longrightarrow Ab_V$$

is an abelian sheaf on X. We denote this sheaf by $\underline{D}(\overline{F},G)$; it is the G-dual of the cosheaf $\overline{F}$. Clearly $\underline{D}(?,G):Cosh^{op}(X) \longrightarrow Shv(X,Ab_V)$ and $\underline{D}(\overline{F},?):Pro_V\text{-}Ab_U \longrightarrow Shv(X,Ab_V)$ are two left exact functors. One extends this definition to complexes of cosheaves as usual: for each complex of cosheaves $\overline{F}^{\cdot}$ an each object $G^{\cdot}$ of $C(Pro_V\text{-}Ab_U)$ we define $\underline{D}^{\cdot}(\overline{F}^{\cdot},G^{\cdot})$ by

$$\underline{D}^p(\overline{F}^{\cdot},G^{\cdot})(U) = \prod_{q\in Z} Hom(\overline{F}^p(U),G^{p+q}).$$

The basic properties of $\underline{D}$ are given in the following proposition:

PROPOSITION 3.2.

a) If $\overline{F}$ *is a cosheaf on* X *and* G *an essentially constant object of* $Pro_V\text{-}Ab_U$ *then*

$$\Gamma_\Phi \underline{D}(\overline{F},G) = \underline{D}(\Gamma^\Phi(\overline{F}),G)$$

for every family of supports Φ *on* X.

b) If $\overline{F}$ *is a cosheaf on* X *and if* G *belongs to* $Ob(Pro_V\text{-}Ab_U)$ *then*

$$f_*\underline{D}(\overline{F},G) = \underline{D}(f_*\overline{F},G)$$

for every continuous map $f:X \longrightarrow Y$.

c) If $\overline{F}$ *is a cosheaf on* Y *and* G *is an essentially constant object of* $Pro_V\text{-}Ab_U$ *then*

$$f^*\underline{D}(\overline{F},G) = \underline{D}(f^*\overline{F},G)$$

for every continuous map $f:X \longrightarrow Y$.

Proof:

a) Since G is essentially constant there exists an U-abelian group H with $G \simeq I(H)$. By definition we have:

$$\Gamma_\Phi\ \underline{D}(\overline{F},G) = \operatorname*{colim}_{F\in\Phi}\ \Gamma_F\underline{D}(\overline{F},I(H))$$

We know that the sequences:

$$0 \longrightarrow \Gamma_F\underline{D}(\overline{F},I(H)) \longrightarrow Hom(\overline{F}(X),I(H)) \longrightarrow Hom(\overline{F}(X\setminus F),I(H))$$

$$0 \longleftarrow \Gamma^F(\overline{F}) \longleftarrow \overline{F}(X) \longleftarrow \overline{F}(X\setminus F)$$

are exact and from this it follows that:

$$Hom(\Gamma^F(\overline{F}),I(H)) \simeq \Gamma_F\ \underline{D}(\overline{F},I(H))$$

Since I(H) is cosmall in $Pro_V\text{-}Ab_U$, we have:

$$\operatorname*{colim}_{F\in\Phi}\ \Gamma_F\underline{D}(\overline{F},I(H)) = Hom(\lim_{F\in\Phi}\ \Gamma^F(\overline{F}),I(H))$$

and this concludes the proof of part (a) since:

$$\Gamma^\Phi(\overline{F}) = \lim_{F\in\Phi}\ \Gamma^F(\overline{F})$$

b) is trivial.

c) As in a) choose an object H of Ab_U such that $G \simeq I(H)$.

From the canonical morphism:

$$f_* f^* \overline{F} \longrightarrow \overline{F}$$

we get a canonical morphism:

$$\underline{D}(\overline{F},G) \longrightarrow \underline{D}(f_* f^* \overline{F},G)$$

Using b) and the basic property of f_* , we get the following canonical morphism:

$$f^* \underline{D}(\overline{F},G) \to \underline{D}(f^* \overline{F},G)$$

To prove that it is an isomorphism we just have to prove it is an isomorphism at the level of fibers. We have the following canonical commutative diagram:

$$\begin{array}{ccc} (f^*(\underline{D}(\overline{F},G)))_x & \longrightarrow & \underline{D}(f^*\overline{F},G)_x \\ \downarrow \wr & & \downarrow \\ \underline{D}(\overline{F},G)_{f(x)} & \longrightarrow & \mathrm{Hom}(\overline{F}_{f(x)},G) \end{array}$$

To conclude, we just have to note that:

$$\underline{D}(\overline{G},G)_z = \mathrm{Hom}(\overline{G}_z ,G)$$

for every cosheaf $\overline{G}$ on a topological space Z since we have the following sequence of isomorphisms :

$$\begin{aligned} \underline{D}(\overline{G},G)_z &= \operatorname*{colim}_{U \ni z} \mathrm{Hom}(\overline{G}(U),G) \\ &= \mathrm{Hom}(\lim_{U \ni z} \overline{G}(U),G) \\ &= \mathrm{Hom}(G_z,G). /// \end{aligned}$$

The basic link between homology and cohomology is given by

PROPOSITION 3.3

If $\overline{F}$ *is an object of* $D^{-}(\mathrm{Cosh}(X))$ *and if* $G^{\cdot}$ *is an object of* $D^{+}(\mathrm{Pro}_V\text{-}\mathrm{Ab}_U)$ *with essentially constant cohomology objects then:*

a) $\mathbb{R}\Gamma_\Phi \mathbb{R}\underline{D}^{\cdot}(\overline{F},G^{\cdot}) = \mathbb{R}\,\mathrm{Hom}(\mathbb{L}\Gamma^{\Phi}(\overline{F}^{\cdot}),G^{\cdot})$

b) There exists a regular spectral sequence $(E_r)_{r\geq 2}$ *such that*

$$E_2^{p,q} = \bigoplus_{r\in\mathbb{Z}} \mathrm{Ext}^q\,(\mathbb{H}_r^{\Phi}(X,\overline{F}),H^{p-r}(G^{\cdot})) \underset{q}{\Longrightarrow} \mathbb{H}_{\Phi}^{p+q}(X,\mathbb{R}\underline{D}(\overline{F},G^{\cdot}))$$

Proof:

a) First we construct a morphism from $\mathbb{R}\Gamma_\Phi \mathbb{R}\underline{D}^{\cdot}(\overline{F},G^{\cdot})$ to $\mathbb{R}\mathrm{Hom}^{\cdot}(\mathbb{L}\Gamma^{\Phi}(\overline{F}^{\cdot},G^{\cdot}))$ as follow: we choose a flabby complex $\overline{R}^{\cdot}$ quasi-isomorphic to $\overline{F}^{\cdot}$ and a complex $J^{\cdot}$ which is isomorphic to $G^{\cdot}$ and which has injective components; it follows from these choices that:

$$\mathbb{R}\Gamma_\Phi\, \mathbb{R}\underline{D}^{\cdot}(\overline{F},G^{\cdot}) = \Gamma_\Phi \underline{D}^{\cdot}(\overline{R}^{\cdot},J^{\cdot})$$

and that:

$$\mathbb{R}\;\mathrm{Hom}^{\cdot}(\mathbb{L}\Gamma^{\Phi}(\overline{F}^{\cdot}),G^{\cdot}) = \mathrm{Hom}\;(\Gamma^{\Phi}(\overline{R}^{\cdot}),J^{\cdot})$$

now there is a canonical morphism

$$\operatorname*{colim}_{F\in\Phi}\;\mathrm{Hom}\;(\Gamma^{F}(\overline{R}^{\cdot}),\;J^{\cdot}) \rightarrow \mathrm{Hom}(\;\lim_{F\in\Phi}\;\Gamma^{F}(\overline{R}^{\cdot}),J^{\cdot}) \qquad (*)$$

so there is a canonical morphism

$$\Gamma_\Phi \underline{D}^{\cdot}(\overline{R}^{\cdot},J^{\cdot}) \longrightarrow \mathrm{Hom}(\Gamma^{\Phi}\,\overline{R}^{\cdot},J^{\cdot})$$

and this gives rise to a morphism

$$\mathbb{R}\Gamma_\Phi \mathbb{R}\underline{D}^{\cdot}(\overline{F}^{\cdot},G^{\cdot}) \longrightarrow \mathbb{R}\mathrm{Hom}^{\cdot}(\mathbb{L}\Gamma^{\Phi}\,(\overline{F}^{\cdot}),G^{\cdot})$$

which is easily seen to be independant of the choice of the complexes $R^{\cdot}$ and $J^{\cdot}$.

Using the "lemma on way out functors" of (6)we see that the morphism we have just constructed is an isomorphism if this is the case when $G^{\cdot}$ is reduced to a single essentially constant object of Pro_V-Ab_U. Such a $G^{\cdot}$ has a quasi-isomorphic complex of the form $I(T^{\cdot})$where $T^{\cdot}$ is a complex of injective objects of Ab_U and for such a complex (*) is clearly an isomorphism, so the proof of (a) is complete.

b) The result is deduced from (a) using a classical result of homological algebra, see (3 ,p.368).///

Taut subspaces have interesting homological properties as shown by the following:

PROPOSITION 3.4.

If A *is a* Φ-*taut subspace of the topological space* X *then*

$$H_p^{\Phi\cap A}(A,\overline{F}) = \lim_{\substack{U\supset A \\ U \text{ open}}} H_p^{\Phi\cap U}(U,\overline{F})$$

for every cosheaf $\overline{F}$ *on* X.

Proof:

Let $\overline{F}$ be a cosheaf on X. For every injective T of Ab_U Proposition 3.3 shows that:

$$H^p_{\Phi\cap S}(S,\underline{D}(\overline{F},I(T))|_S) = \mathrm{Hom}(H_p^{\Phi\cap S}(S,\overline{F}),I(T)) \qquad (*)$$

for every family of supports on X and every subset S of X.

By a basic property of Φ-tautness we know that:

$$H^p_{\Phi\cap A}(A,\underline{D}(\overline{F},I(T))) = \operatorname*{colim}_{\substack{U\supset A \\ U \text{ open}}} H^p_{\Phi\cap U}(U,\underline{D}(\overline{F},I(T)))$$

From (*) we deduce that

$$\mathrm{Hom}(H_p^{\Phi\cap A}(A,\overline{F}),I(T)) \simeq \operatorname*{colim}_{\substack{U\supset A \\ U \text{ open}}} \mathrm{Hom}(H_p^{\Phi\cap U}(U,\overline{F}),I(T))$$

Since I(T) is a cosmall object of Pro_V-Ab_U, one has the following isomorphism:

$$Hom(H_p^{\Phi\cap A}(A,\overline{F}),I(T)) \simeq Hom(\lim_{\substack{U\supset A\\ U\ \text{open}}} H_p^{\Phi\cap U}(U,\overline{F}),I(T))$$

A trivial verification shows that this isomorphism is induced by the canonical morphism

$$H_p^{\Phi\cap A}(A,\overline{F}) \longrightarrow \lim_{\substack{U\supset A\\ U\ \text{open}}} H_p^{\Phi}(U,\overline{F})$$

Since the injective object T is arbitrary, Proposition 1.2 shows tha the preceding morphism is an isomorphism.///

COROLLARY 3.5.

If the family Φ *is A-paracompactifying then the canonical morphism*

$$H_p^{\Phi\cap A}(A,\overline{F}) \simeq \lim_{\substack{U\supset A\\ U\ \text{open}}} H_p^{\Phi\cap U}(U,\overline{F})$$

is an isomorphism for every cosheaf $\overline{F}$ *on* X.

Proof: See (10).///

Using properties of tautness one gets:

PROPOSITION 3.6.

If A *is* Φ*-taut in* X *and if*

$$\overline{F}^{\cdot}\in Ob(D^{-}(Cosh(X)))$$

then one has the following canonical distinguished triangle:

$$\begin{array}{ccc} & \mathbb{L}\Gamma^{\Phi|X\setminus A}(\overline{F}^{\cdot}) & \\ {}^{+1}\swarrow & & \nwarrow \\ \mathbb{L}\Gamma^{\Phi\cap A}(\overline{F}^{\cdot}|_A) & \rightarrow & \mathbb{L}\Gamma^{\Phi}(\overline{F}^{\cdot}) \end{array}$$

Moreover, if A *is a closed subset of* X *then one has the canonical isomorphism:*

$$\mathbb{L}\Gamma^{\Phi|X\setminus A\cap X\setminus A}(\overline{F}^{\cdot}|_{X\setminus A}) \longrightarrow \mathbb{L}\Gamma^{\Phi|X\setminus A}(\overline{F}^{\cdot})$$

Proof:

Let $\overline{R}^{\cdot}$ be a flabby complex quasi-isomorphic to $\overline{F}^{\cdot}$. Using Proposition 3.4, it is easy to prove that the restriction to A of a flabby cosheaf is $\Gamma^{\Phi\cap A}$-acyclic and by consequence one has the following isomorphism:

$$\mathbb{L}\Gamma^{\Phi\cap A}(\overline{F}^{\cdot}_{|A}) \Rightarrow \Gamma^{\Phi\ A}(\overline{R}^{\cdot}_{|A})$$

For every flabby cosheaf $\overline{G}$ on X and every closed set F belonging to Φ and every open neigbourhood U of A, one has the following canonical exact sequence:

$$0 \longrightarrow \Gamma^{F\cap U}(\overline{G}_{|U}) \longrightarrow \Gamma^{F}(\overline{G}) \longrightarrow \Gamma^{F\cap(X\setminus U)}(\overline{G}) \longrightarrow 0$$

Taking the limit of these sequences for $F\in\Phi$ one gets the following exact sequence:

$$0 \longrightarrow \Gamma^{\Phi\cap U}(\overline{G}_{|U}) \longrightarrow \Gamma^{\Phi}(\overline{G}) \longrightarrow \Gamma^{\Phi\cap(X\setminus U)}(\overline{G}) \longrightarrow 0$$

Now, we take the limit of these sequences on the open neighbourhoods U of A in X and we use the Φ-tautness of A to obtain the following exact sequence:

$$0 \longrightarrow \Gamma^{\Phi\cap A}(\overline{G}_{|A}) \longrightarrow \Gamma^{\Phi}(\overline{G}) \longrightarrow \Gamma^{\Phi|(X\setminus A)}(\overline{G}) \longrightarrow 0$$

Using this result for each of the components of $\overline{R}$ we deduce the following exact sequence:

$$0 \longrightarrow \Gamma^{\Phi\cap A}(\overline{R}_{|A}) \longrightarrow \Gamma^{\Phi}(\overline{R}^{\cdot}) \longrightarrow \Gamma^{\Phi|(X\setminus A)}(\overline{R}^{\cdot}) \longrightarrow 0$$

Translating this fact in the language of derived categories gives the first part of the result and this concludes the proof since the second part of the result is a trivial application of the excision theorem.///

COROLLARY 3.6

If F is a Φ-taut closed subset of X and if $\overline{F}$ is a cosheaf then one has the following long exact sequence:

$$\ldots \longrightarrow H_p^{\Phi\cap F}(F;\overline{F}) \longrightarrow H_p^{\Phi}(X;\overline{F}) \longrightarrow H_p^{\Phi|X\setminus F}(X\setminus F;\overline{F}) \longrightarrow H_{p-1}^{\Phi\cap F}(F;\overline{F}) \longrightarrow H_{p-1}^{\Phi}(X;\overline{F}) \longrightarrow H_{p-1}^{\Phi|X\setminus F}(X\setminus F;\overline{F}) \longrightarrow \ldots$$

Proof: Trivial.///

In the following propositions, we study a kind of universal KRONECKER product.

PROPOSITION 3.7

For every cosheaf $\overline{F}$ on the V-topological space X, the functor:

$$\underline{D}(\overline{F},?):\mathrm{Pro}_V\text{-}\mathrm{Ab}_U \longrightarrow \mathrm{Shv}(X,\mathrm{Ab}_V)$$

has a left adjoint denoted by:

$$<?,\overline{F}>: \mathrm{Shv}(X,\mathrm{Ab}_V) \longrightarrow \mathrm{Pro}_V\text{-}\mathrm{Ab}_U$$

Proof:

Let $\underline{G}$ be an object of $\mathrm{Shv}(X,\mathrm{Ab}_V)$ and $\overline{F}$ a cosheaf on X. Consider the functor:

$$\underline{G}\otimes\overline{F} : \mathrm{Open}(X)^{op}\times\mathrm{Open}(X) \longrightarrow \mathrm{Pro}_V\text{-}\mathrm{Ab}_U$$

defined by $\underline{G}\otimes\overline{F}(U,V) = \underline{G}(U)\otimes\overline{F}(V)$. Since $\mathrm{Pro}_V\text{-}\mathrm{Ab}_U$ is cocomplete, we can form the coend of $\underline{G}\otimes\overline{F}$ (see [8]); this coend is denoted by $<\underline{G},\overline{F}>$. From this construction and from PROPOSITION 2.1 we deduce easily that:

$$\mathrm{Hom}_{\mathrm{Pro}_V\text{-}\mathrm{Ab}_U}(<\underline{G},\overline{F}>,H) \simeq \mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab}_V)}(\underline{G},\underline{D}(\overline{F},H))$$

Moreover the preceding isomorphism is natural in $\underline{G}$, $\overline{F}$ and H so that $<?,\overline{F}>$ is left adjoint to $\underline{D}(F,?)$.///

COROLLARY 3.8.

For every cosheaf $\overline{F}$ *on X the functor* $<?,\overline{F}>$ *is colimit preserving.*

Proof: Trivial.///

PROPOSITION 3.9.

For every object $\underline{F}$ *of* $Shv(X,Ab_V)$*, the functor*

$$<\underline{F},?> : Cosh(X) \to Pro_V\text{-}Ab_U$$

has a right adjoint denoted by:

$$\overline{D}(\underline{F},?) : Pro_V\text{-}Ab_U \longrightarrow Cosh(X)$$

Proof:

Let $\underline{F}$ be an object of $Shv(X,Ab_V)$ and let H be an object of $Pro_V\text{-}Ab_U$. We define $\overline{D}(\underline{F},H)$ to be the cosheaf associated to the precosheaf $D(\underline{F}(?),H)$ where D is the functor defined in Proposition 2.3. For every cosheaf $\overline{G}$ on X, we have:

$$Hom_{Cosh(X)}(\overline{G},\overline{D}(\underline{F},H)) \simeq Hom_{Pcosh(X)}(\overline{G},D(\underline{F}(?),H))$$

where Pcosh(X) denotes the category of precosheaves on X. From this formula and from PROPOSITION 2.3 we deduce easily that:

$$Hom_{Cosh(X)}(\overline{G},\overline{D}(\underline{F},H)) \simeq Hom_{Pro_V\text{-}Ab_U}(<\underline{F},\overline{G}>,H) .$$

Moreover this isomorphism is natural in $\overline{G},\underline{F}$ and H so that $\overline{D}(\underline{F},?)$ is a right adjoint of $<\underline{F},?>$.///

COROLLARY 3.10.

For every sheaf $\underline{F}$ *on X the functor* $<\underline{F},?>$ *is colimit preserving.*

Proof: Trivial.///

PROPOSITION 3.11

If $\underline{F}\in Ob(Shv(Y,Ab_V))$, *if* $\overline{G}\in Ob(Cosh(X))$ *and if* $f:X\to Y$ *is a continuous map then we have:*

$$<\underline{F},f_*\overline{G}> = <f^*\underline{F},\overline{G}>$$

Proof: By adjunction,we have successively:

$$Hom_{Pro_V-Ab_U}(<\underline{F},f_*\overline{G}>,?) = Hom_{Shv(Y,Ab_V)}(\underline{F},\underline{D}(f_*\overline{G},?))$$
$$= Hom_{Shv(Y,Ab_V)}(\underline{F},f_*\underline{D}(\overline{G},?))$$
$$= Hom_{Shv(X,Ab_V)}(f^*\underline{F},\underline{D}(\overline{G},?))$$
$$= Hom_{Pro_V-Ab_U}(<f^*\underline{F},\overline{G}>,?)$$

The conclusion follows easily from this chain of isomorphisms.///

In what follows, we study a kind of universal cap-product.

DEFINITION 3.12.

Let $\overline{F}$ and $\overline{G}$ be two cosheaves on X. We denote by $\underline{Hom}(\overline{F},\overline{G})$ the presheaf on X defined by $\underline{Hom}(\overline{F},\overline{G})(U)=Hom_{Cosh(U)}(\overline{F}|_U,\overline{G}|_U)$ A trivial verification shows that $\underline{Hom}(\overline{F},\overline{G})$ is in fact an object of $Shv(X,Ab_V)$.

PROPOSITION 3.13.

If $\overline{G}$ *is a cosheaf on* X *then the functor:*

$$\underline{Hom}(\overline{G},?) : Cosh(X) \longrightarrow Shv(X,Ab_V)$$

has a left adjoint denoted by:

$$?\cap\overline{G} : Shv(X,Ab_V) \longrightarrow Cosh(X)$$

Proof:

Let $\underline{F}$ be an object of $Shv(X,Ab_V)$. By Proposition 3.11 it is clear that

$$<\underline{F}_U,\overline{G}|_U> = <\underline{F},i_{U*}(\overline{G}|_U)>$$

where U is an open subset of X and i_U is the canonical injection from U to X. Since there is canonical morphism from $i_{U*}(\overline{G}_{|U})$ to $i_{V*}(\overline{G}_{|V})$ for every open subset V of X containing U, we have a canonical morphism from $<\underline{F}_{|U},\overline{G}_{|U}>$ to $<\underline{F}_V,\overline{G}_{|V}>$. Thus $<\underline{F}_{|?},G_{|?}>$ turns to be a precosheaf on X. We denote this precosheaf by $\underline{F}\cap\overline{G}$. In fact $\underline{F}\cap\overline{G}$ is a cosheaf since for every covering sieve $\mathcal{W}$ on the open set V of X one has successively:

$$
\begin{aligned}
<\underline{F}_{|V},\overline{G}_{|V}> &= <\underline{F},i_{V*}\overline{G}_{|V}> \\
&= <\underline{F},\operatorname*{colim}_{W\in\mathcal{W}} i_{W*}\overline{G}_{|W}> \\
&= \operatorname*{colim}_{W\in\mathcal{W}} <\underline{F},i_{W*}G_{|W}> \\
&= \operatorname*{colim}_{W\in\mathcal{W}} <\underline{F}_W,\overline{G}_{|W}>
\end{aligned}
$$

The proof of the fact that $?\cap\overline{G}$ is left adjoint to $\underline{\mathrm{Hom}}(\overline{G},?)$ is easy and left to the reader; one just has to play with definitions and to use Proposition 3.7.///

DEFINITION 3.14.

Let $\underline{F}$ be an object of $\mathrm{Shv}(X,\mathrm{Ab}_V)$ and $\overline{G}$ be a cosheaf on X. We define $\overline{\mathrm{Hom}}(\underline{F},\overline{G})$ to be the cosheaf on X associated to the precosheaf $D(\underline{F}(?),\overline{G}(?))$.

PROPOSITION 3.15.

For every object $\underline{F}$ *of* $\mathrm{Shv}(X,\mathrm{Ab}_V)$ *the functor*

$$\underline{F}\cap ? \; : \; \mathrm{Cosh}(X) \longrightarrow \mathrm{Cosh}(X)$$

has for right adjoint the functor:

$$\overline{\mathrm{Hom}}(\underline{F},?) \; : \; \mathrm{Cosh}(X) \longrightarrow \mathrm{Cosh}(X)$$

Proof: The proof is easy and left to the reader;one just has to play with definitions and to use Proposition 2.3.///

DEFINITION 3.16

Let X be a locally compact U-topological space and let $\underline{F}^\cdot$ be a bounded below complex of sheaves. Denote by $\underline{R}^\cdot(\underline{F}^\cdot)$ the canonical flabby complex quasi-isomorphic to $\underline{F}^\cdot$. For each integer p the functor

$$I(\Gamma_{c|?}(?:\underline{R}^p(\underline{F}^\cdot))):\mathrm{Open}(X)\longrightarrow \mathrm{Pro}_V\text{-}\mathrm{Ab}_U$$

is a precosheaf on X that we denote by $\overline{P}^p(\underline{F}^\cdot)$. In fact a standard result of sheaf theory shows that $\overline{P}^p(\underline{F}^\cdot)$ is a cosheaf on X. We define $\overline{P}^\cdot(\underline{F}^\cdot)$ to be the complex of cosheaves obtained by endowing the graded cosheaf $(\overline{P}^p(\underline{F}^\cdot))_{p\in Z}$ with the differential induced by the one of $\underline{R}^\cdot(\underline{F}^\cdot)$. Clearly every arrow $f^\cdot:\underline{F}^\cdot\longrightarrow \underline{G}^\cdot$ in $C^+(\mathrm{Shv}(X,\mathrm{Ab}_U))$ induces canonically an arrow

$$\overline{P}^\cdot(f^\cdot) : \overline{P}^\cdot(\underline{F}^\cdot) \rightarrow \overline{P}^\cdot(G^\cdot)$$

One verifies easily that $\overline{P}^\cdot(g^\cdot\circ f^\cdot) = \overline{P}^\cdot(g^\cdot)\circ\overline{P}^\cdot(f^\cdot)$ when $g^\cdot:\underline{G}^\cdot \longrightarrow \underline{H}^\cdot$ is an arrow in $C^+(\mathrm{Shv}(X,\mathrm{Ab}_U))$. The preceding construction gives thus rise to a functor:

$$\overline{P}^\cdot : C^+(\mathrm{Shv}(X,\mathrm{Ab}_U))\longrightarrow C^+(\mathrm{Cosh}(X)).$$

Moreover it is clear that $\overline{P}^\cdot$ transforms quasi-isomorphisms into quasi-isomorphisms.Thus $\overline{P}^\cdot$ induces a functor

$$\overline{P}^\cdot : D^+(\mathrm{Shv}(X,\mathrm{Ab}_U)) \rightarrow D^+(\mathrm{Cosh}(X))$$

Recall that a bounded complex of sheaves $\underline{F}^\cdot$ is of finite cohomological dimension if there exist an integer q such that

$$\mathbb{H}^p(W,\underline{F}^\cdot) = 0$$

for every $p\geq q$ and every open subset W of X.

We denote by $D^b_{fcd}(\mathrm{Shv}(X,\mathrm{Ab}_U))$ the full triangulated subcategory of $D^b(\mathrm{Shv}(X,\mathrm{Ab}_U))$ formed by the complexes which have finite cohomological dimension. Clearly, $\overline{P}^\cdot$ induces by restriction a functor

$$\overline{P}^\cdot : D^b_{fcd}(\mathrm{Shv}(X,\mathrm{Ab}_U)) \rightarrow D^b(\mathrm{Cosh}(X))$$

PROPOSITION 3.17.

If $\underline{G}^{\cdot} \in Ob(D^b_{fcd}(ShV(X, Ab_U))$ *then*

$$\mathbb{L}\Gamma^F \overline{P}^{\cdot}(\underline{G}^{\cdot}) = \mathbb{R}\Gamma_{c\cap F}(\underline{G}^{\cdot}|_F)$$

for every closed subset F *of* X.

Proof:

Let $\underline{R}^{\cdot}(\underline{G}^{\cdot})$ be as in the preceding definition and let q be an integer such that $\underline{H}^p(\underline{G}^{\cdot}) = 0$ and that $\mathbb{H}^p_c(W, \underline{G}^{\cdot}) = 0$ for every $p \geq q$ and every open subset W of X. A classical result of sheaf theory shows that the complex $\tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))$ has c-soft components and is quasi- isomorphic to $\underline{R}^{\cdot}(\underline{G}^{\cdot})$.

It follows from this fact that

$$I(\Gamma_c|_W(W; \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot})))) \underset{Qis}{\approx} \overline{P}^{\cdot}(\underline{G}^{\cdot})(W)$$

for every open subset W of X. Since the functor

$$\Gamma_c|_?(?; (\tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))^p$$

is a flabby cosheaf for every integer p, it is clear that

$$\mathbb{L}\Gamma^F \overline{P}^{\cdot}(\underline{G}^{\cdot}) \simeq \Gamma^F I(\Gamma_c|_?(?; \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot})))$$

Now, since $\tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))$ has c-soft components, we have the following exact sequence:

$$0 \to \Gamma_c|_{X \setminus F}(X \setminus F; \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))) \to \Gamma_c(X, \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))) \to \Gamma_{c\cap F}(F, \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))) \to 0$$

This shows that

$$\mathbb{L}\Gamma^F \overline{P}^{\cdot}(\underline{G}^{\cdot}) \simeq \Gamma_{c\cap F}(F, \tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot})))$$

The conclusion follows from the fact that $\tau_{\leq q}(\underline{R}^{\cdot}(\underline{G}^{\cdot}))$ is quasi-isomorphic to $\underline{G}^{\cdot}$ and has c-soft components.///

§ 4 Comparison with singular and Borel-Moore homology theories

This paragraph uses in a new way old results, so we shall make it brief.

PROPOSITION 4.1.

If X is a HLC space then

$$H_i(X,\mathbb{Z}^X) \simeq HS_i(X,\mathbb{Z})$$

where $\mathbb{Z}^X$ *denotes the constant cosheaf on X associated to* $\mathbb{Z}$ *and where* HS_i *denotes classical singular* i^{th} *homology functor*

Proof:

Denote by $SL.(X,\mathbb{Z})$ the complex of local singular chains on X defined as the colimit of the direct system

$$(S.(X,\mathbb{Z}),(Sb_X)^{q-p})_{p\in\mathbb{Z}}$$

where $S.(X,\mathbb{Z})$ is the classical complex of singular chains on X with integer coefficients and where Sb_X is the barycentric subdivision chain map associated to X.

The Mayer-Vietoris theorem for two open subsets of X states in fact that the following sequence is exact:

$$0 \to SL(U\cap V,\mathbb{Z}) \to SL(U,\mathbb{Z}) \oplus SL.(V,\mathbb{Z}) \to SL.(U\cup V,\mathbb{Z}) \to 0$$

The cocontinuity theorem states in fact that:

$$SL.(X,\mathbb{Z}) = \operatorname*{colim}_{U\in\mathcal{U}} SL.(U,\mathbb{Z})$$

where $\mathcal{U}$ is a directed set of open subsets of X such that $\cup\mathcal{U} = X$

These two results together show that $I(SL_{-p}(?,\mathbb{Z}))$ is a flabby cosheaf on X which we denote by $\overline{S}^p_X$. Denote by $\overline{S}^{\cdot}_X$ the complex of cosheaves obtained by endowing the graded cosheaf $(\overline{S}^p_X)_{p\in Z}$ with the differential induced by the one of $SL.(?)$. It is clear that

$$\mathbb{H}_i(U;\overline{S}^{\cdot}_X) = I(H_i(SL.(U,\mathbb{Z}))) = I(HS_i(U,\mathbb{Z}))$$

So one has

$$\lim_{U\ni x} \mathbb{H}_i(U,\overline{S}^{\cdot}_X) = \lim_{U\ni x} I(HS_i(U,\mathbb{Z})$$

and since X is HLC we deduce that

$$H_i(\overline{S}^{\bullet}_{X,x}) = \lim_{U\ni x} H_i(U,\overline{S}^{\bullet}_X) = \begin{cases} Z \text{ if } i=0 \\ 0 \text{ if } i\neq 0 \end{cases}$$

It follows that

$$\overline{S}^{\bullet}_X \ \widetilde{\text{Qis}}\ \mathbb{Z}^X$$

and by consequence that

$$\mathbb{H}_i(X;\mathbb{Z}^X) = HS_i(X,\mathbb{Z}).///$$

PROPOSITION 4.2

If the space X *is* $\text{clc}^{\infty}_{\mathbb{Z}}$ *then*

$$H_i(X,\mathbb{Z}^X) = HBM^c_i(X,\mathbb{Z})$$

where HBM_i *denotes the* i^{th} *Borel-Moore homology functor.*

Proof:

It follows from THEOREM 11.6 of (2) together with the fact that the quasi-coresolutions of G.E.BREDON induce real flabby resolutions in our category of cosheaves.///

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