Dendric preserving morphisms

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Introduction

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- A *bi-infinite word* $(x, y, \dots)$ is an element of $\mathcal{A}^{\mathbb{Z}}$.

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- If $w = uv$, then u is a *prefix* of w and v is a *suffix* of w.

Left and right extensions

$$LE_x(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(x)\}, \quad RE_x(w) = \{b \in \mathcal{A} \mid wb \in \mathcal{L}(x)\},$$

$$E_x(w) = \{(a, b) \in LE_x(w) \times RE_x(w) \mid awb \in \mathcal{L}(x)\}$$

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Definition

The *extension graph* of $w \in \mathcal{L}(x)$ is the bipartite graph $\mathcal{E}_x(w)$ with vertices $LE_x(w) \sqcup RE_x(w)$ and edges $E_x(w)$.

Example:

$$^{\omega}(010).(010)^{\omega}$$

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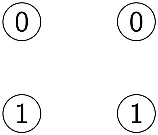

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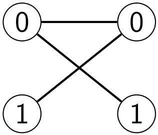

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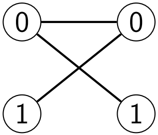

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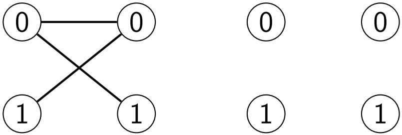

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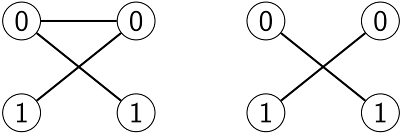

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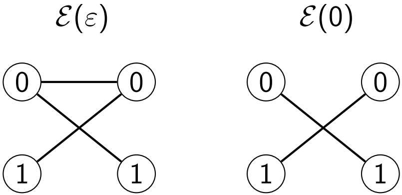

Definition

A word $w \in \mathcal{L}(x)$ is *dendric* (in x) if $\mathcal{E}_x(w)$ is a tree.
A bi-infinite word x is *dendric* if all the words of $\mathcal{L}(x)$ are dendric.

Factor complexity

The factor complexity of $x \in \mathcal{A}^{\mathbb{Z}}$ is the function

$$p_x(n) : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \#\mathcal{L}(x) \cap \mathcal{A}^n.$$

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Proposition

If $x \in \mathcal{A}^{\mathbb{Z}}$ is dendric, then

$$p_x(n) = (\#\mathcal{A} - 1)n + 1.$$

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A *morphism* $(\sigma, \alpha, \tau, \dots)$ is a monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$, i.e. for any $u, v \in \mathcal{A}^*$,

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Assumptions: the image **alphabet is minimal** and the morphism is **non erasing**

Image of a bi-infinite word

$$\sigma : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

$$x : \quad \ldots 2.001210 \ldots$$

$$\sigma(x) : \quad \ldots 0.001\ 001\ 10\ 0\ 10\ 001 \ldots$$

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What are the morphisms such that $\sigma(x)$ is dendric if x is dendric?

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Definition

A morphism $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is *dendric preserving* if $\sigma(x)$ is dendric for all dendric $x \in \mathcal{A}^{\mathbb{Z}}$.

First observations

Unary alphabets

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From now on, we assume that the alphabets are of size at least 2

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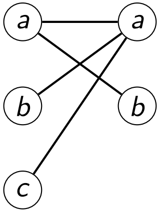

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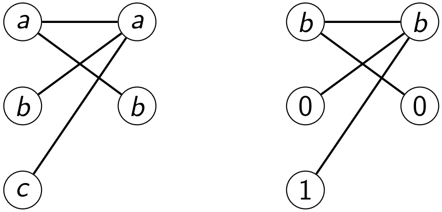
$\mathcal{E}_x(ba)$ $\qquad$ $\mathcal{E}_{\sigma(x)}(0b)$



Arnoux-Rauzy morphisms

The *Arnoux-Rauzy morphisms* are defined by

$$\alpha_\ell^L : \begin{cases} \ell \mapsto \ell \\ a \mapsto \ell a & \text{if } a \neq \ell \end{cases} \qquad \alpha_\ell^R : \begin{cases} \ell \mapsto \ell \\ a \mapsto a\ell & \text{if } a \neq \ell \end{cases}$$

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Proposition

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If σ is an Arnoux-Rauzy morphism, then x is dendric iff $\sigma(x)$ is dendric.
In particular, for any morphism τ, τ is dendric preserving iff $\sigma \circ \tau$ is dendric preserving.

Stability under composition

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If σ and τ are dendric preserving, then $\sigma \circ \tau$ is dendric preserving.

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The morphisms

$$\alpha_{\ell_n}^{s_n} \circ \cdots \circ \alpha_{\ell_1}^{s_1} \circ \pi$$

(where $\pi : \mathcal{A} \to \mathcal{B}$ is a bijection and, for all $i \leq n$, $s_i \in \{L, R\}$ and $\ell_i \in \mathcal{B}$) are dendric preserving.

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Question: Are there other dendric preserving morphisms?

Sizes of the alphabets

Idea for the upper bound

Proposition (Reminder)

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$$p_{\sigma(x)}(n) = (\#\mathcal{B} - 1)n + 1.$$

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And if $\sigma(x)$ is dendric, then

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Goal:

Bound $p_{\sigma(x)}$ by a linear function

$$\sigma : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 10 \\ 2 \mapsto 0 \end{cases}$$

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Coverings

Definition

A *covering* of $u \in \mathcal{B}^n$ is a pair $(w, k) \in \mathcal{L}(x) \times \mathbb{N}$ where $u = \sigma(w)_{[k+1,k+n]}$ and w is minimal, i.e.

$$k + 1 \leq |\sigma(w_1)| \quad \text{and} \quad k + n \geq \left|\sigma(w_{[1,|w|[})\right| + 1$$

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Proposition

We have

$$p_{\sigma(x)}(n) \leq \# C_{x,\sigma}(n).$$

Number of coverings

Proposition

If $x \subseteq \mathcal{A}^{\mathbb{Z}}$ is dendric, then, for all $n \geq 1$,

$$\#C_{x,\sigma}(n) = \sum_{a \in \mathcal{A}} |\sigma(a)| + (\#\mathcal{A} - 1)(n - 1).$$

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Corollary

If $x \subseteq \mathcal{A}^{\mathbb{Z}}$ and $\sigma(x) \subseteq \mathcal{B}^{\mathbb{Z}}$ are dendric, then $\#\mathcal{B} \leq \#\mathcal{A}$.

Context

Definition

A morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is *strongly left proper* if there exists $\ell \in \mathcal{A}$ such that, for all $a \in \mathcal{A}$,

$$\sigma(a) \in \ell(\mathcal{A} \setminus \{\ell\})^*.$$

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Proposition

Let σ be a strongly left proper (for ℓ) and dendric preserving morphism. If p is the longest common prefix of all $\sigma(a)$, $a \in \mathcal{A}$, then for each letter b, there exists exactly one letter a such that pb is a prefix of $\sigma(a)\ell$.

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We have a similar result with the suffixes.

Common prefix

Proposition

The following are equivalent:

1. p *is a prefix of* $\sigma(a)^\omega$ *for all* $a \in \mathcal{A}$;
2. p *is a prefix of* $\sigma(w)p$ *for all* $w \in \mathcal{A}^*$;
3. p *is a prefix of* $\sigma(w)$ *for all long enough* $w \in \mathcal{A}^*$.

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We also define s_σ with suffixes instead of prefixes.

Proposition

If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, then for each $b \in \mathcal{B}$, there exists at most one $a \in \mathcal{A}$ such that $p_\sigma b$ is a prefix of $\sigma(a)p_\sigma$.

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Corollary

If $\sigma : \mathcal{A}^* \to \mathcal{B}^*$ is dendric preserving, then $\#\mathcal{A} \leq \#\mathcal{B}$.

Final results

Initial case

Lemma

If σ is dendric preserving and $s_\sigma p_\sigma = \varepsilon$, then σ is a "coding".

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Lemma

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Proof:

It suffices to prove that the images of the letters are of length 1.

Induction

Lemma

If σ is dendric preserving morphism and $|s_\sigma p_\sigma| = n > 0$, then

1. $(s_\sigma p_\sigma)_1 = (s_\sigma p_\sigma)_n =: \ell$ *and it is such that, for any dendric x,*

$$E_{\sigma(x)}(\varepsilon) = (\ell \times \mathcal{B}) \cup (\mathcal{B} \times \ell);$$

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2. *there exists a morphism* τ *such that* $\sigma \in \{\alpha_\ell^L \circ \tau, \alpha_\ell^R \circ \tau\}$.

Proposition

A morphism is dendric preserving iff it can be decomposed into

$$\alpha_{\ell_n}^{s_n} \circ \cdots \circ \alpha_{\ell_1}^{s_1} \circ \pi$$

(where $\pi : \mathcal{A} \to \mathcal{B}$ is a bijection and, for all $i \leq n$, $s_i \in \{L, R\}$ and $\ell_i \in \mathcal{B}$).

Thank you for your attention!