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A TIME-DEPENDENT DAMAGE LAW IN DEFORMABLE SOLID: A HOMOGENIZATION APPROACH

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Summary. The theoretical developments and the numerical applications of a time-dependent damage law is presented. This law is deduced from considerations at the micro-scale where non-planar growth of micro-cracks, following a subcritical propagation criterion, is assumed. The passage from micro-scale to macro-scale is done through an asymptotic homogenization approach. The model is built in two steps. First, the effective coefficients are calculated at the micro-scale in finite periodical cells, with respect to the micro-cracks length and their orientation. Then, a subcritical damage law is developed in order to establish the evolution of damage. As shown by numerical simulations, the developed model enables to reproduce the long-term behavior encountering relaxation and creep effects.

1 INTRODUCTION

In many engineering applications, the damage evolution of materials must be accurately considered. The modeling of the nucleation and the growth of micro-cracks in solids due to various mechanical loadings is of particular interest in order to assess the stress-strain behaviour of damaged materials. As long as microcracks propagate, the overall stiffness of materials may drastically decrease and provoke failure when coalescence of adjacent micro-cracks occurs.

At micro-scale, materials contain micro-cracks that strongly affect the macroscopic mechanical behavior of the materials. In many materials, the micro-crack distribution may be locally approximated by a periodic one. This periodic structure is characterized by an internal length, that is the distance between two adjacent micro-cracks. In order to find mechanically equivalent homogeneous material at the macro-scale, having relatively similar properties than the heterogeneous medium, the macroscopic description can be obtained by homogenization procedures [1, 2]. The behavior of the macroscopic structure is deduced from the properties of the material at the micro-scale (Fig. 1a). The stiffness of the solid is governed by homogenized coefficients

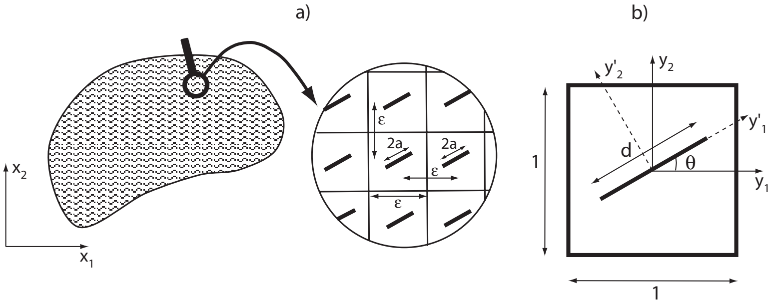

Figure 1: (a) Fissured medium with locally periodic microstructure. (b) Unit cell with rescaled crack of length d. $2a$: length of the micro-crack, ε: distance between two micro-cracks (internal length), θ: orientation of the micro-crack with respect to this (x_1 , x_2) system.

that depend on the elastic properties of the solid matrix, the length and the orientation of micro-cracks.

Moreover, the evolution of damage in many materials is often time-dependent. An important source of time-dependency is the subcritical propagation of micro-cracks. A subcritical criterion is a criterion considering time-dependent crack propagation for energy lower than the critical limit of fracture [3]. The rate of crack propagation may be expressed with respect to stress intensity factor at the crack tips, under tensile mode, with a power law [4]. This paper joins the concepts of asymptotic homogenization and subcritical crack propagation to deduce a time-dependent damage model.

2 MATHEMATICAL FORMULATION

Consider a two-dimensional isotropic elastic medium containing a locally periodic distribution of micro-cracks. Each crack is straight with a length $2a$ and an orientation of angle θ with respect to the x_1 direction (abscissa of the referential system considered at the macro-scale). The length $2a$ and the orientation θ are assumed to vary smoothly almost everywhere in the elastic body. The damage variable d, varying between 0 (for virgin material) and $1/[\max(|\cos(\theta)|;|\sin(\theta)|)]$ (for fully damaged material), is the ratio between the crack length $2a$ and the distance between two micro-cracks ε that also represents the size of the periodicity cell (Fig. 1):

$$d = \frac{2a}{\varepsilon} \tag{1}$$

On the crack faces, traction free opening or frictionless contact conditions are assumed. The way in which the switch from one state to the other is controlled will be described later, in terms of the homogenized solution (Eq. 8).

2.1 Asymptotic homogenization

The locally periodic microstructure is constructed from a unit cell $Y = [-0.5, 0.5] \times [-0.5, 0.5]$ expressed in a (y_1, y_2) orthogonal axis system centered in the middle of the crack, corresponding to the center of the cell. The crack is rotated with an angle θ with respect to y_1 axis. Then this unit cell is rescaled by the parameter ε so that the period of the material is εY. The two distinct scales are represented by the variable $\mathbf{x}$, which is referred to as macroscopic variable and the variable $\mathbf{y} = \mathbf{x}/\varepsilon$, referred to as microscopic variable [5].

Following the method of asymptotic homogenization (e.g. [1]), we look for expansions of $\mathbf{u}^\varepsilon$ and $\boldsymbol{\sigma}^\varepsilon$ in the form

$$\mathbf{u}^\varepsilon(\mathbf{x},t) = \mathbf{u}^{(0)}(\mathbf{x},\mathbf{y},t) + \varepsilon\mathbf{u}^{(1)}(\mathbf{x},\mathbf{y},t) + \varepsilon^2\mathbf{u}^{(2)}(\mathbf{x},\mathbf{y},t) + \ldots \tag{2}$$

$$\boldsymbol{\sigma}^\varepsilon(\mathbf{x},t) = \frac{1}{\varepsilon}\boldsymbol{\sigma}^{(-1)}(\mathbf{x},\mathbf{y},t) + \boldsymbol{\sigma}^{(0)}(\mathbf{x},\mathbf{y},t) + \varepsilon\boldsymbol{\sigma}^{(1)}(\mathbf{x},\mathbf{y},t) + \ldots \tag{3}$$

where $\mathbf{u}^{(i)}(\mathbf{x},\mathbf{y},t), \boldsymbol{\sigma}^{(i)}(\mathbf{x},\mathbf{y},t),\ \mathbf{x} \in \mathcal{B}_s,\ \mathbf{y} \in Y$ are smooth functions and Y-periodic in $\mathbf{y}$.

Based on previous works (e.g. [5, 6, 7]), we can show that the substitution of Eqs. (2) and (3) in the equilibrium equations gives the following relationships for the overall response of the material

$$\frac{\partial}{\partial x_j}\Sigma^{(0)}_{ij} = 0. \tag{4}$$

where

$$\Sigma^{(0)}_{ij} = C_{ijkl}e_{xkl}(\mathbf{u}^{(0)}) \tag{5}$$

$$C^{\pm}_{ijkl}(d,\theta) = \frac{1}{|Y|}\int_{Y_s}(a_{ijkl} + a_{ijmn}e_{ymn}(\xi^{kl}_{\pm}))\ dy \tag{6}$$

are the macroscopic stress and the homogenized coefficients, respectively. Eq. (4) is the homogenized equation of equilibrium. In each regime (opening or closure), we can prove that the microscopic correction $\mathbf{u}^{(1)}$ is

$$\mathbf{u}^{(1)}_{\pm} = \xi^{pq}_{\pm}e_{xpq}(\mathbf{u}^{(0)}) \tag{7}$$

where $\xi^{pq}_{\pm}$ are elementary solutions of $\mathbf{u}^{(1)}$ for particular $e_{xpq}(\mathbf{u}^{(0)}) = \delta_{pq}$. The distinction $\pm$ corresponds to opening (+) or contact (-) conditions of the crack lips. The difference between these microscopic states of contact and opening are obtained from the orientation of the force vector with respect to crack line. These two states induce a separation of the space $\mathbf{R}$ of macroscopic deformations $e_{x11}, e_{x12}, e_{x22}$ into two subregions $\mathbf{R}^{\pm}$ defined by

$$\mathbf{R}^{\pm} = \left\{\mathbf{e}_{\mathbf{x}} |\quad N_i a_{ijkl}e_{xkl}(\mathbf{u}^{(0)})N_j \gtrless 0\right\} \tag{8}$$

The homogenized coefficients C_{ijkl} depend on the state of damage of the material (d and θ) and on the mechanical properties of the solid matrix (E and ν). From the integral (6) and for a given couple (E,ν), the coefficients can be initially computed for a large number of $d^\star \in [0,1]$ ($d^\star = d/\max(|\cos(\theta)|;|\sin(\theta)|)$ being the normalized damage variable) and $\theta \in [0°, 180°]$ and for both opening and closure states from the strain field $e_{ymn}(\xi^{kl})$ obtained by finite element computation on the unit cell. After interpolation, polynomial expressions of $C_{ijkl}(d,\theta)$ can be obtained [7, 8].

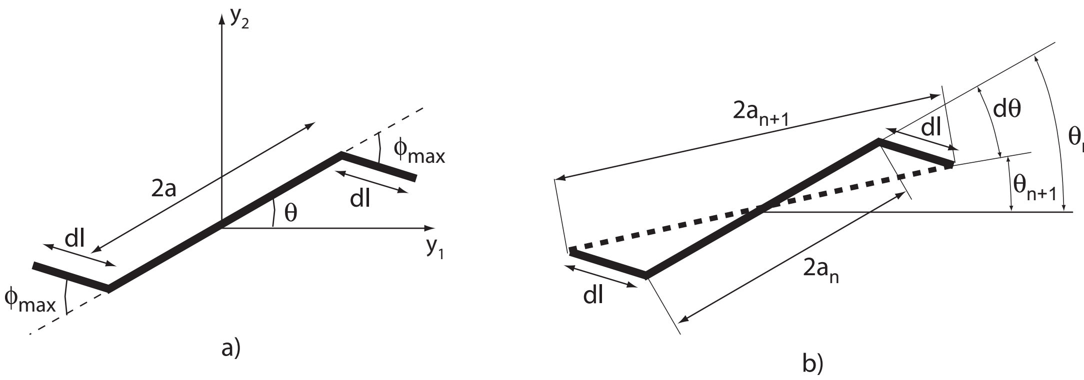

Figure 2: Kinked crack. (a) The out-of-plane crack growth propagates in the direction that maximize the energy release rate. ϕ_{max} is the angle between the crack line and the crack extension. (b) The kinked crack (solid line) and its equivalent replacement crack (dashed line).

2.2 Subcritical growth of micro-crack

The evolution of the micro-crack length (i.e. the propagation of the crack) is described through a subcritical criterion adapted from the Charles' law [4]:

$$\frac{dl}{dt} = v_0 \left(\frac{K_I^\star}{K_0} \right)^n \tag{9}$$

where v_0 is a referential velocity of crack propagation and n is the subcritical growth coefficient. $K_I^\star$ is the stress intensity factor for the tensile mode of rupture (Mode I) of the kinked crack (Fig. 2a). K_0, v_0 and n are material parameters. $K_I^\star$ depends on the stress (or strain) conditions, the internal length ε and the geometry of the micro-cracks. The determination of the stress intensity factors at the crack tips is made through the computation of path-independent J-, L- and M- integrals [9] for straight trajectory of micro-cracks and from the polynoms of Leblond [10] for the kinked cracks (see [11] for more completeness).

The crack is assumed to propagate in the direction that maximizes the energy release rate. This criterion produces a kinking angle between the existing crack and the incrementally propagated crack (Fig. 2a). This kinking angle can be expressed with the following function [12]:

$$\begin{aligned} \phi_{max} = \quad & sgn(K_{II})[0.70966\lambda^3 - 0.097725 sin^2(3.9174\lambda) \\ & -13.1588 tanh(0.15199\lambda)] \end{aligned} \tag{10}$$

where λ is a mode mixity factor that combine the stress intensity factors of mode I, K_I, and mode II, K_{II}, of the straight crack :

$$\lambda = \frac{|K_{II}|}{K_I + K_{II}} \tag{11}$$

3 EQUIVALENT CRACK

The determination of the direction (ϕ_{max}, Eq. 10) and the length (dl, Eq. 9) of the out-of plane crack extension is not enough to compute the whole trajectory of the crack tips. At

E	ν	K_0	v_0	n	ε	d_0	θ_0
$[Pa]$	$[-]$	$[MPa.m^{1/2}]$	$[m/s]$	$[-]$	$[-]$	$[-]$	$[°]$
2.10^9	0.3	0.6	1.10^{-3}	20	1.10^{-4}	0.28	45

Table 1: Material parameters used in the simulations.

each time step, the kinked crack must be replaced by an equivalent straight crack. We adopt an equivalent crack obtained by joining the tips of the real branched crack (Fig. 2b). The equivalent crack is determined by means of geometrical relationships in the form of differential equations, assuming small time increments and using the up-scaling relation (1):

$$\frac{dd}{dt} = \frac{2}{\varepsilon}\cos(\phi_{max})\frac{dl}{dt} \tag{12}$$

$$\frac{d\theta}{dt} = d\frac{\varepsilon}{2}\sin(\phi_{max})\frac{dl}{dt} \tag{13}$$

These two last equations show that the geometry of the equivalent micro-crack, in term of length and orientation, depends on the damage variable d, the propagation rate ($\frac{dl}{dt}$) and the orientation (ϕ_{max}) of the kinked crack. Those values depend on the macroscopic variables d, θ and $\mathbf{e_x}$ which demonstrates that the developed model is a macroscopic damage model starting from considerations at the microscopic scale.

4 NUMERICAL EXAMPLES

The time-dependent behavior of materials can be underlined by means of quasi-static loading tests, creep tests or relaxation tests that point out the same time-dependent properties under different stress and strain conditions. The set of materials parameters, that have been used in the simulations are reported in Table 1. All the simulations presented in the following have been made considering biaxial loading. Plane-strain condition is considered in the third direction.

4.1 Loading at constant strain rate

Figures 3 and 4 illustrate the response of a material submitted to an uniaxial tension loading at constant vertical strain rate (the horizontal direction is free of stress). Figure 3 shows that the developed model is able to reproduce the effect of strain rate on the obtained failure stress. Under low strain rate, the effect of time becomes predominant and the failure appears for a lower strain level than in the case of faster loading. The micro-cracks are not growing in their own plane but a kinked angle is forming in order to propagate in the direction that maximizes the energy release rate (Eq. 10). As the damage increases, the equivalent crack is rotating and the rigidity decreases until coalescence of micro-cracks producing failure of the material. At the end of the loading, the equivalent cracks tend to be perpendicular to the direction of the principal tensile strain (Figure 4).

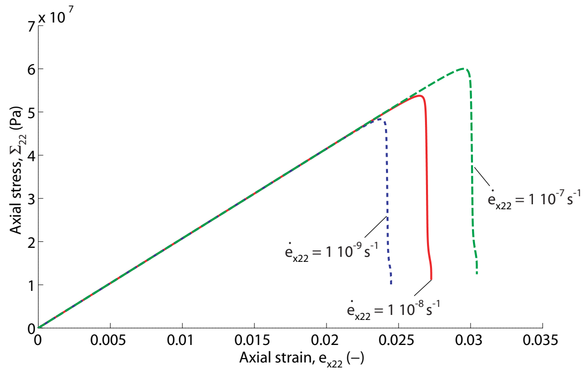

Figure 3: Axial tension tests at various constant strain rate $\dot{e}_{x22}$. The strength increases when the strain rate increases.

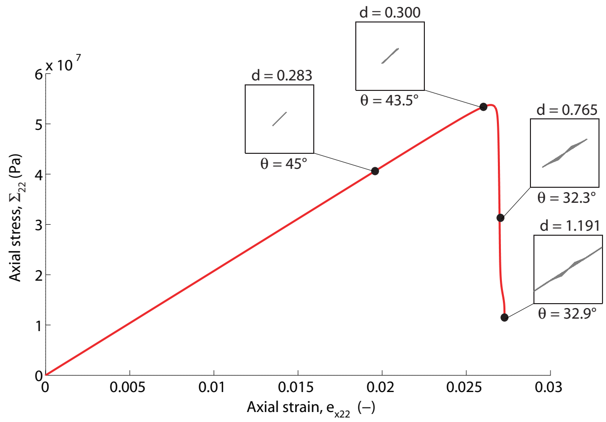

Figure 4: Axial tension test at a constant strain rate $\dot{e}_{x22} = 1.10^{-8}s^{-1}$. Evolution of the micro-crack in the periodic cell.

4.2 Relaxation tests

Applying a constant axial strain, the relaxation test aims at investigating the time-dependent response of materials. Under a constant loading, the subcritical micro-crack growth produces a progressive decrease of the rigidity as long as the damage state increases. As a consequence, the stresses are gradually relaxing upon failure. Under a biaxial combined tensile/compressive constant strain field ($e_{x22} = -0.035$ (compression) and $e_{x11} = 0.035$ (tension)), Figure 5 shows the evolution of horizontal and vertical stresses with time. In parallel is shown the evolution with time of the ratio between horizontal and vertical stress. As long as the crack propagate, the direction of the equivalent crack tends toward vertical. As a consequence, the crack lips being under opening condition, the horizontal rigidity becomes much lower than vertical one. So, the ratio between horizontal and vertical stresses evolves in accordance with the relative lost of horizontal rigidity with respect to vertical one. That is a characteristic of the evolving anisotropic damage law.

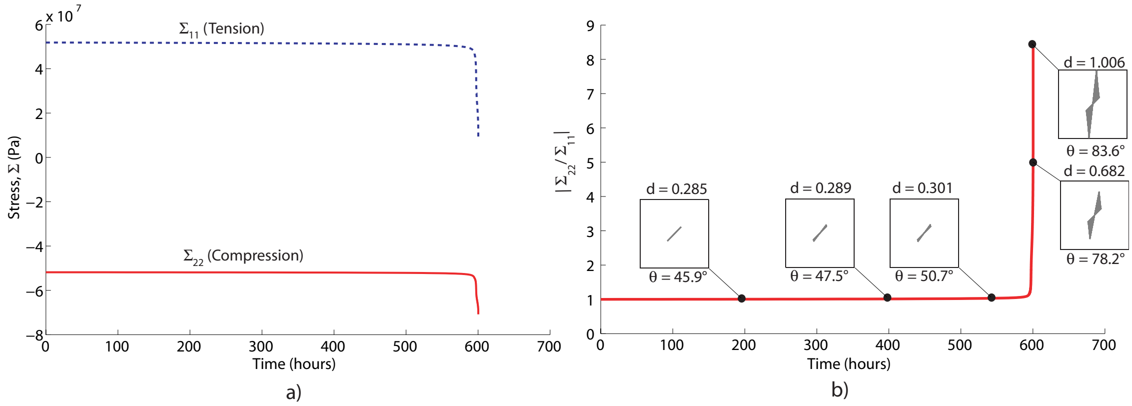

Figure 5: Relaxation test under biaxial conditions. $e_{x22} = -0.035$ (compression) and $e_{x11} = 0.035$ (tension). Evolution with time of (a) the horizontal and vertical stresses and of (b) the ratio of anisotropy, defined as the absolute value of the stress ratio.

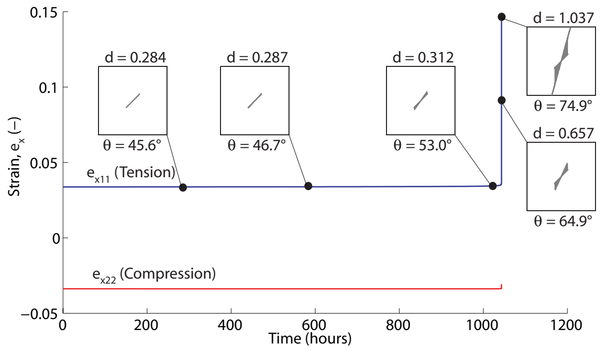

Figure 6: Creep test under biaxial conditions. $\Sigma_{22} = -50MPa$ (compression) and $\Sigma_{11} = 50MPa$ (tension). Evolution of the horizontal and vertical strain with time.

4.3 Creep tests

Figure 6 depicts the creep strain predicted by the model under a biaxial combined tensile/compressive constant stress field ($\Sigma_{22} = -50MPa$ (compression) and $\Sigma_{11} = 50MPa$ (tension)). After an instantaneous strain response corresponding to the short-term behaviour of the material, the time effect makes damage variable increase and micro-cracks rotate. During the first part of the test, the damage evolution is slow. However, the rate of damage is amplifying with time.

5 CONCLUSIONS

The subcritical growth of micro-cracks is responsible of the time-dependent behaviour of many materials. The subcritical criterion has been applied at the micro-scale on a periodic cracked cell and up-scaled by an asymptotic homogenization procedure. So doing, a time-dependent macroscopic damage model has been deduced and the stress-strain response of materials, depending on time, has been computed in a macroscopic point.

As long as the micro-cracks grow, due to the combined effect of time and high stresses at the crack tips, the internal damage increases and the global rigidity of the material decreases. The kinked crack obtained at each step of calculation has been replaced by an equivalent straight crack defined by its length and its orientation. So doing, the homogenized elastic properties of the damaged unit cell can be determined and introduced in the macroscopic stress-strain relationship of the material.

Numerical simulations of loading at constant strain rate, relaxation or creep tests have shown the ability of the developed model to reproduce this time-dependent damage response.

REFERENCES

[1] Sanchez-Palencia, E. (1980). Non-homogeneous Media and Vibration Theory. Lecture Notes in Physics, vol.127, Springer, Berlin.

[2] Dascalu, C. (2009). A two-scale damage model with material length. *C.R. Mécanique* **337**, 645-652.

[3] Atkinson, B., Meredith, P. (1987). The theory of subcritical crack growth with applications to minerals and rocks. In *Fracture Mechanics of Rocks, Academic Press Inc.*, 111-166.

[4] Charles, R. (1958). Dynamic fatigue of glass. *J. Appl. Phys.* **29**, 1657-1662.

[5] Leguillon, D., Sanchez-Palencia, E. (1982). On the behavior of a cracked elastic body with (or without) friction. *J. Mech. Theor. Appl.* **1**, 195-209.

[6] Dascalu, C., Bilbie, G. (2007). A multiscale approach to damage configurational forces. *Int. J. of Fract.* **147**, 285-293.

[7] Dascalu, C., Bilbie, G., Agiasofitou, E. (2008). Damage and size effect in elastic solids: A homogenization approach. *Int. J. Solid Struct.* **45**, 409-430.

[8] Dascalu, C., François B., Keita O. (2010). A two-scale model for subcritical damage propagation. *Int. J. Solid Struct.* **47**, 493-502.

[9] François, B., Dascalu, C. (2010). A two-scale time-dependent damage model based on non-planar growth of micro-cracks. (Submitted)

[10] Kienzler, R., Herrmann, G. (2000). Mechanics in material space with applications to defect and fracture mechanics. Springer-Verlag, Berlin Heidelberg.

[11] Leblond, J.B. (1999). Crack paths in three-dimensional elastic solids. I: two-term expansion of the stress intensity factors - application to crack path stability in hydraulic fracturing. *Int. J. Solids Struct.* **36**, 79-103.

[12] Schütte, H., Bruhns, O.T. (2002). On a geometrically nonlinear damage model based on a multiplicative decomposition of the deformation gradient and the propagation of microcracks. *J. Mech. Phys. Solids* **50**, 827-853.