# WEYL QUANTIZATION OF DEGREE 2 SYMPLECTIC GRADED MANIFOLDS 

MELCHIOR GRÜTZMANN, JEAN-PHILIPPE MICHEL, AND PING XU


#### Abstract

RÉsumé. Soit $S$ un fibré spinoriel d'un fibré vectoriel pseudo-Euclidien ( $E, \mathrm{~g}$ ) de rang pair. Nous introduisons une nouvelle filtration de l'algèbre $\mathcal{D}(M, S)$ des opérateurs différentiels sur $S$. Pour cette filtration, l'algèbre graduée associée $\operatorname{gr} \mathcal{D}(M, S)$ s'avère être isomorphe à l'algèbre $\mathcal{O}(\mathcal{M})$ des fonctions lisses sur $\mathcal{M}$, la variété graduée symplectique de degré 2 canoniquement associée à $(E, \mathrm{~g})$. En conséquence, nous construisons la quantification de Weyl de $\mathcal{M}$ comme une application $\mathcal{W} \mathcal{Q}_{\hbar}: \mathcal{O}(\mathcal{M}) \rightarrow \mathcal{D}(M, S)$, et montrons que $\mathcal{W} \mathcal{Q}_{\hbar}$ satisfait toutes les propriétés voulues d'une quantification. En application, nous obtenons une bijection entre les structures d'algébroïdes de Courant $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$, caractérisées de manière équivalentes par des fonctions hamiltoniennes génératrices sur la variété graduée symplectique $\mathcal{M}$, et les opérateurs de Dirac générateurs anti-symétriques $D \in \mathcal{D}(M, S)$. L'opérateur $D^{2}$ est un nouvel invariant de $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket \rrbracket)$, qui généralise la norme au carré de la 3-forme de Cartan d'une algèbre de Lie quadratique. Nous étudions cet invariant en détail dans le cas particulier où $E$ est le double d'un bi-algébroïde de Lie $\left(A, A^{*}\right)$. Abstract. Let $S$ be a spinor bundle of a pseudo-Euclidean vector bundle ( $E, \mathrm{~g}$ ) of even rank. We introduce a new filtration on the algebra $\mathcal{D}(M, S)$ of differential operators on $S$. As a main property, the associated graded algebra $\operatorname{gr} \mathcal{D}(M, S)$ is proved to be isomorphic to the algebra $\mathcal{O}(\mathcal{M})$ of smooth functions on $\mathcal{M}$, where $\mathcal{M}$ is the degree 2 symplectic graded manifold canonically associated to $(E, \mathrm{~g})$. Accordingly, we establish the Weyl quantization of $\mathcal{M}$ as a map $\mathcal{W} \mathcal{Q}_{\hbar}: \mathcal{O}(\mathcal{M}) \rightarrow \mathcal{D}(M, S)$, and prove that $\mathcal{W} \mathcal{Q}_{\hbar}$ satisfies all the desired properties of quantizations. As an application, we obtain a bijection between Courant algebroid structures ( $E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket$ ) equivalently characterized by Hamiltonian generating functions on $\mathcal{M}$, and skew-symmetric Dirac generating operators $D \in$ $\mathcal{D}(M, S)$. The operator $D^{2}$ gives a new invariant of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ which generalizes the square norm of the Cartan 3-form of a quadratic Lie algebra. We study this invariant in detail in the particular case of $E$ being the double of a Lie bialgebroid $\left(A, A^{*}\right)$.


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## 1. Introduction

This paper is devoted to the study of Weyl quantization of degree 2 symplectic graded manifolds, and its application to Courant algebroids.

In searching for the Lie algebroid analogue of Drinfeld's double of Lie bialgebras [11], Liu-Weinstein- Xu introduced the notion of Courant algebroids [24]. The Courant algebroid axioms were later reformulated by Roytenberg [31] in terms of Dorfman brackets. However these axioms still remain mysterious, and various attempts have been made in order to understand Courant algebroids in a more conceptual and transparent way. One approach was through a degree 3 Hamiltonian function in a degree 2 symplectic graded manifold. When a Courant algebroid $E$ is the double $A \oplus A^{*}$ of a Lie bialgebroid $\left(A, A^{*}\right)$, Roytenberg [31 proved that the Courant algebroid structure on $E$ is indeed equivalent to a degree 3 Hamiltonian generating function $\Theta$ on the symplectic graded manifold $T^{*}[2](A[1])$ satisfying the equation $\{\Theta, \Theta\}=0$. Following an idea of Weinstein, Roytenberg [32] and Ševera 34] extended this result to arbitrary Courant algebroids and proved that there is essentially a bijection between Courant algebroids and degree 3 Hamiltonian generating functions on degree 2 symplectic graded manifolds. By a graded manifold we always mean a $\mathbb{N}$-graded manifold.

Around the same time, in 2001, in an unpublished manuscript [1], Alekseev-Xu took a different approach in terms of Dirac generating operators, an analogue of Kostant's cubic Dirac operators [22]. Alekseev-Xu's approach was motivated by the following basic example due to Cabras-Vinogradov [6]. Let $M$ be a manifold, and $\mathfrak{X}(M) \cong \Gamma(T M)$ be the Lie algebra of vector fields on $M$. The idea is to extend the Lie bracket on $\mathfrak{X}(M)$ to sections of the bundle $E=T M \oplus T^{*} M$. Observe that sections of $E$
act on the space of differential forms $\Omega(M)$ by contraction and by exterior multiplication, respectively,

$$
(X+\alpha) \cdot \mu:=\iota_{X} \mu+\alpha \wedge \mu
$$

where $\alpha \in \Gamma\left(T^{*} M\right)=\Omega^{1}(M), X \in \mathfrak{X}(M)$ and $\mu \in \Omega(M)$. This action turns $\Omega(M)$ into a Clifford module of the Clifford bundle $\mathrm{Cl}(E)$, where $E$ is equipped with the standard bilinear form:

$$
\left(X_{1}+\alpha_{1}, X_{2}+\alpha_{2}\right)=\left\langle\alpha_{1}, X_{2}\right\rangle+\left\langle\alpha_{2}, X_{1}\right\rangle
$$

and the Clifford generating relation is $x y+y x=(x, y)$.
Using the de Rham differential d : $\Omega^{\bullet}(M) \rightarrow \Omega^{\bullet+1}(M)$, one can form the derived bracket [19] on sections of $E$ :

$$
\begin{equation*}
\llbracket e_{1}, e_{2} \rrbracket:=\left[\left[\mathrm{d}, e_{1}\right], e_{2}\right], \quad \forall e_{1}, e_{2} \in \Gamma(E) \tag{1.1}
\end{equation*}
$$

where both sides are viewed as operators on $\Omega(M)$. It is straightforward to check that Eq. (1.1) coincides with the Dorfman bracket of the standard Courant algebroid $T M \oplus T^{*} M$.

Observe that $\Omega(M)$ is indeed a real spinor bundle of $\left(T M \oplus T^{*} M,(\cdot, \cdot)\right)$. For a general Courant algebroid $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$, Alekseev-Xu [1] proved that there exist cubic Dirac type generating operators, acting on a certain spinor bundle of $(E, g)$, which play exactly the same role as the de Rham differential operator d does in the Cabras-Vinogradov's approach to the standard Courant algebroid ${ }^{1}$. Thus a natural question arises: is there any relation between Hamiltonian generating functions and Dirac generating operators for a Courant algebroid?

To answer this question, we are naturally led to the study of Weyl quantization on symplectic graded manifolds of degree 2. The classical Weyl quantization formula is the prototypical example of a quantization map: to each polynomial function on the classical phase space $T^{*} \mathbb{R}^{n}$, it assigns a differential operator on $\mathbb{R}^{n}$. More precisely, the Weyl quantization maps polynomials in the coordinates $\left(x^{i}, p_{i}\right)$ to differential operators on $\mathbb{R}^{n}$ with coordinates $\left(x^{i}\right)$, and is defined as the symmetrization map such that $p_{i} \mapsto \frac{\hbar}{i} \frac{\partial}{\partial x^{i}}$ and $x^{i} \mapsto m_{x^{i}}$ (multiplication by $x^{i}$ ). Weyl quantization has been generalized to various contexts and plays an important role in different branches of mathematics, e.g. harmonic analysis [13], pseudo-differential symbolic calculus [18], formal and strict deformation quantizations [2, [29. More specifically, we are concerned with the extension of Weyl quantization to smooth manifolds and supermanifolds. Using an affine connection on $M$, Underhill built a Weyl quantization on the symplectic manifold $T^{*} M$ [37]. More generally, by choosing an additional linear connection on the vector bundle $V \rightarrow M$, Widom [40] obtained a quantization map of the form $\mathcal{Q}_{\hbar}^{M}: \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R}$ $\Gamma($ End $V) \rightarrow \mathcal{D}(M, V)$, where $\mathcal{D}(M, V)$ is the algebra of differential operators on $V$ (see also [4]). In the case when $V$ is the spinor bundle $S$ of a Riemannian spin manifold ( $M, \mathrm{~g}$ ), Getzler [14 used such a map $\mathcal{Q}_{\hbar}^{M}$ to obtain a Weyl quantization on the symplectic supermanifold $T^{*} M \oplus \Pi T M$ and then proved the index theorem for the Riemannian Dirac operator. See [39] for the case where $S=\Omega(M)$. Other related quantization schemes have been applied to even symplectic supermanifolds. See e.g. [5, 26, 27].

[^1]In our situation, as a first step, by taking a metric preserving linear connection on $E$, we can identify any degree 2 symplectic graded manifold as $T^{*}[2] M \oplus E[1]$, where $E$ is a vector bundle over $M$ equipped with a pseudo-metric $g$. According to [30], the degree 2 symplectic form on $T^{*}[2] M \oplus E[1]$ can be written explicitly in terms of the canonical symplectic structure on $T^{*}[2] M$, the pseudo-metric $g$ and the chosen connection. Our first main result is to construct a Weyl type quantization for this graded symplectic manifold $T^{*}[2] M \oplus E[1]$ by a combination of Clifford quantization and classical Weyl quantization. This generalizes the previous works [14, 39. More precisely, we assume that $(E, \mathrm{~g})$ is of even rank and admits a spinor bundle $S$. That is, End $S \cong \mathbb{C l}(E)$ with $\mathbb{C l}(E)=\mathrm{Cl}(E) \otimes \mathbb{C}$ being the complex Clifford bundle. We introduce an increasing and exhaustive filtration of the algebra $\mathcal{D}(M, S)=\bigcup_{k \in \mathbb{N}} \mathcal{D}_{k}(M, S):$

$$
\begin{equation*}
\mathcal{D}_{0}(M, S) \subset \mathcal{D}_{1}(M, S) \subset \cdots \subset \mathcal{D}_{k}(M, S) \subset \cdots \tag{1.2}
\end{equation*}
$$

satisfying the conditions: for any $k, l \in \mathbb{N}$,

$$
\mathcal{D}_{k}(M, S) \cdot \mathcal{D}_{l}(M, S) \subseteq \mathcal{D}_{k+l}(M, S), \quad\left[\mathcal{D}_{k}(M, S), \mathcal{D}_{l}(M, S)\right] \subseteq \mathcal{D}_{k+l-2}(M, S)
$$

As a consequence, the associated graded algebra $\operatorname{gr} \mathcal{D}(M, S)=\bigoplus_{k \in \mathbb{N}} \mathcal{D}_{k}(M, S) / \mathcal{D}_{k-1}(M, S)$, is a graded commutative Poisson algebra of degree -2 , which we prove to be isomorphic to $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus\right.$ $E[1])$. Such an isomorphism enables us to define the principal symbol $\sigma_{k}: \mathcal{D}_{k}(M, S) \rightarrow \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus\right.$ $E[1]), \forall k \in \mathbb{N}$, exactly in the same way as in the classical case. We prove that the Weyl quantization $\operatorname{map} \mathcal{W} \mathcal{Q}_{\hbar}$ establishes an isomorphism $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right) \xrightarrow{\sim} \mathcal{D}(M, S)$, which is the right inverse of the principal symbol map, and satisfies all the desired properties. See Theorem4.9. Note that the filtration (1.2) on $\mathcal{D}(M, S)$ is different from both the usual filtration by the orders of differential operators and Getzler's filtration [14]. In a certain sense, this is the only filtration which turns $\operatorname{gr} \mathcal{D}(M, S)$ into a graded commutative Poisson algebra, with a non-degenerate Poisson bracket. Roughly speaking, it assigns degree 2 to derivations and degree 1 to sections in $\Gamma(E) \subset \Gamma(\mathbb{C l}(E))$.

The second part of the paper is devoted to the application of Weyl quantization to Courant algebroids. The consideration of a specific spinor bundle $\mathbb{S}$, obtained by twisting $S$ by a certain line bundle, allows us to define a conjugation map and an adjoint operation on $\mathcal{D}(M, \mathbb{S})$. A Dirac generating operator is then defined as a real operator in $\mathcal{D}_{3}(M, \mathbb{S})$, which is odd and squares to a function on the base manifold. Following an idea of Ševera [34], we prove that, for a given Courant algebroid, there exists a unique skew-symmetric Dirac generating operator, and moreover the Weyl quantization $\operatorname{map} \mathcal{W} \mathcal{Q}$ establishes a bijection between Hamiltonian generating functions and skew-symmetric Dirac generating operators. This is our second main result.

As an application, by considering the square of the unique skew-symmetric Dirac generating operator, we obtain a new Courant algebroid invariant. This new invariant, as a function on the base manifold, is a natural extension of the square norm of the Cartan 3-form of a quadratic Lie algebra. As another consequence, we recover a result of Chen-Stiénon [9] regarding Dirac generating operators for Lie bialgebroids, which gives an equivalent description of the Lie bialgebroid compatibility condition. In this case, $E=A \oplus A^{*}$ and there are two natural twisted spinor bundles, each of them admitting a Dirac generating operator. For the Lie bialgebroid arising from a generalized complex structure, they coincide with the $\partial$ and $\bar{\partial}$-operators of the generalized complex structure [8].

Some remarks are in order. We learned recently that Li-Bland and Meinrenken also obtained similar results in their study of Dirac generating operators [23].

Notations. Finally, we list the notations used throughout the paper.
$\mathbb{N}=\{0,1,2, \ldots\}$ denotes the set of non-negative integers, $\mathbb{N}^{\times}=\{1,2, \ldots\}$ the set of positive integers and $\mathrm{i}=\sqrt{-1}$. Tensor products over the algebra of real numbers are denoted by $\otimes$ or $\otimes_{\mathbb{R}}$, whereas tensor products over the algebra $R:=\mathrm{C}^{\infty}(M)$ are denoted by $\otimes_{R}$. For a vector space (or a vector bundle) $V$, symmetric and skew-symmetric tensor products are denoted by $\mathcal{S} V$ and $\wedge V$, respectively. We use the Einstein's summation convention without further comments.

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## 2. Symplectic graded manifolds of degree 2

We recall in this section some standard materials concerning symplectic graded manifolds. Our presentation is mainly based on [32, 33, enriched with the work of Rothstein on symplectic supermanifolds 30.
2.1. Definition. A graded manifold $\mathcal{M}$ is a smooth manifold $M$ endowed with a sheaf of $\mathbb{N}$-graded algebras $\mathcal{O}$ such that, for every contractible open set $U \subset M$, the algebra of functions $\mathcal{O}(U)$ is isomorphic to $\mathrm{C}^{\infty}(U) \otimes \mathcal{S} V$, for a fixed $\mathbb{N}^{\times}$-graded vector space $V=\bigoplus_{i \in \mathbb{N}^{\times}} V_{i}$. Here $\mathcal{S} V$ denotes the graded symmetric tensor algebra of $V$ and the grading of $\mathcal{O}(U)$ is induced by the one of $V$. In particular, the degree 0 component of $\mathcal{O}(U)$ is $\mathrm{C}^{\infty}(U)$. The algebra of global sections of $\mathcal{O}$ is called the algebra of functions on $\mathcal{M}$ and denoted by $\mathcal{O}(\mathcal{M})$. Coordinates on $U$ together with a graded basis of $V$ form a local coordinate system $\left(x^{i}\right)$ on $\mathcal{M}$. The grading on the algebra of functions, $\mathcal{O}(\mathcal{M})=\bigoplus_{k \in \mathbb{N}} \mathcal{O}_{k}(\mathcal{M})$, is then given by the decomposition of $\mathcal{O}(\mathcal{M})$ into the eigenspaces of the Euler vector field on $\mathcal{M}$ :

$$
\epsilon=w\left(x^{i}\right) x^{i} \frac{\partial}{\partial x^{i}}
$$

where $w\left(x^{i}\right) \in \mathbb{N}$ denotes the degree of the coordinate $x^{i}$, i.e., $w\left(x^{i}\right)=0$ if $x^{i}$ is a coordinate on $U$ and $w\left(x^{i}\right)=j$ if $x^{i} \in V_{j}$.

Given a smooth vector bundle $E \rightarrow M$ and a positive integer $k \in \mathbb{N}^{\times}$, by $E[k]$, we denote the graded manifold with base $M$, whose algebra of functions is

$$
\mathcal{O}(E[k])= \begin{cases}\Gamma\left(\wedge E^{*}\right) & \text { for } k \text { odd } \\ \Gamma\left(\mathcal{S} E^{*}\right) & \text { for } k \text { even }\end{cases}
$$

Here sections of $E^{*}$ are assigned the degree $k$. The graded manifold $E[k]$ is also called a shifted vector bundle. We will mostly consider graded manifolds built out of shifted vector bundles.

A symplectic graded manifold of degree $n$ is a graded manifold $\mathcal{M}$ endowed with a symplectic form of degree $n$, i.e. a closed non-degenerate 2 -form $\omega$, whose Lie derivative along the Euler vector field $\epsilon$ satisfies $\mathrm{L}_{\epsilon} \omega=n \omega$. The algebra of functions $\mathcal{O}(\mathcal{M})$ admits a Poisson bracket of degree $-n$, i.e. $\left\{\mathcal{O}_{k}(\mathcal{M}), \mathcal{O}_{l}(\mathcal{M})\right\} \subset \mathcal{O}_{k+l-n}(\mathcal{M})$ for all $k, l \in \mathbb{N}$. The Poisson bracket is graded skew-symmetric and satisfies the graded Jacobi identity, namely

$$
\begin{aligned}
\{F, G\} & =-(-1)^{(k-n)(l-n)}\{G, F\}, \\
\{F,\{G, H\}\} & =\{\{F, G\}, H\}+(-1)^{(k-n)(l-n)}\{G,\{F, H\}\},
\end{aligned}
$$

for all $F \in \mathcal{O}_{k}(\mathcal{M}), G \in \mathcal{O}_{l}(\mathcal{M})$ and $H \in \mathcal{O}(\mathcal{M})$.
2.2. Example. In [30], Rothstein gives a description of symplectic supermanifolds in terms of the following data: a pseudo-Euclidean vector bundle ( $E, \mathrm{~g}$ ) over an ordinary symplectic manifold and a metric connection $\nabla$ on $(E, g)$, i.e., a (linear) connection on $E$ satisfying $\nabla \mathrm{g}=0$. In the sequel, we will adapt his construction to the graded context and obtain a symplectic structure of degree 2 on the Whitney sum $T^{*}[2] M \oplus E[1]$ over $M$.

Proposition 2.1 (30]). Let $E \rightarrow M$ be a smooth vector bundle, endowed with a pseudo-Euclidean metric g and a metric connection $\nabla$. Then, the graded manifold $T^{*}[2] M \oplus E[1]$ admits an exact symplectic 2 -form of degree 2 :

$$
\begin{equation*}
\omega_{\mathrm{g}, \nabla}:=\mathrm{d} \alpha \quad \text { where } \quad \alpha=\pi_{1}^{*} \alpha_{0}+\pi_{2}^{*} \beta \tag{2.1}
\end{equation*}
$$

Here $\pi_{1}$ and $\pi_{2}$ are the canonical projections on $T^{*}[2] M$ and $E[1]$ respectively, $\alpha_{0}$ is the Liouville 1-form on $T^{*}[2] M$ and $\beta \in \Omega^{1}(E[1])$ is the 1-form on $E[1]$ which annihilates the horizontal subspace of $T(E[1])$, corresponding to the connection $\nabla$, and satisfies $\beta_{e}\left(v_{e}\right)=\frac{1}{2} \mathrm{~g}_{\pi(e)}\left(v_{e}, e\right)$ for all vertical tangent vectors $v_{e} \in T_{e}(E[1])$.

In what follows, we make repeated use of the identification $E \cong E^{*}$, induced by the metric g , and of the identifications below, without mentioning them explicitly:

$$
\begin{aligned}
& \mathcal{O}_{0}\left(T^{*}[2] M \oplus E[1]\right) \cong \mathrm{C}^{\infty}(M), \\
& \mathcal{O}_{1}\left(T^{*}[2] M \oplus E[1]\right) \cong \Gamma(E) \\
& \mathcal{O}_{2}\left(T^{*}[2] M \oplus E[1]\right) \cong \Gamma\left(T M \oplus \wedge^{2} E\right)
\end{aligned}
$$

The spaces $\mathrm{C}^{\infty}(M), \Gamma(E)$ and $\mathfrak{X}(M) \cong \mathcal{O}_{2}\left(T^{*}[2] M\right)$ generate the entire algebra of functions on $T^{*}[2] M \oplus$ $E[1]$. Hence, by the Leibniz rule, the symplectic structure on $T^{*}[2] M \oplus E[1]$ can be characterized in terms of the following Poisson brackets:

$$
\begin{align*}
\{X, f\} & =X(f), & & \{h, f\}=0 \\
\{X, \xi\} & =\nabla_{X} \xi, & & \{f, \xi\}=0  \tag{2.2}\\
\{X, Y\} & =[X, Y]+R^{E}(X, Y), & & \{\xi, \eta\}=\mathrm{g}(\xi, \eta)
\end{align*}
$$

where $X, Y \in \mathfrak{X}(M), \xi, \eta \in \Gamma(E), f, h \in \mathrm{C}^{\infty}(M)$, and $R^{E}$ denotes the curvature of the connection $\nabla$. Since $\nabla$ is a metric connection, $R^{E}(X, Y)$ defines an element in $\Gamma\left(\wedge^{2} E\right)$, i.e. a degree 2 function on $T^{*}[2] M \oplus E[1]$. From (2.2), it is simple to see that the bracket $\{\cdot, \cdot\}$ is indeed of degree -2 .

Remark 2.2. According to (2.2), E[1] is a Poisson submanifold of $T^{*}[2] M \oplus E[1]$, whose Poisson bracket is induced by the pseudo-metric $g$.

For a vector bundle $E$ over $M$, the cotangent bundle $T^{*}[2](E[1])$ is naturally a degree 2 symplectic graded manifold. Let

$$
\pi_{E}: T^{*}[2](E[1]) \rightarrow E[1]
$$

and

$$
\pi_{E^{*}}: T^{*}[2](E[1]) \cong T^{*}[2]\left(E^{*}[1]\right) \rightarrow E^{*}[1]
$$

be the natural projections. They combine into a map:

$$
\begin{equation*}
\tilde{\pi}: T^{*}[2](E[1]) \longrightarrow\left(E \oplus E^{*}\right)[1] \tag{2.3}
\end{equation*}
$$

A connection $\nabla$ on $E$ gives rise to a horizontal distribution on the tangent bundle $T E$, which in turn induces a surjective submersion

$$
\begin{equation*}
\pi_{\nabla}: T^{*}[2](E[1]) \longrightarrow T^{*}[2] M \tag{2.4}
\end{equation*}
$$

It is simple to see that pro $\tilde{\pi}=\operatorname{pr} \circ \pi_{\nabla}$, where, by abuse of notation, pr denotes both natural projections pr : $\left(E \oplus E^{*}\right)[1] \rightarrow M$ and $T^{*}[2] M \rightarrow M$. Putting together the maps $\tilde{\pi}$ and $\pi_{\nabla}$, we obtain a map:

$$
\begin{equation*}
\widetilde{\Xi}_{\nabla}: T^{*}[2](E[1]) \longrightarrow T^{*}[2] M \oplus\left(E \oplus E^{*}\right)[1] \tag{2.5}
\end{equation*}
$$

It is simple to check that $\widetilde{\Xi}_{\nabla}$ is indeed a diffeomorphism. Note that $E \oplus E^{*}$ is a pseudo-Euclidean bundle over $M$ with the duality pairing. The connection $\nabla$ on $E$ induces a connection on $E \oplus E^{*}$, which is compatible with the duality pairing. According to Proposition 2.1, these data induce a degree 2 symplectic structure on $T^{*}[2] M \oplus\left(E \oplus E^{*}\right)[1]$.
Lemma 2.3. The map $\widetilde{\Xi}_{\nabla}$ in (2.5) is a symplectic diffeomorphism.
Proof. Since $\widetilde{\Xi}_{\nabla}$ is a diffeomorphism, it suffices to prove that it is a Poisson map. As $\mathcal{O}\left(T^{*}[2] M \oplus\right.$ $\left.\left(E \oplus E^{*}\right)[1]\right)$ is generated by $\mathrm{C}^{\infty}(M), \Gamma\left(E \oplus E^{*}\right)$ and $\mathfrak{X}(M)$, it is thus sufficient to check that $\widetilde{\Xi}_{\nabla}^{*}$ preserves the Poisson brackets between elements of these spaces.

It is simple to see that

$$
\begin{align*}
\left(\widetilde{\Xi}_{\nabla}\right)^{*} f=\left(\pi_{E}\right)^{*} f, & \left(\widetilde{\Xi}_{\nabla}\right)^{*} \eta=\left(\pi_{E}\right)^{*} \eta \\
\left(\widetilde{\Xi}_{\nabla}\right)^{*} X=\left(\pi_{\nabla}\right)^{*} X, & \left(\widetilde{\Xi}_{\nabla}\right)^{*} \xi=\left(\pi_{E^{*}}\right)^{*} \xi \tag{2.6}
\end{align*}
$$

for any $f \in \mathrm{C}^{\infty}(M), \eta \in \Gamma(E), \xi \in \Gamma\left(E^{*}\right)$ and $X \in \mathfrak{X}(M)$. Here, both $\left(\pi_{E}\right)^{*} f$ and $\left(\pi_{E}\right)^{*} \eta$ are fiberwise constant functions on $T^{*}[2](E[1])$, while both $\left(\pi_{\nabla}\right)^{*} X$ and $\left(\pi_{E^{*}}\right)^{*} \xi$ are fiberwise linear functions on $T^{*}[2](E[1])$, corresponding to the vector fields $\nabla_{X}$ and $\iota_{\xi}=\mathrm{g}(\xi, \cdot)$ on $E[1]$, respectively. Hence, the Poisson brackets in $T^{*}[2](E[1])$ of the four types of functions in (2.6) are easily computed. On the other hand, for the graded symplectic manifold $T^{*}[2] M \oplus\left(E \oplus E^{*}\right)[1]$, the Poisson brackets of $f, \eta$, $\xi$ and $X$ are given by (2.2), with $E$, g and $\nabla$ being replaced by $E \oplus E^{*}$, the duality pairing and the induced connection, respectively. The conclusion follows from a straightforward verification.
2.3. Classification of symplectic graded manifolds of degree 2. According to 32, any pseudoEuclidean vector bundle ( $E, \mathrm{~g}$ ) determines a degree 2 symplectic graded manifold $\mathcal{M}$, which is defined to be the fiber product $\left(T^{*}[2](E[1])\right) \times_{\left(E \oplus E^{*}\right)[1]} E[1]$. Thus, we have the following commutative diagram:


Here $\tilde{\pi}$ is defined as in (2.3), and $i$ is the diagonal-like embedding $\psi \mapsto \psi \oplus \frac{1}{2} \mathrm{~g}(\psi, \cdot)$. Since $i$ is an isometric embedding, $i_{\mathcal{M}}$ must be an embedding as well. It is simple to see that the restriction to $\mathcal{M}$ of the canonical symplectic form on $T^{*}[2](E[1])$ is non-degenerate. Therefore $\mathcal{M}$ is a symplectic submanifold of $T^{*}[2](E[1])$. It is known that, up to an isomorphism, every degree 2 symplectic graded manifold indeed arises in this way. We refer the interested reader to 32 for details.

Let $\nabla$ be a connection on $E$. Then composing $i_{\mathcal{M}}$ with the map $\pi_{\nabla}$, as defined in (2.4), we obtain a map

$$
\pi_{\nabla} \circ i_{\mathcal{M}}: \mathcal{M} \longrightarrow T^{*}[2] M
$$

Together with the natural projection $\pi: \mathcal{M} \rightarrow E[1]$, we obtain a map

$$
\Xi_{\nabla}: \mathcal{M} \longrightarrow T^{*}[2] M \oplus E[1]
$$

It is simple to check that $\Xi_{\nabla}$ is a diffeomorphism. The following result is known to experts and sketched in 32.

Theorem 2.4. Let $\nabla$ be a metric connection on ( $E, \mathrm{~g}$ ). Then the map $\Xi_{\nabla}$ is a symplectic diffeomorphism, where $T^{*}[2] M \oplus E[1]$ is endowed with the symplectic structure (2.1).

Proof. Consider the map

$$
i_{T}=\operatorname{id}_{T^{*}[2] M} \oplus i: T^{*}[2] M \oplus E[1] \longrightarrow T^{*}[2] M \oplus\left(E \oplus E^{*}\right)[1]
$$

It is simple to check that $i_{T}$ is a symplectic embedding. By definition, we have

$$
\begin{equation*}
\widetilde{\Xi}_{\nabla}=\pi_{\nabla} \oplus \tilde{\pi} \quad \text { and } \quad \Xi_{\nabla}=\pi_{\nabla} \circ i_{\mathcal{M}} \oplus \pi \tag{2.8}
\end{equation*}
$$

According to the relation $\tilde{\pi} \circ i_{\mathcal{M}}=i \circ \pi$ (see diagram (2.7)), we have the following commutative diagram:


Since both $i_{\mathcal{M}}$ and $i_{T}$ are symplectic embeddings and $\widetilde{\Xi}_{\nabla}$ is a symplectic diffeomorphism, it follows that $\Xi_{\nabla}$ must be a symplectic diffeomorphism.

As a consequence, the symplectic graded manifolds $\left(T^{*}[2] M \oplus E[1], \omega_{\mathrm{g}, \nabla}\right)$, associated to different metric connections, are all isomorphic. They provide minimal symplectic realization of the Poisson manifold $E[1]$ [32].

## 3. The algebra $\mathcal{D}(M, S)$ of spinor differential operators

In this section, after recalling some basic materials regarding Clifford algebras and spinor bundles (see e.g. [3, 25, 36]), we introduce a new filtration on the algebra $\mathcal{D}(M, S)$ of spinor differential operators and determine its associated graded Poisson algebra. By considering a specific spinor bundle $\mathbb{S}$, we obtain, in addition, two involutions on $\mathcal{D}(M, \mathbb{S})$.
3.1. Clifford and spinor bundles. Let $(E, g)$ be a pseudo-Euclidean vector bundle of even rank over a smooth manifold $M$. The real Clifford bundle $\mathrm{Cl}(E)$ is a bundle of associative algebras, whose fiber at $x \in M$ is isomorphic to the real Clifford algebra

$$
\mathrm{Cl}\left(E_{x}\right):=\left(\bigotimes E_{x} / \mathcal{I}\right)
$$

where $\mathcal{I}$ is the ideal generated by $\xi_{1}(x) \otimes \xi_{2}(x)+\xi_{2}(x) \otimes \xi_{1}(x)-\mathrm{g}\left(\xi_{1}(x), \xi_{2}(x)\right), \forall \xi_{1}, \xi_{2} \in \Gamma(E)$. In the sequel, we mainly use the complex Clifford bundle $\mathbb{C l}(E):=\mathrm{Cl}(E) \otimes \mathbb{C}$ and refer to it simply as the Clifford bundle.

The Clifford bundle inherits a natural filtration, $\mathbb{C l}(E)=\bigcup_{k \in \mathbb{N}} \mathbb{C l}_{k}(E)$, and a natural $\mathbb{Z}_{2}$-grading,

$$
\mathbb{C l}(E)=\mathbb{C l}^{+}(E) \oplus \mathbb{C l}^{-}(E)
$$

where $\Gamma\left(\mathbb{C l}_{k}(E)\right)$ is spanned by products of at most $k$ sections of $E$, and $\mathbb{C l}^{+}(E)$ (resp. $\mathbb{C l}^{-}(E)$ ) is spanned by products of even (resp. odd) number of sections of $E$. Through the metric g , the algebra $\Gamma(\wedge E \otimes \mathbb{C})$ can be identified with $\mathcal{O}^{\mathbb{C}}(E[1])$, the algebra of complex valued functions on $E[1]$. Moreover, g induces a Poisson bracket $\{\cdot, \cdot\}$ on $E[1]$ (see Remark [2.2). There is a standard $\mathbb{C}$-linear isomorphism, called the Clifford quantization map:

$$
\begin{equation*}
\gamma: \mathcal{O}^{\mathbb{C}}(E[1]) \longrightarrow \Gamma(\mathbb{C l}(E)) . \tag{3.1}
\end{equation*}
$$

It extends the canonical embedding $\Gamma(E) \hookrightarrow \Gamma(\mathbb{C l}(E))$ by skew-symmetrization and satisfies

$$
\begin{equation*}
\gamma(\{\mu, \cdot\})=[\gamma(\mu), \gamma(\cdot)], \quad \forall \mu \in \Gamma\left(E \oplus \wedge^{2} E\right) \tag{3.2}
\end{equation*}
$$

We refer the interested reader to [3, 25, 27] for more details.
By a spinor bundle, we mean a complex vector bundle $S$ over $M$ such that End $S \cong \mathbb{C l}(E)$.
Example 3.1. Let $A$ be a vector bundle, and $E=A \oplus A^{*}$. Let g be the following pairing on $E$ :

$$
\mathrm{g}\left(\zeta_{1}+\eta_{1}, \zeta_{2}+\eta_{2}\right)=\left\langle\zeta_{1}, \eta_{2}\right\rangle+\left\langle\zeta_{2}, \eta_{1}\right\rangle, \quad \forall \zeta_{1}, \zeta_{2} \in \Gamma(A), \eta_{1}, \eta_{2} \in \Gamma\left(A^{*}\right)
$$

where $\langle\cdot, \cdot\rangle$ denotes the duality pairing. Then $\wedge A^{*} \otimes \mathbb{C}($ or $\wedge A \otimes \mathbb{C})$ is a spinor bundle of $(E, \mathrm{~g})$, and the Clifford action is given by

$$
\begin{equation*}
\gamma(\zeta) \phi=\iota_{\zeta} \phi \quad \text { and } \quad \gamma(\eta) \phi=\eta \wedge \phi, \quad \forall \zeta \in \Gamma(A), \eta \in \Gamma\left(A^{*}\right), \phi \in \Gamma\left(\wedge A^{*} \otimes \mathbb{C}\right) \tag{3.3}
\end{equation*}
$$

Note that $\wedge A^{*}$ is a real spinor bundle, i.e., $\operatorname{End}\left(\wedge A^{*}\right) \cong \operatorname{Cl}(E)$.

Remark 3.2. There are certain topological obstructions to the existence of a spinor bundle, which depend on the signature of the metric $g$. The existence of a real spinor bundle imposes a further topological constraint on the pseudo-Euclidean vector bundle $(E, \mathrm{~g})$.

In what follows, we will always assume that a spinor bundle exists, and use the algebra isomorphisms below without mentioning them explicitly

$$
\Gamma(\wedge E \otimes \mathbb{C}) \cong \mathcal{O}^{\mathbb{C}}(E[1]) \quad \text { and } \quad \Gamma(\operatorname{End} S) \cong \Gamma(\mathbb{C l}(E))
$$

Let $\nabla^{E}$ be a connection on $E$. It induces a connection on the exterior bundle $\wedge E$ compatible with the wedge product, i.e., $\nabla_{X}^{E}\left(\xi_{1} \wedge \xi_{2}\right)=\left(\nabla_{X}^{E} \xi_{1}\right) \wedge \xi_{2}+\xi_{1} \wedge\left(\nabla_{X}^{E} \xi_{2}\right), \forall X \in \mathfrak{X}(M), \xi_{1}, \xi_{2} \in \Gamma(\wedge E)$. If $\nabla^{E}$ is a metric connection, it also induces a connection on the Clifford bundle $\mathbb{C l}(E)$ compatible with the Clifford multiplication. The quantization map (3.1) intertwines the induced connections on $\wedge E \otimes \mathbb{C}$ and $\mathbb{C l}(E)$. That is,

$$
\gamma\left(\nabla_{X}^{E} \eta\right)=\nabla_{X}^{E} \gamma(\eta), \quad \forall \eta \in \mathcal{O}^{\mathbb{C}}(E[1]), X \in \mathfrak{X}(M)
$$

Definition 3.3. Assume that $(E, \mathrm{~g})$ admits a spinor bundle $S$. A connection $\nabla^{S}$ on $S$ is called a spinor connection if there exists a metric connection $\nabla^{E}$ on $E$ satisfying the following compatibility condition

$$
\begin{equation*}
\left[\nabla_{X}^{S}, \gamma(\eta)\right]=\gamma\left(\nabla_{X}^{E} \eta\right) \tag{3.4}
\end{equation*}
$$

for all $X \in \mathfrak{X}(M)$ and $\eta \in \mathcal{O}^{\mathbb{C}}(E[1])$. In this case, $\left(\nabla^{E}, \nabla^{S}\right)$ is called a compatible pair of connections.
For compatible connections $\left(\nabla^{E}, \nabla^{S}\right)$, the induced connections on $\mathbb{C l}(E)$ and End $S$ are intertwined by the isomorphism $\mathbb{C l}(E) \cong \operatorname{End} S$.

Lemma 3.4. Assume that $(E, \mathrm{~g})$ admits a spinor bundle $S$. Then,
(i) there always exist compatible connections $\left(\nabla^{E}, \nabla^{S}\right)$;
(ii) there exists $r \in \Omega^{2}(M) \otimes \mathbb{C}$ such that the curvature $R^{E}$ of $\nabla^{E}$ and the curvature $R^{S}$ of $\nabla^{S}$ satisfy

$$
\begin{equation*}
R^{S}=\gamma\left(R^{E}\right)+r \mathrm{id}_{S} \tag{3.5}
\end{equation*}
$$

(iii) any two pairs of compatible connections $\left(\tilde{\nabla}^{E}, \tilde{\nabla}^{S}\right)$ and $\left(\nabla^{E}, \nabla^{S}\right)$ are related as follows

$$
\tilde{\nabla}^{E}-\nabla^{E}=\{\varpi, \cdot\} \quad \text { and } \quad \tilde{\nabla}^{S}-\nabla^{S}=\gamma(\varpi)+\nu \operatorname{id}_{S}
$$

where $\varpi \in \Omega^{1}\left(M, \wedge^{2} E\right)$ and $\nu \in \Omega^{1}(M) \otimes \mathbb{C}$. Here, $\{\cdot, \cdot\}$ stands for the Poisson bracket on $E[1]$.

Proof. The existence of a metric connection $\nabla^{E}$ is classical. Such a connection induces a connection on $\mathbb{C l}(E) \cong \operatorname{End} S$. According to [36, the latter is always induced by a connection $\nabla^{S}$ on $S$. Thus $\left(\nabla^{E}, \nabla^{S}\right)$ is a pair of compatible connections.

Let $X, Y \in \mathfrak{X}(M)$. Since $\nabla^{E}$ preserves the metric, its curvature can be identified with $R^{E} \in$ $\Omega^{2}\left(M, \wedge^{2} E\right)$, via the following relation

$$
\left(\nabla_{X}^{E} \nabla_{Y}^{E}-\nabla_{Y}^{E} \nabla_{X}^{E}-\nabla_{[X, Y]}^{E}\right) \xi=\left\{R^{E}(X, Y), \xi\right\}, \quad \forall \xi \in \Gamma(E)
$$

The same relation holds by replacing $\xi \in \Gamma(E)$ with $\eta \in \Gamma(\wedge E \otimes \mathbb{C})$. Using the compatibility condition (3.4), we deduce that

$$
\left[R^{S}(X, Y), \gamma(\eta)\right]-\gamma\left(\left\{R^{E}(X, Y), \eta\right\}\right)=0, \quad \forall \eta \in \Gamma(\wedge E \otimes \mathbb{C})
$$

Hence, by Eq. (3.2), we have

$$
\left[R^{S}(X, Y)-\gamma\left(R^{E}(X, Y)\right), \gamma(\eta)\right]=0, \quad \forall \eta \in \Gamma(\wedge E \otimes \mathbb{C})
$$

The center of $\Gamma(\mathbb{C l}(E))$ being the algebra $\mathrm{C}^{\infty}(M) \otimes \mathbb{C}$, Eq. (3.5) thus follows.
We now compare two pairs of compatible connections $\left(\tilde{\nabla}^{E}, \tilde{\nabla}^{S}\right)$ and $\left(\nabla^{E}, \nabla^{S}\right)$. First, we have $\tilde{\nabla}^{E}-\nabla^{E}=\mu$, for some $\mu \in \Omega^{1}(M$, End $E)$. As both connections preserve the metric g, so does $\mu$ and we have $\mu=\{\varpi, \cdot\}$ with $\varpi \in \Omega^{1}\left(M, \wedge^{2} E\right)$. Similarly, we have $\tilde{\nabla}^{S}-\nabla^{S}=A$ with $A \in \Omega^{1}(M, \operatorname{End} S)$. The compatibility condition (3.4) implies that $[A, \cdot]=[\gamma(\varpi), \cdot]$ on $\Gamma(\operatorname{End} S)$. This yields $A=\gamma(\varpi)+\nu$ for some $\nu \in \Omega^{1}(M) \otimes \mathbb{C}$.
3.2. The filtered algebra $\mathcal{D}(M, S)$ and its associated graded algebra. Assume that ( $E, \mathrm{~g}$ ) admits a spinor bundle $S$. The algebra $\mathcal{D}(M, S)$ of differential operators on $S$ is a subalgebra of $\operatorname{End}(\Gamma(S))$ generated by $\Gamma(\operatorname{End} S) \cong \Gamma(\mathbb{C l}(E))$ and the covariant derivatives $\nabla_{X}$, where $\nabla$ is any connection on $S$ and $X$ ranges over all vector fields on $M$. There is a $\mathbb{Z}_{2}$-grading on $\mathcal{D}(M, S)$, inherited from $\mathbb{C l}(E)$,

$$
\begin{equation*}
\mathcal{D}(M, S)=\mathcal{D}^{+}(M, S) \oplus \mathcal{D}^{-}(M, S) \tag{3.6}
\end{equation*}
$$

where $\mathcal{D}^{ \pm}(M, S)$ is generated by the covariant derivatives $\nabla_{X}$ and $\Gamma\left(\mathbb{C l}^{ \pm}(E)\right)$. By $[\cdot, \cdot]$, we denote the graded commutator:

$$
[A, B]=A \circ B-(-1)^{|A||B|} B \circ A
$$

for any $A, B \in \mathcal{D}(M, S)$ of the parity $|A|,|B| \in\{0,1\}$, respectively.
We now introduce a sequence of subspaces of $\mathcal{D}(M, S)$ inductively as follows. Set $\mathcal{D}_{-1}(M, S):=$ $\{0\}$, and

$$
\begin{equation*}
\mathcal{D}_{k}(M, S):=\left\{D \in \mathcal{D}(M, S) \mid \forall \xi \in \Gamma(E),[D, \gamma(\xi)] \in \mathcal{D}_{k-1}(M, S)\right\} \tag{3.7}
\end{equation*}
$$

Note that the ordinary filtration on $\mathcal{D}(M, S)$, given by the differential order, can be defined in a similar fashion replacing $\Gamma(E)$ by $\mathrm{C}^{\infty}(M)$.

Proposition 3.5. Let $\nabla^{S}$ be a spinor connection on $S$. For all $k \in \mathbb{N}$, we have

$$
\begin{equation*}
\mathcal{D}_{k}(M, S)=\operatorname{span}\left\{\gamma(\eta) \circ \nabla_{X_{1}}^{S} \cdots \nabla_{X_{m}}^{S} \mid 2 m+\operatorname{deg}(\eta) \leq k\right\} \tag{3.8}
\end{equation*}
$$

where $X_{1}, \ldots, X_{m} \in \mathfrak{X}(M)$ and $\operatorname{deg}(\eta)$ denotes the degree of $\eta \in \mathcal{O}^{\mathbb{C}}(E[1])$.
Proof. The proof is by induction. Set

$$
\widetilde{\mathcal{D}_{k}}(M, S)=\operatorname{span}\left\{\gamma(\eta) \circ \nabla_{X_{1}}^{S} \cdots \nabla_{X_{m}}^{S} \mid 2 m+\operatorname{deg}(\eta) \leq k\right\}
$$

By definition, $\mathcal{D}_{-1}(M, S)=\widetilde{\mathcal{D}_{-1}}(M, S)=\{0\}$. Assume that $\mathcal{D}_{k}(M, S)=\widetilde{\mathcal{D}_{k}}(M, S)$ holds for a given $k \geq-1$. Since $\nabla^{S}$ is a spinor connection, there exists a metric connection $\nabla^{E}$ on $E$ such that the following identities hold:

$$
\begin{equation*}
\left[\nabla_{X}^{S}, \gamma(\xi)\right]=\gamma\left(\nabla_{X}^{E} \xi\right) \quad \text { and } \quad[\gamma(\eta), \gamma(\xi)]=\mathrm{g}(\eta, \xi), \quad \forall \xi, \eta \in \Gamma(E), X \in \mathfrak{X}(M) \tag{3.9}
\end{equation*}
$$

Hence, if $D \in \widetilde{\mathcal{D}_{k+1}}(M, S)$, we have $[D, \gamma(\xi)] \in \widetilde{\mathcal{D}_{k}}(M, S)$ for all $\xi \in \Gamma(E)$. The inclusion $\widetilde{\mathcal{D}_{k+1}}(M, S) \subset$ $\mathcal{D}_{k+1}(M, S)$ thus follows. To prove the converse inclusion we need a lemma.

Let $\left(x^{i}\right)$ be a local coordinate system on $M$. For any multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right), \nabla_{\alpha}^{S}$ stands for the composition $\nabla_{\alpha_{1}}^{S} \circ \cdots \circ \nabla_{\alpha_{k}}^{S}$, where $\nabla_{i}^{S}:=\nabla_{\frac{\partial}{\partial x^{i}}}^{S}$.
Lemma 3.6. Any operator $D \in \mathcal{D}(M, S)$ locally admits a unique linear decomposition

$$
D=\sum_{\kappa, \alpha} \gamma\left(\eta_{\kappa}^{\alpha}\right) \nabla_{\alpha}^{S}
$$

where $\eta_{\kappa}^{\alpha} \in \Gamma\left(\wedge^{\kappa} E \otimes \mathbb{C}\right)$. In the sum above, $\kappa$ runs over $\mathbb{N}$ and $\alpha$ runs over ordered multi-indices of arbitrary length $|\alpha|=k \in \mathbb{N}$, i.e. $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ and $1 \leq \alpha_{1} \leq \cdots \leq \alpha_{k} \leq \operatorname{dim} M$.

Proof. It is well-known that $\mathcal{D}(M, S)$ is a locally free $\mathrm{C}^{\infty}(M)$-module, generated as an algebra by $\nabla_{i}^{S}$ and $\gamma(\eta)$, with $\eta \in \Gamma\left(\wedge^{\kappa} E \otimes \mathbb{C}\right), \kappa \in \mathbb{N}$. The result follows from Eqns (3.4)-(3.5).

We assume now that $D \in \mathcal{D}_{k+1}(M, S)$. There exists a minimum $l \in \mathbb{N}$ such that $D \in \widetilde{\mathcal{D}_{l}}(M, S)$. We prove that $l-1 \leq k$. By Lemma 3.6 we have $D=\sum_{\kappa+2|\alpha| \leq l} \gamma\left(\eta_{\kappa}^{\alpha}\right) \nabla_{\alpha}^{S}$. From the definition of $\mathcal{D}_{k+1}(M, S)$, we deduce that $[D, \gamma(\xi)] \in \widetilde{D_{k}}(M, S)$ and $\gamma(\xi) \cdot[D, f]=[D, \gamma(\xi) f]-[D, \gamma(\xi)] f \in$ $\widetilde{D_{k}}(M, S)$. These relations read respectively as

$$
\begin{align*}
& \sum_{\kappa+2|\alpha| \leq l}\left[\gamma\left(\eta_{\kappa}^{a}\right), \gamma(\xi)\right] \cdot \nabla_{\alpha}^{S}+\gamma\left(\eta_{\kappa}^{\alpha}\right) \cdot\left[\nabla_{\alpha}^{S}, \gamma(\xi)\right] \in \widetilde{D_{k}}(M, S),  \tag{3.10}\\
& \sum_{\kappa+2|\alpha| \leq l} \gamma(\xi) \gamma\left(\eta_{\kappa}^{\alpha}\right) \cdot\left[\nabla_{\alpha}^{S}, f\right] \in \widetilde{D_{k}}(M, S), \quad \forall \xi \in \Gamma(E), \tag{3.11}
\end{align*}
$$

Consider $\kappa$ and $\alpha$ such that $\eta_{\kappa}^{\alpha}$ is non-vanishing, $\kappa+2|\alpha|=l$ and $\alpha$ is maximal. The term $\left[\gamma\left(\eta_{\kappa}^{a}\right), \gamma(\xi)\right]$. $\nabla_{\alpha}^{S}$ is then clearly independent of the other terms in (3.10), hence it pertains to $\widetilde{D_{k}}(M, S)$. If $\kappa \neq 0$, there exists $\xi \in \Gamma(E)$ such that $0 \neq\left[\gamma\left(\eta_{\kappa}^{a}\right), \gamma(\xi)\right] \in \Gamma\left(\wedge^{\kappa-1} E\right)$, and we deduce that $l-1 \leq k$. If $\kappa=0$, the same conclusion follows from (3.11).

According to Proposition [3.5, we have $\gamma(\xi) \in \mathcal{D}_{1}(M, S)$ and $\nabla_{X}^{S} \in \mathcal{D}_{2}(M, S), \forall \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Moreover, the operators in $\mathcal{D}_{0}(M, S)$ are multiplication by functions in $\mathrm{C}^{\infty}(M) \otimes \mathbb{C}$. The following proposition is a direct consequence of (3.7)-(3.8).

Proposition 3.7. The subspaces $\left\{\mathcal{D}_{k}(M, S)\right\}_{k \in \mathbb{N}}$ define an increasing and exhaustive filtration of the algebra $\mathcal{D}(M, S)$ :

$$
\begin{aligned}
\mathcal{D}_{0}(M, S) \subset \mathcal{D}_{1}(M, S) & \subset \cdots \subset \mathcal{D}_{k}(M, S) \subset \cdots \\
\mathcal{D}(M, S) & =\bigcup_{k \in \mathbb{N}} \mathcal{D}_{k}(M, S)
\end{aligned}
$$

Moreover, for any $k, l \in \mathbb{N}$, we have

$$
\mathcal{D}_{k}(M, S) \cdot \mathcal{D}_{l}(M, S) \subseteq \mathcal{D}_{k+l}(M, S), \quad\left[\mathcal{D}_{k}(M, S), \mathcal{D}_{l}(M, S)\right] \subseteq \mathcal{D}_{k+l-2}(M, S)
$$

Remark 3.8. Note that in [15, 26, 27], the same choice of filtration on $\mathcal{D}(M, S)$ was made, whereas in [14] and [39, a different choice was used: both Clifford generators $\gamma(\xi), \xi \in \Gamma(E)$, and covariant derivatives $\nabla_{X}^{S}, X \in \mathfrak{X}(M)$, are of order 1 . In the more standard filtration, however, the Clifford generators normally have order 0 .

As a consequence of Proposition 3.7, the graded algebra

$$
\operatorname{gr} \mathcal{D}(M, S)=\bigoplus_{k \in \mathbb{N}} \mathcal{D}_{k}(M, S) / \mathcal{D}_{k-1}(M, S)
$$

is a graded commutative Poisson algebra of degree -2 . By $\varsigma$, we denote the canonical projection $\varsigma: \mathcal{D}(M, S) \rightarrow \operatorname{gr} \mathcal{D}(M, S)$.

Theorem 3.9. Let $S$ be a spinor bundle of $(E, g)$ and $\left(\nabla^{E}, \nabla^{S}\right)$ a pair of compatible connections. There exists a unique isomorphism of graded commutative Poisson $\mathbb{C}$-algebras:

$$
\mathfrak{S}_{\nabla}: \operatorname{gr} \mathcal{D}(M, S) \xrightarrow{\sim} \mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)
$$

satisfying

$$
\begin{equation*}
\left(\mathfrak{S}_{\nabla} \circ \varsigma\right)(f)=f, \quad\left(\mathfrak{S}_{\nabla} \circ \varsigma\right)(\gamma(\xi))=\xi, \quad\left(\mathfrak{S}_{\nabla} \circ \varsigma\right)\left(\nabla_{X}^{S}\right)=X \tag{3.12}
\end{equation*}
$$

for all $f \in \mathrm{C}^{\infty}(M), \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Moreover, the isomorphism $\mathfrak{S}_{\nabla}$ only depends on the connection $\nabla^{E}$.

Proof. Both algebras $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$ and $\operatorname{gr} \mathcal{D}(M, S)$ are generated by $\Gamma(E)$ and $\mathfrak{X}(M)$ over $\mathrm{C}^{\infty}(M)$ (see Proposition (3.5). Hence, a $\mathbb{C}$-algebra morphism $\mathfrak{S}_{\nabla}$ satisfying (3.12) is an isomorphism and must be unique if it exists. As a result, it suffices to prove the existence of $\mathfrak{S}_{\nabla}$ locally. We use canonical coordinates $\left(x^{i}, p_{i}\right)$ on $T^{*} M$, and write $\nabla_{i}^{S}:=\nabla_{\frac{\partial}{\partial x^{i}}}^{S}$. We define a map $\mathfrak{S}_{\nabla}$ locally by setting

$$
\mathfrak{S}_{\nabla} \circ \varsigma\left(\gamma(\eta) \nabla_{\alpha}^{S}\right)=\eta p_{\alpha}
$$

where $\eta \in \Gamma(\wedge E \otimes \mathbb{C}), \alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ is an ordered multi-index and $p_{\alpha}=p_{\alpha_{1}} \cdots p_{\alpha_{k}}$. According to Lemma 3.6 the map $\mathfrak{S}_{\nabla}$ is well-defined. Moreover, by (3.9), this is indeed a $\mathbb{C}$-algebra morphism which satisfies (3.12).

To prove that $\mathfrak{S}_{\nabla}$ is a Poisson map, it suffices to check that $\mathfrak{S}_{\nabla}$ preserves the Poisson brackets on generators, i.e. on elements in $\mathrm{C}^{\infty}(M) \oplus \Gamma(E) \oplus \mathfrak{X}(M)$. For any $f, g \in \mathrm{C}^{\infty}(M), \xi, \eta \in \Gamma(E)$, and $X, Y \in \mathfrak{X}(M)$, we have the following graded commutators of differential operators

$$
\begin{array}{ll}
{\left[\nabla_{X}^{S}, f\right]=X(f),} & {[g, f]=0} \\
{\left[\nabla_{X}^{S}, \gamma(\xi)\right]=\gamma\left(\nabla_{X}^{E} \xi\right),} & {[g, \gamma(\xi)]=0}  \tag{3.13}\\
{\left[\nabla_{X}^{S}, \nabla_{Y}^{S}\right]=\nabla_{[X, Y]}^{S}+R^{S}(X, Y),} & {[\gamma(\xi), \gamma(\eta)]=\mathrm{g}(\xi, \eta)}
\end{array}
$$

where we have used Eq. (3.4). Writing the curvature $R^{S}$ as in Eq. (3.5) and applying the map $\mathfrak{S}_{\nabla} \circ \varsigma$, we immediately see that these commutators coincide with the generating relations of the Poisson bracket as given in (2.2). Therefore, $\mathfrak{S}_{\nabla}$ is indeed a Poisson map.

According to Lemma [3.4, two spinor connections, compatible with the same metric connection $\nabla^{E}$, satisfy $\widetilde{\nabla}^{S}-\nabla^{S}=\nu \in \Omega^{1}(M) \otimes \mathbb{C}$. However, different $\nu$ do not modify the defining relations for $\mathfrak{S}_{\nabla}$ in (3.12). This concludes the proof of the theorem.

Definition 3.10. We define the principal symbol map $\sigma_{k}: \mathcal{D}_{k}(M, S) \rightarrow \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right), \forall k \in \mathbb{N}$, as the composition $\mathfrak{S}_{\nabla} \circ \varsigma$.

Note that the principal symbol map $\sigma_{k}$ depends on the choice of the connection $\nabla^{E}$.

Proposition 3.11. Let $k, l \in \mathbb{N}$. The principal symbol maps satisfy the following properties:

$$
\begin{align*}
\sigma_{k+l}(A B) & =\sigma_{k}(A) \sigma_{l}(B) \\
\sigma_{k+l-2}([A, B]) & =\left\{\sigma_{k}(A), \sigma_{l}(B)\right\}, \tag{3.14}
\end{align*}
$$

for all $A \in \mathcal{D}_{k}(M, S)$ and $B \in \mathcal{D}_{l}(M, S)$.
Proof. This is a direct consequence of Theorem 3.9,
Remark 3.12. It is helpful to compare our filtration with those in [14, 39, as shown in the table below:

| Filtration | order $\gamma(\xi)$ | order $\nabla_{X}^{S}$ | $\operatorname{gr} \mathcal{D}(M, S)$ |
| :---: | :---: | :---: | :---: |
| standard filtration | 0 | 2 | $\mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma(\operatorname{End} S)$ |
| filtration in [14, [39] | 2 | 2 | $\left(\Gamma(\otimes T M) \otimes_{R} \mathcal{O}^{\mathbb{C}}(E[1])\right) / \mathcal{J}$ |
| filtration in (3.7) | 1 | 2 | $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$ |

where $\xi \in \Gamma(E), X, Y \in \mathfrak{X}(M)$ and $\mathcal{J}=\left(X \otimes Y-Y \otimes X-R^{E}(X, Y)\right)$. As before, we have $R^{E}(X, Y) \in \Gamma\left(\wedge^{2} E\right)$. The filtration (3.7) is the only one for which $\operatorname{gr} \mathcal{D}(M, S)$ is a graded commutative algebra.

Recall that, according to Theorem [2.4, there exists a symplectic diffeomorphism $\Xi_{\nabla}: \mathcal{M} \rightarrow$ $T^{*}[2] M \oplus E[1]$. Let $\rho_{\nabla}=\left(\Xi_{\nabla}\right)^{*} \circ \mathfrak{S}_{\nabla}$. We have the following

Proposition 3.13. The map $\rho_{\nabla}: \operatorname{gr} \mathcal{D}(M, S) \rightarrow \mathcal{O}^{\mathbb{C}}(\mathcal{M})$ is an isomorphism of graded Poisson algebras, which is independent of the metric connection $\nabla^{E}$ used in defining $\Xi_{\nabla}$ and $\mathfrak{S}_{\nabla}$.

Proof. Let $\left(\nabla^{E}, \nabla^{S}\right)$ be a pair of compatible connections, and $\Xi_{\nabla}, \mathfrak{S}_{\nabla}$ their associated maps as defined in Theorems 2.4 and 3.9. Since both $\left(\Xi_{\nabla}\right)^{*}$ and $\mathfrak{S}_{\nabla}$ are isomorphisms of graded Poisson algebras, so is their composition $\rho_{\nabla}$. It remains to prove that $\rho_{\nabla}$ is independent of the choice of the connections $\left(\nabla^{E}, \nabla^{S}\right)$. Since $\rho_{\nabla}$ is an algebra isomorphism, it suffices to prove the assertion on generators of $\operatorname{gr} \mathcal{D}(M, S)$, which are of the following three types: $f, \varsigma \circ \gamma(\xi)$ and $\varsigma\left(\nabla_{X}^{S}\right), \forall f \in \mathrm{C}^{\infty}(M), \xi \in \Gamma(E)$, $X \in \mathfrak{X}(M)$. By definition, both maps $\left(\Xi_{\nabla}\right)^{*}$ and $\mathfrak{S}_{\nabla}$ are independent of the choice of connections when applying to elements $f$ and $\varsigma \circ \gamma(\xi)$. Thus it remains to show that $\rho_{\nabla}$ is also independent of the choice of connections on the elements $\varsigma\left(\nabla_{X}^{S}\right), \forall X \in \mathfrak{X}(M)$.

Assume that $\left(\widetilde{\nabla}^{E}, \widetilde{\nabla}^{S}\right)$ is another pair of compatible connections. From (2.8) and (3.12), it follows that

$$
\begin{align*}
\rho_{\nabla} \circ \varsigma\left(\nabla_{X}^{S}\right) & =\left(\pi_{\nabla} \circ i_{\mathcal{M}}\right)^{*} l_{X} \\
\rho_{\nabla} \circ \varsigma\left(\widetilde{\nabla}_{X}^{S}\right) & =\left(\pi_{\widetilde{\nabla}} \circ i_{\mathcal{M}}\right)^{*} l_{X} \tag{3.15}
\end{align*}
$$

where $l_{X}$ denotes the fiberwise linear function on $T^{*}[2] M$ corresponding to $X \in \mathfrak{X}(M)$. According to Lemma 3.4 there exists $\varpi \in \Omega^{1}\left(M, \wedge^{2} E\right)$ and $\nu \in \Omega^{1}(M) \otimes \mathbb{C}$ such that

$$
\begin{align*}
& \widetilde{\nabla}_{X}^{S}-\nabla_{X}^{S}=\gamma(\varpi(X))+\nu(X),  \tag{3.16}\\
& \widetilde{\nabla}_{X}^{E}-\nabla_{X}^{E}=\{\varpi(X), \cdot\} \tag{3.17}
\end{align*}
$$

Eq. (3.17) implies that

$$
\begin{equation*}
\left(\pi_{\widetilde{\nabla}} \circ i_{\mathcal{M}}\right)^{*} l_{X}-\left(\pi_{\nabla} \circ i_{\mathcal{M}}\right)^{*} l_{X}=\pi^{*} \varpi(X) \tag{3.18}
\end{equation*}
$$

Therefore, by Eqns (3.15) and (3.18), we have

$$
\rho_{\widetilde{\nabla}} \circ \varsigma\left(\widetilde{\nabla}_{X}^{S}\right)=\rho_{\nabla} \circ \varsigma\left(\nabla_{X}^{S}\right)+\pi^{*} \varpi(X)
$$

On the other hand, Eq. (3.16) implies that

$$
\rho_{\nabla} \circ \varsigma\left(\widetilde{\nabla}_{X}^{S}\right)=\rho_{\nabla} \circ \varsigma\left(\nabla_{X}^{S}\right)+\pi^{*} \varpi(X) .
$$

As a consequence, we conclude that the equality $\rho_{\widetilde{\nabla}}=\rho_{\nabla}$ holds on every element $\varsigma\left(\widetilde{\nabla}_{X}^{S}\right), \forall X \in \mathfrak{X}(M)$. This concludes the proof of the proposition.
3.3. Two involutions on $\mathcal{D}(M, \mathbb{S})$. Assume that the line bundle ( $\left.\operatorname{det} S^{*}\right)^{1 / N}$ exists, with $N$ being the rank of $S$. Consider the twisted spinor bundle $\mathbb{S}:=S \otimes\left(\operatorname{det} S^{*}\right)^{1 / N} \otimes\left|\wedge^{\text {top }} T^{*} M\right|^{1 / 2}$, where $\left|\wedge^{\text {top }} T^{*} M\right|^{1 / 2}$ denotes the half-density line bundle of $M$. A connection on $T M$ induces a connection on $\left|\wedge^{\text {top }} T^{*} M\right|^{1 / 2}$. Together with a spinor connection $\nabla^{S}$ on $S$, it yields an induced spinor connection on the twisted spinor bundle $\mathbb{S}$, denoted by $\nabla^{\mathbb{S}}$. Since $\mathbb{S}$ is a spinor bundle, the algebra $\mathcal{D}(M, \mathbb{S})$ of differential operators on $\mathbb{S}$, satisfies in particular all the results proved in Section 3.2 for the algebra of spinor differential operators. Considering $\mathbb{S}$ rather than an arbitrary spinor bundle allows us to have additional structures on $\mathcal{D}(M, \mathbb{S})$.

First, we introduce an adjoint operation on the algebra $\mathcal{D}(M, \mathbb{S})$. Let $U \subset M$ be a contractible open subset. A pseudo-Hermitian pairing on $\left.\mathbb{S}\right|_{U}$ is a fiberwise non-degenerate sesquilinear map

$$
\begin{equation*}
\langle\cdot, \cdot\rangle_{U}: \Gamma\left(\left.\mathbb{S}\right|_{U}\right) \times \Gamma\left(\left.\mathbb{S}\right|_{U}\right) \longrightarrow \Gamma\left(\left|\wedge^{\mathrm{top}} T^{*} U\right|\right) \otimes \mathbb{C} \tag{3.19}
\end{equation*}
$$

satisfying $\overline{\langle\phi, \psi\rangle_{U}}=\langle\psi, \phi\rangle_{U}$ for all $\phi, \psi \in \Gamma\left(\left.\mathbb{S}\right|_{U}\right)$.
Proposition 3.14. Up to multiplication by a scalar $a \in \mathbb{R}^{\times}$, there exists a unique smooth pseudoHermitian pairing $\langle\cdot, \cdot\rangle_{U}$ on $\mathbb{S}_{U}$ satisfying the following properties:
(i) $\nabla_{X}\left(\langle\phi, \psi\rangle_{U}\right)=\left\langle\nabla_{X}^{\mathbb{S}} \phi, \psi\right\rangle_{U}+\left\langle\phi, \nabla_{X}^{\mathbb{S}} \psi\right\rangle_{U}$,
(ii) $\langle\gamma(\xi) \phi, \psi\rangle_{U}=\langle\phi, \gamma(\xi) \psi\rangle_{U}$,
for any $X \in \Gamma(T U), \xi \in \Gamma\left(\left.E\right|_{U}\right)$ and $\phi, \psi \in \Gamma\left(\left.\mathbb{S}\right|_{U}\right)$.
Moreover, $\langle\cdot, \cdot\rangle_{U}$ is independent of the choice of connections $\nabla$ and $\nabla^{\mathbb{S}}$ involved in (i).
Proof. Choosing a local frame $\left(\xi^{i}\right)$ of $\left.E\right|_{U}$ such that $\mathrm{g}\left(\xi^{i}, \xi^{j}\right)= \pm \delta_{i j}$, we obtain a trivialization of the bundle $\left.E\right|_{U} \cong U \times \mathbb{R}^{n}$, on which g becomes a constant metric. This in turn induces a trivialization of $\mathbb{C l}\left(\left.E\right|_{U}\right) \cong \operatorname{End}\left(\left.S\right|_{U}\right)$, for which $\gamma\left(\xi^{i}\right) \in \Gamma\left(\operatorname{End}\left(\left.S\right|_{U}\right)\right) \cong C^{\infty}\left(U, \operatorname{End} \mathbb{C}^{N}\right)$ are constant sections, for all $i=1, \cdots n$. Let $\left.S\right|_{U} \cong U \times \mathbb{C}^{N}$ be a trivialization defined by a local frame $\left(\phi_{a}\right)$ of $S$. The action of $\operatorname{End}\left(\left.S\right|_{U}\right)$ on $\left.S\right|_{U}$ translates into a family of faithful representations $\rho_{x}: \operatorname{End} \mathbb{C}^{N} \times \mathbb{C}^{N} \longrightarrow \mathbb{C}^{N}$, depending smoothly on $x \in U$. Each representation defines an automorphism of End $\mathbb{C}^{N}$. Since automorphisms of End $\mathbb{C}^{N}$ are all inner automorphisms, there exists a unique element $\alpha_{x} \in \operatorname{PGL}(N, \mathbb{C})$ such that $\rho_{x}=\operatorname{Ad}_{\alpha_{x}}$. It is clear that $x \rightarrow \alpha_{x}$ is a smooth map from $U$ to $\operatorname{PGL}(N, \mathbb{C})$. Shrinking $U$ if necessary, we can choose a smooth lift $x \rightarrow A_{x}$ from $U$ to GL $(N, \mathbb{C})$. Then $\rho_{x}(M) \cdot \phi_{0}=A_{x} M A_{x}^{-1} \cdot \phi_{0}$, for all $x \in U, M \in \operatorname{End} \mathbb{C}^{N}$ and $\phi_{0} \in \mathbb{C}^{N}$. Thus $\left(A_{x} \phi_{a}\right)$ is another local frame of $S$, under which the
trivialization of $\left.S\right|_{U}$ is consistent with the trivialization of $\operatorname{End}\left(\left.S\right|_{U}\right)$ induced by the trivialization of $\mathbb{C l}\left(\left.E\right|_{U}\right)$ at the beginning of the proof. We pick up such a trivialization of $\left.S\right|_{U}$ for the rest of the proof, so that the operators $\gamma\left(\xi^{i}\right) \in \Gamma\left(\operatorname{End}\left(\left.S\right|_{U}\right)\right) \cong C^{\infty}\left(U, \operatorname{End} \mathbb{C}^{N}\right)$ acts as constant endomorphisms on $\left.S\right|_{U} \cong U \times \mathbb{C}^{N}$, for all $i=1, \cdots n$. Consider the induced trivialization of the line bundle $\left(\left.\operatorname{det} S^{*}\right|_{U}\right)^{1 / N}$ and pick up a trivialization of $\left|\wedge^{\text {top }} T^{*} U\right|$. We can identify the space of sections of $\left|\wedge^{\text {top }} T^{*} U\right|$ with $C^{\infty}(U)$, and the space of sections of $\mathbb{S}$ with $C^{\infty}\left(U, \mathbb{C}^{N}\right)$. Under such identifications, the connections $\nabla$ and $\nabla^{\mathbb{S}}$ can be written as follows:

$$
\begin{align*}
& \nabla_{X} v=X v+\nu_{0}(X) v, \quad \forall v \in \Gamma\left(\left|\wedge^{\mathrm{top}} T^{*} U\right|\right) \cong C^{\infty}(U) \\
& \nabla_{X}^{\mathbb{S}} \phi=X \phi+\gamma(\varpi(X)) \phi+\frac{1}{2} \nu_{0}(X) \phi, \quad \forall \phi \in \Gamma\left(\left.\mathbb{S}\right|_{U}\right) \cong C^{\infty}\left(U, \mathbb{C}^{N}\right) \tag{3.20}
\end{align*}
$$

where $\nu_{0} \in \Omega^{1}(U)$ and $\varpi \in \Omega^{1}\left(U,\left.\wedge^{2} E\right|_{U}\right)$ (see Lemma 3.4). If Property (ii) holds, it is then simple to check that Property (i) is equivalent to

$$
\begin{equation*}
X\left(\langle\phi, \psi\rangle_{U}\right)=\langle X \phi, \psi\rangle_{U}+\langle\phi, X \psi\rangle_{U}, \quad \forall \phi, \psi \in C^{\infty}\left(U, \mathbb{C}^{N}\right), X \in \Gamma(T U) \tag{3.21}
\end{equation*}
$$

The latter equation means that $\langle\cdot, \cdot\rangle_{U}$ is a constant pairing in the trivialization $\left.\mathbb{S}\right|_{U} \cong U \times \mathbb{C}^{N}$.
Let $x \in U$ and $\beta$ be the semi-linear involution of the complex Clifford algebra $\mathbb{C l}\left(E_{x}\right) \cong$ End $\mathbb{S}_{x}$ defined by

$$
\beta\left(\gamma\left(\xi^{i_{1}}\right) \gamma\left(\xi^{i_{2}}\right) \cdots \gamma\left(\xi^{i_{\kappa}}\right)\right)=\gamma\left(\xi^{i_{\kappa}}\right) \cdots \gamma\left(\xi^{i_{2}}\right) \gamma\left(\xi^{i_{1}}\right), \quad \forall i_{1}, i_{2}, \ldots, i_{\kappa} \in\{1,2, \ldots, n\}
$$

Then, $\langle\cdot, \cdot\rangle_{U}$ satisfies Property (ii) over $x$ if and only if its adjoint operation on End $\mathbb{S}_{x} \cong \mathbb{C l}\left(E_{x}\right)$ is given by $\beta$. The existence of a pseudo-Hermitian pairing on the complex spinor space $\mathbb{S}_{x}$ with adjoint operation $\beta$ is classical. See e.g. [28, Theorem 7.14]. Two such pairings differ by multiplication by a non-vanishing scalar, because they have the same adjoint operation. Using the trivialization $\mathbb{S}_{U} \cong U \times \mathbb{S}_{x}$, we deduce that there exists a constant pseudo-Hermitian pairing $\langle\cdot, \cdot\rangle_{U}$ satisfying Property (ii) over $U$, and that any other such pairing is of the form $f\langle\cdot, \cdot\rangle_{U}$ with $f \in \mathrm{C}^{\infty}(U)$. According to Eq. (3.21), the pairing $f\langle\cdot, \cdot\rangle_{U}$ satisfies Property (i) if and only if $f$ is a constant function.

As Property (ii) and Eq. (3.21) are independent of the choice of connections $\nabla$ and $\nabla^{\mathbb{S}}$, so is the pseudo-Hermitian pairing $\langle\cdot, \cdot\rangle_{U}$. This concludes the proof.

The pairing $\langle\cdot, \cdot\rangle_{U}$ in Proposition 3.14 is called a spinor pairing. Since one can integrate 1densities, a spinor pairing yields a pseudo-Hermitian scalar product on the space of compactly supported sections $\Gamma_{c}\left(\left.\mathbb{S}\right|_{U}\right)$ :

$$
\begin{equation*}
(\phi, \psi)_{U}=\int_{U}\langle\phi, \psi\rangle_{U}, \quad \forall \phi, \psi \in \Gamma_{c}\left(\left.\mathbb{S}\right|_{U}\right) \tag{3.22}
\end{equation*}
$$

This is called a spinor scalar product. It is not clear if $(\cdot, \cdot)_{U}$ can be extended to a global scalar product on $M$. However, as we see below, the spinor scalar products over $\mathbb{S}_{\mid U}$ enable us to introduce a globally defined adjoint operation on $\mathcal{D}(M, \mathbb{S})$.

Lemma 3.15. For any $D \in \mathcal{D}(M, \mathbb{S})$, there exists a unique differential operator $D^{*} \in \mathcal{D}(M, \mathbb{S})$ such that

$$
\begin{equation*}
(D \phi, \psi)_{U}=\left(\phi, D^{*} \psi\right)_{U}, \quad \forall \phi, \psi \in \Gamma_{c}\left(\left.\mathbb{S}\right|_{U}\right) \tag{3.23}
\end{equation*}
$$

where $U$ is any contractible open subset of $M$ and $(\cdot, \cdot)_{U}$ is any spinor scalar product on $\mathbb{S}_{\mid U}$.
Proof. Let $D \in \mathcal{D}(M, \mathbb{S})$. Denote by $U \subset M$ a contractible open subset and by $\left.D\right|_{U} \in \mathcal{D}\left(U,\left.\mathbb{S}\right|_{U}\right)$ the restriction of the operator $D$ to $U$. Choose a spinor scalar product $(\cdot, \cdot)_{U}$. As is well-known, there exists a unique operator $\left(\left.D\right|_{U}\right)^{*} \in \mathcal{D}\left(U,\left.\mathbb{S}\right|_{U}\right)$ satisfying (3.23). Clearly, this operator also satisfies (3.23) for the spinor scalar product $a(\cdot, \cdot)_{U}$, with $a \in \mathbb{R}^{\times}$. By Proposition 3.14) ( $\left.\left.D\right|_{U}\right)^{*}$ satisfies (3.23) for any choice of spinor scalar products on $\left.\mathbb{S}\right|_{U}$.

Since the operators $\left(\left.D\right|_{U}\right)^{*}$ are uniquely defined by (3.23), they must glue together into a globally defined differential operator $D^{*} \in \mathcal{D}(M, \mathbb{S})$.

The map $D \mapsto D^{*}$ is called the adjoint operation, and admits the usual properties of the standard adjoint operation.

Proposition 3.16. The adjoint operation $D \mapsto D^{*}$ satisfies the following properties:
(i) it is an involutive map on $\mathcal{D}(M, \mathbb{S})$ preserving the filtration;
(ii) it is a $\mathbb{C}$-antilinear antiautomorphism:

$$
\left\{\begin{array}{l}
\left(\lambda_{1} D_{1}+\lambda_{2} D_{2}\right)^{*}=\overline{\lambda_{1}} D_{1}^{*}+\overline{\lambda_{2}} D_{2}^{*},  \tag{3.24}\\
\left(D_{1} D_{2}\right)^{*}=D_{2}^{*} D_{1}^{*},
\end{array} \quad \forall \lambda_{1}, \lambda_{2} \in \mathbb{C}, D_{1}, D_{2} \in \mathcal{D}(M, \mathbb{S})\right.
$$

uniquely determined by the following relations

$$
f^{*}=\bar{f}, \quad \gamma(\xi)^{*}=\gamma(\xi) \quad \text { and } \quad\left(\nabla_{X}^{\mathbb{S}}\right)^{*}=-\nabla_{X}^{\mathbb{S}}-\operatorname{Tr} \nabla X
$$

for all $f \in \mathrm{C}^{\infty}(M) \otimes \mathbb{C}, \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. Here, $\operatorname{Tr} \nabla X \in \mathrm{C}^{\infty}(M)$ is the trace of $\nabla X \in \Gamma\left(T^{*} M \otimes T M\right) ;$
(iii) it does not depend on the choice of connections $\nabla$ on $T M$ and $\nabla^{\mathbb{S}}$ on $\mathbb{S}$.

Proof. It suffices to prove all the properties above over any contractible open subset $U \subset M$.
Choose a spinor scalar product $(\cdot, \cdot)_{U}$. Eq. (3.23) uniquely characterizes the operation $*$. As a consequence, Eq. (3.24) holds as well as the relations $\left(D^{*}\right)^{*}=D, f^{*}=\bar{f}$. Using in addition Proposition 3.14 we obtain that $\gamma(\xi)^{*}=\gamma(\xi)$ and that the operation $*$ is independent of any choice of connections.

Any $v \in \Gamma_{c}\left(\left|\wedge^{\text {top }} T^{*} M\right|\right)$ satisfies the identities $L_{X} v=\nabla_{X} v+(\operatorname{Tr} \nabla X) v$ and $\int_{U} L_{X} v=0$. For any $\phi$ and $\psi \in \Gamma_{c}\left(\left.\mathbb{S}\right|_{U}\right)$, their pairing $\langle\phi, \psi\rangle_{U}$ pertains to $\Gamma_{c}\left(\left|\wedge^{\text {top }} T^{*} M\right|\right)$, so the above identities hold for $v=\langle\phi, \psi\rangle_{U}$. Using in addition Proposition 3.14, we obtain that

$$
\int_{U} L_{X}\langle\phi, \psi\rangle_{U}=\int_{U}\left\langle\nabla_{X}^{\mathbb{S}} \phi, \psi\right\rangle_{U}+\left\langle\phi, \nabla_{X}^{\mathbb{S}} \psi+(\operatorname{Tr} \nabla X) \psi\right\rangle_{U}=0
$$

Hence, the equation $\left(\nabla_{X}^{\mathbb{S}}\right)^{*}=-\nabla_{X}^{\mathbb{S}}-\operatorname{Tr} \nabla X$ holds.
Since $\mathcal{D}\left(M,\left.\mathbb{S}\right|_{U}\right)$ is generated by $\mathrm{C}^{\infty}(U), \Gamma\left(\left.E\right|_{U}\right)$ and $\Gamma(T U)$, Eqns (3.24)-(3.25) completely determine the adjoint operation $*$. Finally, one easily sees that if $D \in \mathcal{D}_{k}\left(M,\left.\mathbb{S}\right|_{U}\right)$ then $D^{*} \in$ $\mathcal{D}_{k}\left(M,\left.\mathbb{S}\right|_{U}\right)$. This concludes the proof of the proposition.

Denote by - the complex conjugation on $\mathbb{C}$ and its natural extension to any complexified $\mathbb{R}$-vector space $V \otimes \mathbb{C}$. In what follows, we will extend this operation to the algebra $\mathcal{D}(M, \mathbb{S})$.

Proposition 3.17. There exists a unique $\mathbb{C}$-antilinear algebra morphism

$$
\div: \mathcal{D}(M, \mathbb{S}) \longrightarrow \mathcal{D}(M, \mathbb{S})
$$

which coincides with the complex conjugation on $\mathcal{D}_{0}(M, \mathbb{S}) \cong \mathrm{C}^{\infty}(M) \otimes \mathbb{C}$ and satisfies the following properties:

$$
\begin{equation*}
\overline{\gamma(\xi)}=\gamma(\bar{\xi}) \quad \text { and } \quad \overline{\nabla_{X}^{\mathbb{S}}}=\nabla \frac{\mathbb{S}}{X}, \quad \forall \xi \in \Gamma(E) \otimes \mathbb{C}, X \in \mathfrak{X}(M) \otimes \mathbb{C}, \tag{3.26}
\end{equation*}
$$

for any induced spinor connection $\nabla^{\mathbb{S}}$.
Proof. Choose a spinor connection on $S$ and a connection on $T M$. Let $\nabla^{\mathbb{S}}$ be the induced connection on $\mathbb{S}$. Since $\mathcal{D}(M, \mathbb{S})$ is generated by $f, \gamma(\xi), \nabla_{X}^{\mathbb{S}}$, with $f \in \mathbb{C}^{\infty}(M) \otimes \mathbb{C}, \xi \in \Gamma(E), X \in \mathfrak{X}(M)$, a $\mathbb{C}$-antilinear algebra morphism - satisfying (3.26) must be unique if it exists. Hence, it suffices to prove the existence of - locally. Over a contractible open subset of $M$, we extend the conjugation of complex functions by setting

$$
\overline{f \gamma(\eta) \nabla_{\alpha}^{\mathbb{S}}}:=\bar{f} \gamma(\eta) \nabla_{\alpha}^{\mathbb{S}}
$$

with $f \in \mathrm{C}^{\infty}(M) \otimes \mathbb{C}, \eta \in \Gamma(\wedge E)$ and $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$ an ordered multi-index. By Lemma 3.6, the map - is well-defined. It is obvious to see that $\div$ is indeed a $\mathbb{C}$-antilinear involutive map satisfying (3.26). To check that - is an algebra morphism, one may use the commutation relations (3.13), and note that the curvature 2 -form of $\nabla^{\mathbb{S}}$ is valued in the real Clifford bundle $\mathrm{Cl}(E)$ (see (3.20)).

According to (3.20), different choices of connections on $T M$ and $S$ induce connections on $\mathbb{S}$ which differ by a real term. Therefore the map ${ }^{-}$is independent of choices of connections. This concludes the proof.

The map - introduced above is called the conjugation map. By $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, we denote the real subalgebra of $\mathcal{D}(M, \mathbb{S})$ generated by sections of the real Clifford bundle $\mathrm{Cl}(E)$ and the covariant derivatives $\nabla_{X}^{\mathbb{S}}$, with $X \in \mathfrak{X}(M)$. The next proposition gives an intrinsic description of this real subalgebra.

Proposition 3.18. The real subalgebra $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ is the fixed point set of - . Therefore we have $\mathcal{D}(M, \mathbb{S}) \cong \mathcal{D}(M, \mathbb{S})_{\mathbb{R}} \otimes \mathbb{C}$.

Proof. The proof is straightforward and is left to the reader.
Remark 3.19. Assume there exists a real spinor bundle $\mathbb{S}_{\mathbb{R}}$ such that $\mathbb{S} \cong \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$. Then, the conjugation map on $\mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}\right)$ is induced by the natural conjugation on $\mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$, so that $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}} \cong$ $\mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}}\right)$. However, $\mathbb{S}_{\mathbb{R}}$ may not exist.

The following result is obvious.
Proposition 3.20. For any $k, l \in \mathbb{N}$, let $m=2 k+l$. Assume that the principal symbol of $D \in$ $\mathcal{D}_{m}(M, \mathbb{S})$ satisfies $\sigma_{m}(D) \in \mathcal{O}_{k}\left(T^{*}[2] M\right) \otimes_{R} \mathcal{O}_{l}^{\mathbb{C}}(E[1])$. Then we have

$$
\sigma_{m}(\bar{D})=\overline{\sigma_{m}(D)} \quad \text { and } \quad \sigma_{m}\left(D^{*}\right)=(-1)^{k}(-1)^{\frac{l(l-1)}{2}}\left(\overline{\sigma_{m}(D)}\right)
$$

## 4. Quantization of symplectic graded manifolds of degree 2

First, we recall some basic materials on the Weyl quantization on $T^{*} \mathbb{R}^{n}$ and its extension to arbitrary cotangent bundles $T^{*} M$ [37, 40]. Then, we introduce a similar construction of quantizations on symplectic graded manifolds of degree 2 .
4.1. Weyl quantization on $T^{*} \mathbb{R}^{n}$. Let $V \rightarrow \mathbb{R}^{n}$ be a complex vector bundle over $\mathbb{R}^{n}$, and denote by $\mathcal{D}\left(\mathbb{R}^{n}, V\right)$ the algebra of differential operators on $V$. Choosing a trivialization of $V$, one readily gets that $\mathcal{D}\left(\mathbb{R}^{n}, V\right) \cong \mathcal{D}\left(\mathbb{R}^{n}\right) \otimes_{R} \Gamma($ End $V)$. Recall that the normal order quantization is defined, in terms of the canonical coordinate system $\left(x^{i}, p_{i}\right)$ on $T^{*}[2] \mathbb{R}^{n}$, by

$$
\begin{aligned}
\mathcal{N}_{\hbar}: \mathcal{O}\left(T^{*}[2] \mathbb{R}^{n}\right) \otimes_{R} \Gamma(\text { End } V) & \longrightarrow \mathcal{D}\left(\mathbb{R}^{n}, V\right) \\
F^{i_{1} \cdots i_{k}}(x) p_{i_{1}} \cdots p_{i_{k}} & \longmapsto F^{i_{1} \cdots i_{k}}(x)\left(\frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x^{i_{1}}}\right) \cdots\left(\frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x^{i_{k}}}\right),
\end{aligned}
$$

where $F^{i_{1} \cdots i_{k}} \in \Gamma($ End $V)$ and $\hbar \in \mathbb{C}^{\times}$is a parameter 2 . Let Div $=\frac{\partial^{2}}{\partial x^{i} \partial p_{i}}$ be the divergence-like operator acting on the symbol algebra $\mathcal{O}\left(T^{*}[2] \mathbb{R}^{n}\right)$.

Definition 4.1. On $T^{*}[2] \mathbb{R}^{n}$, the $\mathcal{D}\left(\mathbb{R}^{n}, V\right)$-valued Weyl quantization is a $\mathbb{C}$-linear isomorphism, $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}: \mathcal{O}\left(T^{*}[2] \mathbb{R}^{n}\right) \otimes_{R} \Gamma($ End $V) \rightarrow \mathcal{D}\left(\mathbb{R}^{n}, V\right)$, indexed by $\hbar \in \mathbb{C}^{\times}$and defined by

$$
\begin{equation*}
\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}:=\mathcal{N}_{\hbar} \circ \exp \left(\frac{\hbar}{2 \mathrm{i}} \operatorname{Div} \otimes \mathrm{id}\right) \tag{4.1}
\end{equation*}
$$

In particular, the Weyl quantization satisfies $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}\left(p_{i}\right)=\frac{\hbar}{i} \frac{\partial}{\partial x^{i}}$ and $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}(F)=F$, for any $F \in$ $\Gamma($ End $V)$. In the classical case when $V$ is a trivial line bundle, $\Gamma($ End $V)$ is identified with $\mathrm{C}^{\infty}\left(\mathbb{R}^{n}\right)$, and $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}(F)$ is the multiplication operator by $F$. For polynomials in the coordinates $\left(x^{i}, p_{i}\right)$, the Weyl quantization is just the symmetrization map satisfying $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}\left(p_{i}\right)=\frac{\hbar}{i} \frac{\partial}{\partial x^{i}}$ and $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}\left(x^{i}\right)=x^{i}$.

The Weyl quantization can also be defined by the integral formula (4.2) below [13].
Proposition 4.2. The Weyl quantization satisfies

$$
\begin{equation*}
\left(\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}(F) \psi\right)(x)=\frac{1}{(2 \pi \hbar)^{n}} \int_{T_{x}^{*} \mathbb{R}^{n} \oplus T_{x} \mathbb{R}^{n}} \mathrm{e}^{-\frac{i}{\hbar}\langle p, v\rangle} F(x+v / 2, p) \cdot \psi(x+v) \mathrm{d} p \mathrm{~d} v \tag{4.2}
\end{equation*}
$$

for all $\psi \in \Gamma(V)$ and $F \in \mathcal{O}\left(T^{*}[2] \mathbb{R}^{n}\right) \otimes_{R} \Gamma($ End $V)$.
4.2. Exponential map and parallel transport. To generalize the above Weyl quantization to vector bundles over any smooth manifolds, we will need connections. We fix some notations concerning exponential maps and parallel transports below.

A connection $\nabla$ on $T M$ induces an exponential map, indexed by $x \in M$,

$$
\exp _{x}: U_{x} \longrightarrow M
$$

where $U_{x}$ is an open neighborhood of $0 \in T_{x} M$. We choose $U_{x}$ such that $\exp _{x}$ is a diffeomorphism onto its image. In addition, let $\nabla^{V}$ be a connection on a vector bundle $V \rightarrow M$. Then we have parallel transport maps:

$$
\mathcal{T}_{x, y}^{V}: V_{x} \longrightarrow V_{y}
$$

[^2]for any pairs of points $x, y \in M$ such that $y=\exp _{x} v$ with $v \in U_{x}$. Exponential map and parallel transport together induce a local isomorphism of vector bundles
\[

$$
\begin{align*}
\mathcal{T}_{x}: U_{x} \times V_{x} & \longrightarrow V \\
(v, \phi) & \longmapsto\left(y, \mathcal{T}_{x, y}^{V} \phi\right) \tag{4.3}
\end{align*}
$$
\]

where $y=\exp _{x} v$. Consider a cut-off function $\chi \in \mathrm{C}^{\infty}(T M)$, i.e. a function which equals to 1 in a neighborhood of the zero section of $T M$ such that the support of $\chi(x, \cdot)$ is included in $U_{x}$. Define a map

$$
\begin{equation*}
\chi(x, \cdot) \mathcal{T}_{x}^{*}: \Gamma(V) \longrightarrow \mathrm{C}^{\infty}\left(T_{x} M\right) \otimes_{\mathbb{R}} V_{x} \tag{4.4}
\end{equation*}
$$

by setting

$$
\left(\chi(x, \cdot) \mathcal{T}_{x}^{*} \psi\right)(v)=\chi(x, v)\left(\mathcal{T}_{x}^{*} \psi\right)(v), \quad \forall v \in T_{x} M, \psi \in \Gamma(V)
$$

Note that by definition, if $v \in U_{x}$ and $y=\exp _{x} v$, we have

$$
\begin{equation*}
\left(\mathcal{T}_{x}^{*} \psi\right)(v)=\mathcal{T}_{y, x}^{V}(\psi(y)) \tag{4.5}
\end{equation*}
$$

The connections $\nabla$ and $\nabla^{V}$ induce a connection on the vector bundle $\mathcal{S} T M \otimes$ End $V$, and therefore we have a map analogous to (4.4). Since $\mathcal{O}\left(T^{*}[2] M\right) \cong \Gamma(\mathcal{S T M})$, this map reads as

$$
\chi(x, \cdot) \mathcal{T}_{x}^{*}: \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma(\text { End } V) \longrightarrow \mathcal{O}\left(T^{*}[2]\left(T_{x} M\right)\right) \otimes_{\mathbb{R}} \operatorname{End}\left(V_{x}\right)
$$

If $(v, p) \in T^{*} U_{x}$ and $y=\exp _{x} v$, for any $F \in \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma($ End $V)$, we have the following explicit formula

$$
\begin{equation*}
\left(\mathcal{T}_{x}^{*} F\right)(v, p):=\mathcal{T}_{y, x}^{V} \circ F\left(y, \mathcal{T}_{x, y} p\right) \circ \mathcal{T}_{x, y}^{V} \tag{4.6}
\end{equation*}
$$

where $\mathcal{T}_{x, y}: T_{x}^{*} M \rightarrow T_{y}^{*} M$ is the parallel transport map induced by $\nabla$.
4.3. Weyl quantization on $T^{*}[2] M$. The Weyl quantization integral formula (4.2) has been extended to arbitrary cotangent bundles $T^{*} M$ with the help of a connection on $T M$ [37]. It has been further generalized to differential operators acting on any vector bundle $V$ over $M$, using an additional connection on the vector bundle $V$ 40]. We recall the construction briefly below.

Definition 4.3. By a Weyl quantization map, we mean a map

$$
\mathcal{Q}_{\hbar}^{M}: \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma(\text { End } V) \rightarrow \operatorname{End}(\Gamma(V))
$$

defined by

$$
\begin{equation*}
\left(\mathcal{Q}_{\hbar}^{M}(F) \psi\right)(x)=\frac{1}{(2 \pi \hbar)^{n}} \int_{T_{x}^{*} M \oplus T_{x} M} \mathrm{e}^{-\frac{i}{\hbar}\langle p, v\rangle}\left(\mathcal{T}_{x}^{*} F\right)(v / 2, p) \cdot\left(\mathcal{T}_{x}^{*} \psi\right)(v) \chi(x, v) \mathrm{d} p \mathrm{~d} v \tag{4.7}
\end{equation*}
$$

for all $x \in M, \psi \in \Gamma(V)$ and $F \in \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma($ End $V)$.
The map $\mathcal{Q}_{\hbar}^{M}$ depends on the choice of connections on $T M$ and $V$ defining the pull-backs $\mathcal{T}_{x}^{*} \psi$ and $\mathcal{T}_{x}^{*} F$. Note that a priori, $\mathcal{Q}_{\hbar}^{M}$ also depends on the cut-off function $\chi \in \mathrm{C}^{\infty}(T M)$.

For any $x \in M$, the projection $\mathbb{R}^{n} \times V_{x} \rightarrow \mathbb{R}^{n}$ defines a trivial vector bundle on $\mathbb{R}^{n}$. Denote by

$$
\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}: \mathcal{O}\left(T^{*}[2] \mathbb{R}^{n}\right) \otimes_{\mathbb{R}} \Gamma\left(\text { End } V_{x}\right) \longrightarrow \mathcal{D}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \times V_{x}\right)
$$

the corresponding Weyl quantization map, given by (4.1) or equivalently (4.2). The maps $\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}$ and $\mathcal{Q}_{\hbar}^{M}$ are related by parallel transportation as indicated below.

Lemma 4.4. For any $x \in M, \psi \in \Gamma(V)$ and $F \in \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma(\operatorname{End} V)$, we have

$$
\mathcal{T}_{x}^{*}\left(\mathcal{Q}_{\hbar}^{M}(F) \psi\right)(w)=\left(\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*} F\right) \mathcal{T}_{x}^{*} \psi\right)(w), \quad \forall w \in U_{x}
$$

where $U_{x}$ is an open neighborhood of $0 \in T_{x} M$, on which $\exp _{x}$ is a diffeomorphism onto its image.
Proof. For all $w \in U_{x}$, Eqns (4.5) and (4.7) imply that

$$
\begin{equation*}
\mathcal{T}_{x}^{*}\left(\mathcal{Q}_{\hbar}^{M}(F) \psi\right)(w)=\frac{1}{(2 \pi \hbar)^{n}} \int_{T_{y}^{*} M \oplus T_{y} M} \mathrm{e}^{-\frac{i}{\hbar}\langle p, v\rangle} \mathcal{T}_{y, x}^{V}\left[\left(\mathcal{T}_{y}^{*} F\right)(v / 2, p) \cdot\left(\mathcal{T}_{y}^{*} \psi\right)(v)\right] \chi(y, v) \mathrm{d} p \mathrm{~d} v \tag{4.8}
\end{equation*}
$$

where $y=\exp _{x} w$. The equality $\exp _{y} v=\exp _{x}\left(w+\mathcal{T}_{y, x} v\right)$ and Eq. (4.5) lead to

$$
\begin{equation*}
\left(\mathcal{T}_{y}^{*} \psi\right)(v)=\mathcal{T}_{x, y}^{V}\left[\left(\mathcal{T}_{x}^{*} \psi\right)\left(w+\mathcal{T}_{y, x} v\right)\right] \tag{4.9}
\end{equation*}
$$

Similarly, using the identity $\exp _{y} \frac{v}{2}=\exp _{x}\left(w+\frac{1}{2} \mathcal{T}_{y, x} v\right)$ and Eq. (4.6), we have

$$
\begin{equation*}
\mathcal{T}_{y, x}^{V} \circ\left[\left(\mathcal{T}_{y}^{*} F\right)(v / 2, p)\right]=\left[\left(\mathcal{T}_{x}^{*} F\right)\left(w+\frac{1}{2} \mathcal{T}_{y, x} v, \mathcal{T}_{y, x} p\right)\right] \circ \mathcal{T}_{y, x}^{V} \tag{4.10}
\end{equation*}
$$

As parallel transport in the fibers of $T M \oplus T^{*} M$ preserves the duality pairing $\langle p, v\rangle$ and the measure $\mathrm{d} p \mathrm{~d} v$, by Eqns (4.8)-(4.10) and the change of variables $\left(\mathcal{T}_{y, x} v, \mathcal{T}_{y, x} p\right) \mapsto(v, p)$, we obtain

$$
\begin{aligned}
\mathcal{T}_{x}^{*}\left(\mathcal{Q}_{\hbar}^{M}(F) \psi\right)(w)=\frac{1}{(2 \pi \hbar)^{n}} \int_{T_{x}^{*} M \oplus T_{x} M} \mathrm{e}^{-\frac{i}{\hbar}\langle p, v\rangle}\left(\mathcal{T}_{x}^{*} F\right)(w+v / 2, p) \cdot\left(\mathcal{T}_{x}^{*} \psi\right)(w+v) \\
\chi\left(\exp _{x} w, \mathcal{T}_{x, \exp _{x} w} v\right) \mathrm{d} p \mathrm{~d} v
\end{aligned}
$$

Since $F$ is polynomial in $p$, its Fourier transform w.r.t. $p$ is a distribution supported at 0 . As the function $v \mapsto \chi(x, v)$ is equal to 1 in a neighborhood of the zero section, the right hand side in the above equation reduces to $\left(\mathcal{Q}_{\hbar}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*} F\right) \mathcal{T}_{x}^{*} \psi\right)(w)$ (see formula (4.2)). The conclusion thus follows.

As a straightforward consequence, we have
Proposition 4.5. Given any connections on $T M$ and $V$, there is a unique Weyl quantization map:

$$
\mathcal{Q}_{\hbar}^{M}: \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma(\operatorname{End} V) \rightarrow \mathcal{D}(M, V)
$$

That is, Formula (4.7) does not depend on the choice of cut-off functions $\chi$. Moreover, $\mathcal{Q}_{\hbar}^{M}$ is a linear isomorphism.

In what follows, we assume a choice of connections on $T M$ and $V$ is made and we refer to the corresponding map $\mathcal{Q}_{\hbar}^{M}$ as the Weyl quantization map.

Let $V$ be a vector bundle endowed with a pseudo-Hermitian pairing $\langle\cdot, \cdot\rangle_{V}: \Gamma(V) \times \Gamma(V) \rightarrow$ $\Gamma\left(\left|\wedge^{\text {top }} T^{*} M\right|\right) \otimes \mathbb{C}$ as in (3.19). Then $(\phi, \psi):=\int_{M}\langle\phi, \psi\rangle_{V}$ defines a pseudo-Hermitian scalar product on $\Gamma_{c}(V)$. Denote by $*: \mathcal{D}(M, V) \rightarrow \mathcal{D}(M, V)$ the adjoint operation associated to $(\cdot, \cdot)$ and by $*_{V}: \Gamma($ End $V) \rightarrow \Gamma($ End $V)$ the adjoint operation associated to $\langle\cdot, \cdot\rangle_{V}$. Abusing notation, we also denote by $*_{V}$ the obvious extension to $\mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma($ End $V)$ defined by $\left(F_{0} \otimes F_{1}\right)^{* V}=F_{0} \otimes F_{1}^{* V}$, $\forall F_{0} \otimes F_{1} \in \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma($ End $V)$.

Proposition 4.6. Assume that the following equality holds

$$
\begin{equation*}
\nabla_{X}\langle\phi, \psi\rangle_{V}=\left\langle\nabla_{X}^{V} \phi, \psi\right\rangle_{V}+\left\langle\phi, \nabla_{X}^{V} \psi\right\rangle_{V}, \quad \forall X \in \mathfrak{X}(M), \phi, \psi \in \Gamma(V) . \tag{4.11}
\end{equation*}
$$

Then the Weyl quantization $\mathcal{Q}_{\hbar}^{M}$ satisfies the property

$$
\mathcal{Q}_{\hbar}^{M}(F)^{*}=\mathcal{Q}_{\hbar}^{M}\left(F^{* V}\right)
$$

for all $F \in \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R} \Gamma($ End $V)$.
Proof. Eqns (4.5)-(4.7) yield the formula

$$
\left(\mathcal{Q}_{\hbar}^{M}(F) \psi\right)(x)=\frac{1}{(2 \pi \hbar)^{n}} \int_{T_{x}^{*} M \oplus T_{x} M} \mathrm{e}^{-\frac{i}{\hbar}\langle p, v\rangle} \mathcal{T}_{y, x}^{V}\left[F\left(y, \mathcal{T}_{x, y} p\right) \cdot \mathcal{T}_{z, y}^{V}(\psi(z))\right] \chi(x, v) \mathrm{d} p \mathrm{~d} v
$$

where $y=\exp _{x}(v / 2)$ and $z=\exp _{x}(v)$. By Eq. (4.11), the parallel transports $\mathcal{T}$ and $\mathcal{T}^{V}$ satisfy the relation

$$
\mathcal{T}_{y, x}\left(\langle\phi(y), \psi(y)\rangle_{V}\right)=\left\langle\mathcal{T}_{y, x}^{V}(\phi(y)), \mathcal{T}_{y, x}^{V}(\psi(y))\right\rangle_{V}, \quad \forall \phi, \psi \in \Gamma(V)
$$

which implies

$$
\left\langle\mathcal{T}_{y, x}^{V}\left[F\left(y, \mathcal{T}_{x, y} p\right)^{* V} \cdot \mathcal{T}_{z, y}^{V}(\phi(z))\right], \psi(x)\right\rangle_{V}=\mathcal{T}_{z, x}\left\langle\phi(z), \mathcal{T}_{y, z}^{V}\left[F\left(y, \mathcal{T}_{x, y} p\right) \cdot \mathcal{T}_{x, y}^{V}(\psi(x))\right]\right\rangle_{V}
$$

We deduce that

$$
\begin{align*}
& \left(\mathcal{Q}_{\hbar}^{M}\left(F^{* V}\right) \phi, \psi\right)=\frac{1}{(2 \pi \hbar)^{n}} \times  \tag{4.12}\\
& \quad \int_{M} \int_{T_{x} M \oplus T_{x}^{*} M} \mathcal{T}_{z, x}\left\langle\phi(z), \mathcal{T}_{y, z}^{V}\left[F\left(y, \mathcal{T}_{x, y} p\right) \cdot \mathcal{T}_{x, y}^{V}(\psi(x))\right]\right\rangle_{V} \mathrm{e}^{\frac{i}{\hbar}\langle p, v\rangle} \chi(x, v) \mathrm{d} p \mathrm{~d} v .
\end{align*}
$$

Consider the change of variables $(x, v, p) \mapsto\left(z, v^{\prime}, p^{\prime}\right)$ in Eq. (4.12), where $z=\exp _{x} v, v^{\prime}=\mathcal{T}_{x, z} v$ and $p^{\prime}=\mathcal{T}_{x, z} p$. Since $\mathrm{d} p \mathrm{~d} v=\mathcal{T}_{z, x}\left(\mathrm{~d} p^{\prime} \mathrm{d} v^{\prime}\right)$ and $\langle p, v\rangle=\left\langle p^{\prime}, v^{\prime}\right\rangle$, we obtain

$$
\begin{aligned}
\left(\mathcal{Q}_{\hbar}^{M}\left(F^{*}\right) \phi, \psi\right) & =\frac{1}{(2 \pi \hbar)^{n}} \times \\
& \int_{M}\left\langle\phi(z), \int_{T_{z} M \oplus T_{z}^{*} M} \mathcal{T}_{y, z}^{V}\left[F\left(y, \mathcal{T}_{z, y} p^{\prime}\right) \cdot \mathcal{T}_{x, y}^{V}(\psi(x))\right] \mathrm{e}^{-\frac{i}{\hbar}\left\langle p^{\prime}, v^{\prime}\right\rangle} \chi\left(x, \mathcal{T}_{z, x} v^{\prime}\right) \mathrm{d} p^{\prime} \mathrm{d} v^{\prime}\right\rangle_{V}
\end{aligned}
$$

where $y=\exp _{z}\left(v^{\prime} / 2\right)$ and $x=\exp _{z}\left(v^{\prime}\right)$. Using Eq. (4.7), we conclude that $\left(\mathcal{Q}_{\hbar}^{M}\left(F^{* V}\right) \phi, \psi\right)=$ $\left(\phi, \mathcal{Q}_{\hbar}^{M}(F) \psi\right), \forall \phi, \psi \in \Gamma_{c}(V)$, and the result follows.

Finally, let us describe an explicit formula for the Weyl quantization $\mathcal{Q}_{\hbar}^{M}$ in low degrees. By abuse of notation, we also denote by $\nabla$ the induced connection on $T M \otimes$ End $V$. For a vector bundle $W$ over $M$, let

$$
\begin{equation*}
\operatorname{Tr}: \Gamma\left(W \otimes T M \otimes T^{*} M\right) \longrightarrow \Gamma(W) \tag{4.13}
\end{equation*}
$$

be the trace map.
Proposition 4.7. The Weyl quantization $\mathcal{Q}_{\hbar}^{M}$ satisfies

$$
\begin{equation*}
\mathcal{Q}_{\hbar}^{M}(F)=F \quad \text { and } \quad \mathcal{Q}_{\hbar}^{M}(X \otimes F)=\frac{\hbar}{\mathrm{i}}\left[F \nabla_{X}^{V}+\frac{1}{2} \operatorname{Tr} \nabla(X \otimes F)\right] \tag{4.14}
\end{equation*}
$$

for all $F \in \Gamma($ End $V)$ and $X \in \mathcal{O}_{2}\left(T^{*}[2] M\right) \cong \mathfrak{X}(M)$.

Proof. The first case is trivial. For the second equation, choose an open neighborhood $U_{x}$ of $0 \in T_{x} M$ such that $\exp _{x}: U_{x} \rightarrow \exp _{x}\left(U_{x}\right)$ is a diffeomorphism. Pulling back a Cartesian coordinate system on $T_{x} M \times V_{x}$ by $\mathcal{T}_{x}^{-1}$, we obtain normal coordinates $\left(x^{i}\right)$ centered at $x$ and a trivialization of $V$ over $\exp _{x}\left(U_{x}\right) \subset M$. Denote by $\left(x^{i}, p_{i}\right)$ the induced canonical coordinates on $T_{x}^{*}\left(\exp _{x}\left(U_{x}\right)\right)$. Under such coordinates, we can write $X(x)=X^{i}(x) p_{i}$, with $X^{i} \in \mathrm{C}^{\infty}\left(\exp _{x}\left(U_{x}\right)\right)$. By Lemma 4.4 and Eq. (4.1), we have

$$
\begin{equation*}
\left(\mathcal{Q}_{\hbar}^{M}(X \otimes F) \psi\right)(x)=\left(F(x) X^{i}(x) \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x^{i}}+\frac{\hbar}{2 \mathrm{i}} \frac{\partial}{\partial x^{i}}\left(F(x) X^{i}(x)\right)\right) \psi(x), \quad \forall \psi \in \Gamma(V) \tag{4.15}
\end{equation*}
$$

Since $\left(x^{i}\right)$ are normal coordinates at $x$, the partial derivatives $\frac{\partial}{\partial x^{i}}$ coincide with covariant derivatives at the point $x$. Hence, the second equation of (4.14) holds at $x$. The point $x$ being arbitrary, this implies the result.
4.4. Weyl quantization on symplectic graded manifolds of degree 2. Let ( $E, \mathrm{~g}$ ) be a pseudoEuclidean vector bundle equipped with a metric connection $\nabla^{E}$. Assume that ( $E, \mathrm{~g}$ ) admits a spinor bundle $S$. Choose a connection $\nabla$ on $T M$, and choose a spinor connection on $S$, compatible with $\nabla^{E}$ (see Eq. (3.4)). They induce a Weyl quantization $\mathcal{Q}_{\hbar}^{M}$, valued in $\mathcal{D}(M, S)$.

Definition 4.8. The Weyl quantization $\mathcal{W} \mathcal{Q}_{\hbar}$ on the symplectic graded manifold $\left(T^{*}[2] M \oplus E[1], \omega_{\mathrm{g}, \nabla^{E}}\right)$ is defined by the following commutative diagram


The vertical map reads as

$$
\begin{equation*}
\gamma_{\hbar}:=\operatorname{id} \otimes\left(\frac{\hbar}{\mathrm{i}}\right)^{\kappa / 2} \gamma, \quad \text { on } \mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M\right) \otimes_{R} \Gamma\left(\wedge^{\kappa} E\right) \tag{4.17}
\end{equation*}
$$

where $\gamma$ is the standard Clifford quantization map (see (3.1)).
The Weyl quantization $\mathcal{W} \mathcal{Q}_{\hbar}$ extends those studied in [14, 39, with $E$ being $T M$ and $T M \oplus T^{*} M$, respectively.

Theorem 4.9. For any $F \in \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right), k \in \mathbb{N}$, and $G \in \mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$, we have
(i) $\mathcal{W} \mathcal{Q}_{\hbar}(F)=(\hbar / \mathrm{i})^{k / 2} \mathcal{W} \mathcal{Q}_{\mathrm{i}}(F)$;
(ii) $\sigma_{k} \circ \mathcal{W} \mathcal{Q}_{\hbar}(F)=(\hbar / \mathrm{i})^{k / 2} F$ (see Definition 3.10 for the principal symbol map $\sigma_{k}$ );
(iii) $\mathcal{W} \mathcal{Q}_{\hbar}(F) \in \mathcal{D}^{+}(M, S)$ if $k$ is even and $\mathcal{W} \mathcal{Q}_{\hbar}(F) \in \mathcal{D}^{-}(M, S)$ if $k$ is odd (see (3.6));
(iv) $\left[\mathcal{W} \mathcal{Q}_{\hbar}(F), \mathcal{W} \mathcal{Q}_{\hbar}(G)\right]=\frac{\hbar}{\mathrm{i}} \mathcal{W} \mathcal{Q}_{\hbar}(\{F, G\})+O\left(\hbar^{2}\right)$.

Proof. As $\mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)=\bigoplus_{2 \ell+\kappa=k} \mathcal{O}_{\ell}\left(T^{*}[2] M\right) \otimes_{R} \mathcal{O}_{\kappa}(E[1])$, it suffices to check Properties (i) (ii) and (iii) on homogeneous functions $F \in \mathcal{O}_{\ell}\left(T^{*}[2] M\right) \otimes_{R} \mathcal{O}_{\kappa}(E[1])$. This can be easily checked using Lemma 4.4 and Eqns (4.1), (4.17).

Let $F \in \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$ and $G \in \mathcal{O}_{l}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$. Set

$$
H=\mathcal{W} \mathcal{Q}_{\mathrm{i}}^{-1}\left(\left[\mathcal{W} \mathcal{Q}_{\mathbf{i}}(F), \mathcal{W} \mathcal{Q}_{\mathrm{i}}(G)\right]-\mathcal{W} \mathcal{Q}_{\mathrm{i}}(\{F, G\})\right)
$$

Using Proposition 3.11 together with (ii) and (iii), we deduce that $H$ has degree $k+l-4$. The claim (iv) thus follows from (i).

Next we provide an explicit expression for the Weyl quantization $\mathcal{W} \mathcal{Q}_{\hbar}$ in low degrees. By abuse of notation, we denote by $\nabla$ the connection on $\wedge E \otimes T M$ induced by the connections on $E$ and $T M$. From the definition of $\mathcal{W} \mathcal{Q}_{\hbar}$ and Proposition 4.7, we deduce

Proposition 4.10. The Weyl quantization $\mathcal{W} \mathcal{Q}_{\hbar}$ satisfies

$$
\begin{align*}
\mathcal{W} \mathcal{Q}_{\hbar}(F) & =\left(\frac{\hbar}{\mathrm{i}}\right)^{\kappa / 2} \gamma(F), \\
\mathcal{W} \mathcal{Q}_{\hbar}(F X) & =\left(\frac{\hbar}{\mathrm{i}}\right)^{1+\kappa / 2}\left[\gamma(F) \nabla_{X}^{S}+\frac{1}{2} \gamma(\operatorname{Tr} \nabla(X F))\right] \tag{4.18}
\end{align*}
$$

for all $F \in \mathcal{O}_{\kappa}^{\mathbb{C}}(E[1])$ and $X \in \mathcal{O}_{2}\left(T^{*}[2] M\right) \cong \mathfrak{X}(M)$. Here, the trace map $\operatorname{Tr}$ is defined as in (4.13), with $W=\wedge^{\kappa} E \otimes \mathbb{C}$.

To proceed further, we assume that $(E, \mathrm{~g})$ admits a twisted spinor bundle

$$
\mathbb{S}:=S \otimes\left(\operatorname{det} S^{*}\right)^{1 / N} \otimes\left|\wedge^{\mathrm{top}} T^{*} M\right|^{1 / 2}
$$

with $N$ being the rank of $S$. A connection $\nabla$ on $T M$ and a spinor connection on $S$ induce a spinor connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}$ and a Weyl quantization map

$$
\begin{equation*}
\mathcal{W} \mathcal{Q}_{\hbar}: \mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right) \longrightarrow \mathcal{D}(M, \mathbb{S}) \tag{4.19}
\end{equation*}
$$

The latter satisfies in particular Theorem 4.9 and Proposition 4.10, Obviously, one should replace $\nabla^{S}$ by $\nabla^{\mathbb{S}}$ in Eq. (4.18). Considering $\mathbb{S}$ allows for further structures on $\mathcal{D}(M, \mathbb{S})$ (see Section 3.3). We study below their interplay with the map $\mathcal{W} \mathcal{Q}_{\hbar}$. For all $f \in \mathrm{C}^{\infty}(M), \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, let

$$
\tau(f)=f, \quad \tau(\xi)=\mathrm{i} \xi, \quad \tau(X)=X
$$

It is easy to see that $\tau$ extends to a $\mathbb{C}$-antilinear involutive antiautomorphism of $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$. On $\mathcal{O}_{\kappa}^{\mathbb{C}}(E[1])$, we have $\tau\left(\xi^{i_{1}} \cdots \xi^{i_{\kappa}}\right)=i^{\kappa}(-1)^{\frac{\kappa(\kappa-1)}{2}} \xi^{i_{1}} \cdots \xi^{i_{\kappa}}$. Hence, for a homogeneous element $F \in \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$ of degree $k$, the involution $\tau$ satisfies

$$
\tau(F)= \begin{cases}F & \text { for } k \text { even }  \tag{4.20}\\ \mathrm{i} F & \text { for } k \text { odd }\end{cases}
$$

Proposition 4.11. Let $\hbar \in \mathbb{R}^{\times}$. The Weyl quantization map as in (4.19) satisfies

$$
\mathcal{W} \mathcal{Q}_{\hbar} \circ \tau(\cdot)=\mathcal{W} \mathcal{Q}_{\hbar}(\cdot)^{*}
$$

where $*$ is the adjoint operation defined in Lemma 3.15.
Proof. Given the local nature of the maps $\mathcal{W} \mathcal{Q}_{\hbar}, \tau$ and $*$, it suffices to work over a contractible open subset $U \subset M$. Let $\langle\cdot, \cdot\rangle_{U}$ be a spinor pairing (see Proposition 3.14) and $*_{\mathbb{S}}: \Gamma\left(\right.$ End $\left.\left.\mathbb{S}\right|_{U}\right) \rightarrow \Gamma\left(\right.$ End $\left.\left.\mathbb{S}\right|_{U}\right)$ its adjoint operation. As $\hbar$ is real, we have $\gamma_{\hbar}(\xi)^{* s}=\mathrm{i} \gamma_{\hbar}(\xi)=\gamma_{\hbar}(\tau(\xi))$ for all $\xi \in \Gamma\left(\left.E\right|_{U}\right)$. Since both maps $*_{\mathbb{S}}$ and $\tau$ are $\mathbb{C}$-antilinear antiautomorphisms, we have $\gamma_{\hbar}(\tau(F))=\left(\gamma_{\hbar}(F)\right)^{* s}$, for all $F \in \mathcal{O}^{\mathbb{C}}\left(\left.E[1]\right|_{U}\right)$. Extending $*_{\mathbb{S}}$ to the tensor product $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] U\right) \otimes_{R} \mathcal{O}^{\mathbb{C}}\left(\left.E[1]\right|_{U}\right)$ in an obvious way, we obtain that

$$
\gamma_{\hbar}(\tau(F))=\left(\gamma_{\hbar}(F)\right)^{* s}
$$

for all $F \in \mathcal{O}^{\mathbb{C}}\left(\left.T^{*}[2] U \oplus E[1]\right|_{U}\right)$. According to Proposition 3.14 the pair of connections $\left(\nabla, \nabla^{\mathbb{S}}\right)$ satisfies Eq. (4.11), with $V=\mathbb{S}$. Applying Proposition 4.6 yields the result.

The case $\hbar=\mathrm{i}$ plays a peculiar role.
Proposition 4.12. Setting $\mathcal{W Q}:=\mathcal{W} \mathcal{Q}_{\mathrm{i}}$, we have
(i) $\mathcal{W Q}$ restricts to an $\mathbb{R}$-linear isomorphism $\mathcal{W} \mathcal{Q}: \mathcal{O}\left(T^{*}[2] M \oplus E[1]\right) \rightarrow \mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, where the real algebra $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ is defined in Proposition 3.18,
(ii) $\mathcal{W} \mathcal{Q}(F)^{*}=(-1)^{\lfloor k / 2\rfloor} \mathcal{W} \mathcal{Q}(F)$, for all $F \in \mathcal{O}_{k}\left(T^{*}[2] M \oplus E[1]\right)$, with $k \in \mathbb{N}$ and $\lfloor k / 2\rfloor$ the integer part of $k / 2$.

Proof. Complex conjugation in $\mathbb{C}$ extends naturally to the algebras $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right), \mathcal{O}\left(T^{*}[2] M\right) \otimes_{R}$ $\Gamma(\mathbb{C l}(E))$ and $\mathcal{D}(M, \mathbb{S})$ (see Proposition 3.17). To establish (i), it suffices to show that

$$
\mathcal{W} \mathcal{Q}(\bar{F})=\overline{\mathcal{W} \mathcal{Q}(F)}
$$

for all $F \in \mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$. Given the local nature of this property, we can restrict ourselves to a contractible open neighborhood $U$ of a point $x \in M$. Then we have

$$
\begin{array}{rlr}
\mathcal{W} \mathcal{Q}(\bar{F}) & =\left(\mathcal{T}_{x}^{-1}\right)^{*} \circ \mathcal{Q}_{i}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*}\left(\gamma_{\mathrm{i}}(\bar{F})\right)\right) \circ \mathcal{T}_{x}^{*}, \\
& =\left(\mathcal{T}_{x}^{-1}\right)^{*} \circ \mathcal{Q}_{\mathrm{i}}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*}\left(\overline{\gamma_{\mathrm{i}}(F)}\right)\right) \circ \mathcal{T}_{x}^{*}, & \quad \text { (by Lemma 4.4) } \\
\end{array}
$$

Since $\mathcal{T}_{x}$ is built from connections on the real vector bundles $T M$ and $E$, we deduce that $\mathcal{T}_{x}^{*}\left(\overline{\gamma_{i}(F)}\right)=$ $\overline{\mathcal{T}_{x}^{*}}\left(\gamma_{\mathrm{i}}(F)\right)$. Therefore, from (4.1), it follows that

$$
\mathcal{Q}_{\mathrm{i}}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*}\left(\overline{\gamma_{\mathrm{i}}(F)}\right)\right)=\overline{\mathcal{Q}_{\mathrm{i}}^{\mathbb{R}^{n}}\left(\mathcal{T}_{x}^{*}\left(\gamma_{\mathrm{i}}(F)\right)\right)}
$$

Thus

$$
\mathcal{W} \mathcal{Q}(\bar{F})=\left(\mathcal{T}_{x}^{-1}\right)^{*} \circ\left[\overline{\mathcal{T}_{x}^{*} \circ \mathcal{W} \mathcal{Q}(F) \circ\left(\mathcal{T}_{x}^{-1}\right)^{*}}\right] \circ \mathcal{T}_{x}^{*}
$$

From the equation above, we deduce that the $\operatorname{map} \mathcal{W} \mathcal{Q}(F) \mapsto \mathcal{W} \mathcal{Q}(\bar{F})$ is an algebra automorphism of $\mathcal{D}(M, \mathbb{S})$. By Proposition4.10, we see that this map indeed satisfies all the conditions of the map - as in Proposition 3.17. Hence, according to the uniqueness in Proposition 3.17, the map $\mathcal{W} \mathcal{Q}(F) \mapsto \mathcal{W} \mathcal{Q}(\bar{F})$ must be of the form $\mathcal{W Q}(F) \mapsto \overline{\mathcal{W Q}(F)}$. Therefore, we have $\mathcal{W} \mathcal{Q}(\bar{F})=\overline{\mathcal{W} \mathcal{Q}(F)}$.

Let $\hbar \in \mathbb{R}^{\times}$and $F \in \mathcal{O}_{k}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$. Using Theorem4.9 (i) and Proposition4.11, we have

$$
\mathcal{W} \mathcal{Q}(F)^{*}=\left(\left(\frac{\hbar}{\mathrm{i}}\right)^{-\frac{k}{2}} \mathcal{W} \mathcal{Q}_{\hbar}(F)\right)^{*}=\left(\frac{\hbar}{\mathrm{i}}\right)^{-\frac{k}{2}} \mathrm{i}^{-k} \mathcal{W} \mathcal{Q}_{\hbar}(\tau(F))=\mathrm{i}^{-k} \mathcal{W} \mathcal{Q}(\tau(F))
$$

The second assertion of the proposition thus follows from (4.20).
The Weyl quantization map $\mathcal{W} \mathcal{Q}_{\hbar}$ in (4.19) induces a star-product $\star_{\hbar}$ on the symplectic manifold $T^{*}[2] M \oplus E[1]$. The properties of the map $\mathcal{W} \mathcal{Q}_{\hbar}$, exhibited above, can be rephrased in terms of properties of $\star_{\hbar}$.

Corollary 4.13. The product defined on $\mathcal{O}^{\mathbb{C}}\left(T^{*}[2] M \oplus E[1]\right)$ by

$$
F \star_{\hbar} G:=\left(\mathcal{W} \mathcal{Q}_{\hbar}\right)^{-1}\left(\mathcal{W} \mathcal{Q}_{\hbar}(F) \circ \mathcal{W} \mathcal{Q}_{\hbar}(G)\right)
$$

is a symmetric star-product, explicitly given by

$$
F \star_{\hbar} G=\sum_{k=0}^{\infty}\left(\frac{\hbar}{\mathrm{i}}\right)^{k} B_{2 k}(F, G),
$$

such that, for each $k \in \mathbb{N}$,
(i) $B_{2 k}$ is a real bidifferential operator of degree $-2 k$ independent of $\hbar$,
(ii) $B_{2 k}$ is symmetric if $k$ is even and skew-symmetric if $k$ is odd,
(iii) $B_{0}$ is the multiplication and $B_{2}$ is the Poisson bracket on $\left(T^{*}[2] M \oplus E[1], \omega_{\mathrm{g}, \nabla^{E}}\right)$.

Proof. From Theorem 4.9 (i), we deduce that

$$
F \star_{\hbar} G=\sum_{k=0}^{\infty}\left(\frac{\hbar}{\mathrm{i}}\right)^{k / 2} B_{k}(F, G),
$$

where, for each $k \geq 0, B_{k}$ is a bilinear operator of degree $-k$. By Lemma 4.4 and Eq. (4.1), we see that $B_{k}$ is a bidifferential operator. Since $\mathcal{W} \mathcal{Q}_{\hbar}$ preserves the parity, we deduce that $B_{k}$ vanishes if $k$ is odd. From Theorem 4.9 (ii) and (iv), it follows that $B_{0}(F, G)=F G$ and $B_{1}(F, G)=\{F, G\}$.

Set $\hbar=\mathrm{i}$. According to Proposition4.12, the operator $B_{2 k}$ is real and (skew-)symmetric according to the parity of $k$. This concludes the proof of the corollary.

## 5. Applications to Courant algebroids

5.1. Definition of Courant algebroids. A pre-Courant algebroid is a pseudo-Euclidean vector bundle $(E, \mathrm{~g})$ over a smooth manifold $M$, together with a vector bundle morphism $\rho: E \rightarrow T M$, called the anchor, and an $\mathbb{R}$-bilinear operation $\llbracket \cdot, \cdot \rrbracket$ on $\Gamma(E)$, called the Dorfman bracket, subject to the following rules:

$$
\begin{align*}
& \llbracket \xi, f \cdot \eta \rrbracket=\rho(\xi)[f] \cdot \eta+f \cdot \llbracket \xi, \eta \rrbracket, \\
& \llbracket \xi, \xi \rrbracket=\frac{1}{2} \rho^{*} \mathrm{~d}(\mathrm{~g}(\xi, \xi)),  \tag{5.1}\\
& \rho(\xi)[\mathrm{g}(\eta, \eta)]=2 \mathrm{~g}(\llbracket \xi, \eta \rrbracket, \eta),
\end{align*}
$$

for all $f \in \mathrm{C}^{\infty}(M)$ and $\xi, \eta \in \Gamma(E)$. In the second equation above, d stands for the de Rham differential and $\rho^{*}: T^{*} M \rightarrow E^{*} \cong E$ is the dual map of $\rho$. Moreover, if the bracket satisfies the Jacobi identity

$$
\begin{equation*}
\llbracket \xi, \llbracket \eta_{1}, \eta_{2} \rrbracket \rrbracket=\llbracket \llbracket \xi, \eta_{1} \rrbracket, \eta_{2} \rrbracket+\llbracket \eta_{1}, \llbracket \xi, \eta_{2} \rrbracket \rrbracket, \tag{5.2}
\end{equation*}
$$

for all $\xi, \eta_{1}, \eta_{2} \in \Gamma(E)$, then $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ is called a Courant algebroid [24, 31].
Example 5.1. For any smooth manifold $M$, the vector bundle $E=T M \oplus T^{*} M$ admits a standard Courant algebroid structure, where the anchor is the projection onto the first component and the pairing and Dorfman bracket are given, respectively by

$$
\begin{aligned}
\mathrm{g}(X+\alpha, Y+\beta) & =\langle\alpha, Y\rangle+\langle\beta, X\rangle \\
\llbracket X+\alpha, Y+\beta \rrbracket & =[X, Y]+\mathrm{L}_{X} \beta-\iota_{Y} \mathrm{~d} \alpha
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(M) \cong \Gamma(T M)$ and $\alpha, \beta \in \Gamma\left(T^{*} M\right)$. One can also twist the above bracket by a closed 3 -form $H \in \Omega^{3}(M)$ [34, 35],

$$
\llbracket X+\alpha, Y+\beta \rrbracket=[X, Y]+\mathrm{L}_{X} \beta-\iota_{Y} \mathrm{~d} \alpha+H(X, Y, \cdot)
$$

For more examples, see e.g. [24].
5.2. Courant algebroids and symplectic graded manifolds of degree 2. We assume, from now on, that $(E, \mathrm{~g})$ is endowed with a metric connection $\nabla^{E}$ so that the minimal symplectic realization of the Poisson manifold $E[1]$ is given by $\left(T^{*}[2] M \oplus E[1], \omega_{\mathrm{g}, \nabla^{E}}\right)$ as in Proposition 2.1 (see Section 2.3). The Poisson bracket on $T^{*}[2] M \oplus E[1]$ is denoted by $\{\cdot, \cdot\}$ which is of degree -2 . We will use the following identifications without further comments:

$$
\begin{aligned}
& \mathcal{O}_{0}\left(T^{*}[2] M \oplus E[1]\right) \cong \mathrm{C}^{\infty}(M), \\
& \mathcal{O}_{1}\left(T^{*}[2] M \oplus E[1]\right) \cong \Gamma(E), \\
& \mathcal{O}_{2}\left(T^{*}[2] M \oplus E[1]\right) \cong \Gamma\left(T M \oplus \wedge^{2} E\right)
\end{aligned}
$$

Every degree 3 function $\Theta \in \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$ induces a pre-Courant algebroid structure on $(E, \mathrm{~g})$ by setting

$$
\begin{align*}
\rho(\xi)[f] & :=\{\{\Theta, \xi\}, f\}, \quad \forall f \in \mathrm{C}^{\infty}(M), \xi, \eta \in \Gamma(E) . \\
\llbracket \xi, \eta \rrbracket & :=\{\{\Theta, \xi\}, \eta\}, \quad
\end{align*}
$$

The structural identities (5.1) follow from the fact that $\{\cdot, \cdot\}$ is a Poisson bracket which satisfies the relation $\{\xi, \eta\}=\mathrm{g}(\xi, \eta)$ for all $\xi, \eta \in \Gamma(E)$. Moreover, if $\{\Theta, \Theta\}=0$, then $\llbracket \cdot, \cdot \rrbracket$ satisfies the Jacobi identity (5.2) and therefore $E$ becomes a Courant algebroid.

Conversely, one can construct a degree 3 function $\Theta \in \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$ out of a pre-Courant algebroid $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ as follows. The anchor map $\rho$, being a bundle map, can be identified with a section in $\Gamma\left(T M \otimes E^{*}\right)$, which is a function of degree 3 on $T^{*}[2] M \oplus E[1]$. From g and $\llbracket \cdot, \cdot \rrbracket$, one can define the torsion map $\mathrm{C}_{\nabla}: \Gamma\left(\wedge^{3} E\right) \rightarrow \mathrm{C}^{\infty}(M)$ by

$$
\begin{equation*}
\mathrm{C}_{\nabla}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=\frac{1}{2} \operatorname{cycl}_{123} \mathrm{~g}\left(\frac{1}{3}\left(\llbracket \xi_{1}, \xi_{2} \rrbracket-\llbracket \xi_{2}, \xi_{1} \rrbracket\right)-\left(\nabla_{\rho\left(\xi_{1}\right)}^{E} \xi_{2}-\nabla_{\rho\left(\xi_{2}\right)}^{E} \xi_{1}\right), \xi_{3}\right), \tag{5.4}
\end{equation*}
$$

where $\xi_{1}, \xi_{2}, \xi_{3} \in \Gamma(E)$ and cycl $_{123}$ denotes the sum over cyclic permutations. As proved in [1], the identities (5.1) ensure that $\mathrm{C}_{\nabla}$ is $\mathrm{C}^{\infty}(M)$-multilinear, so that it can be identified with a section in $\Gamma\left(\wedge^{3} E\right)$. Set $\Theta=\rho-C_{\nabla}$. Then, $\Theta$ is a degree 3 function which satisfies Eq. (5.3). Moreover, if $\llbracket \cdot, \cdot \rrbracket$ satisfies the Jacobi identity (5.2), we have $\{\Theta, \Theta\}=0$. Thus we recover the following

Theorem 5.2 ([32]). Let $(E, \mathrm{~g})$ be a pseudo-Euclidean vector bundle over $M$. There is a bijection between pre-Courant algebroids $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ and degree 3 functions $\Theta \in \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$. They are related via $\Theta=\rho-\mathrm{C}_{\nabla}$.

Moreover, $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ is a Courant algebroid if and only if $\{\Theta, \Theta\}=0$.
The above function $\Theta$ is called the Hamiltonian generating function of the Courant algebroid.
Remark 5.3. Both $\Theta$ and the Poisson bracket on $T^{*}[2] M \oplus E[1]$ depend on the choice of a metric connection on $E$. Via the symplectic diffeomorphism $\Xi_{\nabla}: \mathcal{M} \rightarrow T^{*}[2] M \oplus E[1]$, both of them admit an intrinsic version on $\mathcal{M}$. They satisfy again Eq. (5.3) (see [32]).
5.3. Dirac generating operators of Courant algebroids. There is another approach to generate a Courant algebroid, via the so-called Dirac generating operators [1, 1].

From now on, we assume that $(E, g)$ admits a spinor bundle $S$ and that $\operatorname{det}\left(S^{*}\right)^{1 / N}$ exists, with $N$ being the rank of $S$. Then, the twisted spinor bundle, $\mathbb{S}:=S \otimes\left(\operatorname{det} S^{*}\right)^{1 / N} \otimes\left|\wedge^{\text {top }} T^{*} M\right|^{1 / 2}$, is well-defined. The algebra of real differential operators $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, defined in Proposition 3.18, inherits a $\mathbb{Z}_{2}$-grading and a filtration from $\mathcal{D}(M, \mathbb{S})$ (cf. (3.6) and (3.8)). In particular, we have

$$
\begin{gathered}
\mathcal{D}_{0}(M, \mathbb{S})_{\mathbb{R}} \cong \mathrm{C}^{\infty}(M) \cong \mathcal{O}_{0}\left(T^{*}[2] M \oplus E[1]\right) \\
\mathcal{D}_{1}^{-}(M, \mathbb{S})_{\mathbb{R}} \cong \Gamma(E) \cong \mathcal{O}_{1}\left(T^{*}[2] M \oplus E[1]\right)
\end{gathered}
$$

Since the Weyl quantization map $\mathcal{W Q}: \mathcal{O}\left(T^{*}[2] M \oplus E[1]\right) \rightarrow \mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ (see Proposition4.12) preserves the parity (see Theorem4.9), it induces the following isomorphism:

$$
\mathcal{W} \mathcal{Q}^{-1}: \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}} \xrightarrow{\sim} \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right) \oplus \mathcal{O}_{1}\left(T^{*}[2] M \oplus E[1]\right)
$$

The Dorfman bracket can be obtained as a derived bracket of the commutator in $\mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$, for a well-chosen generating operator $D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$, as shown in [1]. This approach provides a quantum analog to the Hamiltonian picture for Courant algebroids, presented in the previous section. Namely, as the commutator lowers the order by 2 and preserves the parity, we can define

$$
\begin{align*}
\rho(\xi)[f] & :=[[D, \gamma(\xi)], f] \in \mathcal{D}_{0}(M, \mathbb{S})_{\mathbb{R}} \cong \mathrm{C}^{\infty}(M), \\
\llbracket \xi, \eta \rrbracket: & :[[D, \gamma(\xi)], \gamma(\eta)] \in \mathcal{D}_{1}^{-}(M, \mathbb{S})_{\mathbb{R}} \cong \Gamma(E) \tag{5.5}
\end{align*}
$$

We have the following
Proposition 5.4. Let $D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$ such that $\sigma_{3}(D) \neq 0$. Then
(i) the map $\rho$ and the bracket $\llbracket \cdot, \cdot \rrbracket$ given by (5.5) define a pre-Courant algebroid structure on $(E, \mathrm{~g})$;
(ii) the pre-Courant algebroid defined by $D$ as in (5.5) coincides with the one defined by its principal symbol $\sigma_{3}(D)$ as in (5.3);
(iii) the operator $D$ generates a Courant algebroid if and only if $D^{2} \in \mathcal{D}_{2}(M, \mathbb{S})_{\mathbb{R}}$.

Proof. Using Eq. (3.14), we obtain that the map $\rho$ and the bracket $\llbracket \cdot, \cdot \rrbracket$, defined by $D$ via (5.5), indeed coincide with the ones determined by $\sigma_{3}(D)$ via (5.3). Hence, they define a pre-Courant algebroid structure on $(E, \mathrm{~g})$.

It remains to prove the last assertion. Since $D$ is odd, we have $D^{2}=\frac{1}{2}[D, D]$. As the commutator lowers the order by 2 and preserves the parity, the operator $[D, D]$ is of order 4,2 or 0 . By Eq. (3.14), its principal symbol satisfies $\sigma_{4}([D, D])=\left\{\sigma_{3}(D), \sigma_{3}(D)\right\}$. Hence, $D^{2} \in \mathcal{D}_{2}(M, \mathbb{S})_{\mathbb{R}}$ if and only if $\left\{\sigma_{3}(D), \sigma_{3}(D)\right\}=0$. The result thus follows from Theorem 5.2,

In [1], a stronger condition of Dirac generating operators was introduced.
Definition 5.5. [1] $A$ Dirac generating operator is an operator $D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$, such that $\sigma_{3}(D) \neq 0$ and $D^{2} \in \mathcal{D}_{0}(M, \mathbb{S})_{\mathbb{R}}$.

According to Proposition 5.4 a Dirac generating operator indeed generates a Courant algebroid structure on $(E, \mathrm{~g})$ via (5.5).

The existence of a Dirac generating operator for a given Courant algebroid is nontrivial and was one of the main results of [1]. Note that Dirac generating operators are not unique: two Dirac generating operators $D$ and $D^{\prime}$ defining the same Courant algebroid have the same principal symbol but may differ by an element $\gamma(\xi) \in \mathcal{D}_{1}^{-}(M, \mathbb{S})_{\mathbb{R}}$ satisfying $\left\{\sigma_{3}(D), \xi\right\}=0$. As an application of our theory, in what follows, we construct a Dirac generating operator $D$ via the Weyl quantization map $\mathcal{W Q}$. It turns out that such a Dirac generating operator $D$ is independent of any choice of geometric data and is therefore unique. Below, following an idea of Ševera [34, we describe a completely intrinsic characterization of such an operator $D$ in terms of the adjoint operation defined in (3.23).

Theorem 5.6. Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ be a Courant algebroid admitting a twisted spinor bundle $\mathbb{S}$. There exists a unique Dirac generating operator $D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$ satisfying:
(i) $D$ generates the given Courant algebroid, and
(ii) $D^{*}=-D$, where $*$ is the adjoint operation defined in (3.23).

In fact, we have $D=\mathcal{W} \mathcal{Q}(\Theta)$, where $\Theta \in \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$ is the Hamiltonian generating function of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$.

Note that the adjoint operation $*$ does not depend on any choice of connections according to Proposition 3.16, We need a lemma first.

Lemma 5.7. The Weyl quantization map $\mathcal{W Q}$ induces a linear isomorphism

$$
\begin{equation*}
\mathcal{W Q}: \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right) \xrightarrow{\sim}\left\{D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}} \mid D^{*}=-D\right\} \tag{5.6}
\end{equation*}
$$

which establishes a bijection between Hamiltonian generating functions and skew-symmetric Dirac generating operators.

Proof. According to Proposition4.12, we have $\mathcal{W} \mathcal{Q}(F)^{*}=(-1)^{\lfloor k / 2\rfloor} \mathcal{W} \mathcal{Q}(F)$ for all $F \in \mathcal{O}_{k}\left(T^{*}[2] M \oplus\right.$ $E[1])$. Hence, the map $\mathcal{W} \mathcal{Q}$ sends $\mathcal{O}_{1}\left(T^{*}[2] M \oplus E[1]\right)$ to the space of symmetric (or self-adjoint) operators in $\mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$ and $\mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$ to the space of skew-symmetric operators in $\mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$. Thus, the map (5.6) is indeed an isomorphism.

Let $\Theta$ be a function of degree 3 and $D=\mathcal{W} \mathcal{Q}(\Theta)$. By Corollary 4.13, we have

$$
2(\mathcal{W} \mathcal{Q})^{-1}\left(D^{2}\right)=\{\Theta, \Theta\}+B_{4}(\Theta, \Theta)+B_{6}(\Theta, \Theta)
$$

where $B_{6}(\Theta, \Theta)$ is of degree 0 and $B_{4}(\cdot, \cdot)$ is a symmetric bidifferential operator. As $\Theta$ is of odd degree, we have $B_{4}(\Theta, \Theta)=0$. As a consequence, we conclude that $\{\Theta, \Theta\}=0$ is equivalent to $D^{2} \in \mathcal{D}_{0}(M, \mathbb{S})_{\mathbb{R}}$.

Proof of Theorem [5.6. Let $\Theta \in \mathcal{O}_{3}\left(T^{*}[2] M \oplus E[1]\right)$ be the Hamiltonian generating function of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$. By Proposition [5.4, a Dirac generating operator $D \in \mathcal{D}_{3}^{-}(M, \mathbb{S})_{\mathbb{R}}$ of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ must satisfy $\sigma_{3}(D)=\Theta$. According to Lemma [5.7, there exists a unique such $D$ satisfying in addition $D^{*}=-D$, which is indeed given by $D=\mathcal{W} \mathcal{Q}(\Theta)$.

Next we will describe an explicit formula for the skew-symmetric Dirac generating operator $D$. By uniqueness, $D$ does not depend on any choice of connections. However, the Weyl quantization map $\mathcal{W} \mathcal{Q}$ does. This is reflected in the formula (5.7) below, which is written in terms of the connections $\nabla^{\mathbb{S}}$
on $\mathbb{S}$ and $\nabla$ on $E \otimes T M$. The latter are induced by a connection on $T M$ and compatible connections on $S$ and $E$. In addition, the formula (5.7) involves the torsion $\mathrm{C}_{\nabla} \in \Gamma\left(\wedge^{3} E\right)$ (see Eq. (5.4)), a local frame $\left(\xi^{a}\right)$ of $E$ and the metric components $\mathrm{g}_{a b}$, defined by $\mathrm{g}^{b c}=\mathrm{g}\left(\xi^{b}, \xi^{c}\right)$ and $\mathrm{g}_{a b} \mathrm{~g}^{b c}=\delta_{a}^{c}$.

Corollary 5.8. Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$ be a Courant algebroid. The skew-symmetric Dirac generating operator is given by the following formula

$$
\begin{equation*}
D=\mathrm{g}_{a b} \gamma\left(\xi^{a}\right) \nabla_{\rho\left(\xi^{b}\right)}^{\mathbb{S}}-\gamma\left(\mathrm{C}_{\nabla}\right)+\frac{1}{2} \gamma(\operatorname{Tr} \nabla \rho), \tag{5.7}
\end{equation*}
$$

where $\nabla \rho \in \Gamma\left(E \otimes T M \otimes T^{*} M\right)$ and $\operatorname{Tr} \nabla \rho \in \Gamma(E)$ denotes its trace as defined in (4.13).
Proof. By Theorem 5.6 and Theorem 5.2, we know that $D=\mathcal{W} \mathcal{Q}(\Theta)$ with $\Theta=\rho-\mathrm{C}_{\nabla}$. The result follows from Proposition 4.10.
5.4. An alternative formula for Dirac generating operators. We now prove that the Dirac generating operator constructed by Alekseev and Xu in [1] coincides with the operator given by Eq. (5.7). We need to introduce several notations, directly borrowed from [1].

Definition $5.9(\boxed{1})$. Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ be a Courant algebroid and $V$ a vector bundle over $M$. An $\mathbb{R}$-bilinear map $\nabla: \Gamma(E) \times \Gamma(V) \rightarrow \Gamma(V)$ is called an $E$-connection on $V$ if it satisfies the following conditions:

$$
\begin{aligned}
\nabla_{f \xi} v & =f \nabla_{\xi} v, \quad \text { and } \\
\nabla_{\xi}(f v) & =f \nabla_{\xi} v+\rho(\xi)[f] \cdot v,
\end{aligned}
$$

for all $\xi \in \Gamma(E), v \in \Gamma(V)$ and $f \in \mathrm{C}^{\infty}(M)$.
Remark 5.10. An ordinary linear connection $\nabla$ on the vector bundle $V$ induces an $E$-connection as follows:

$$
\begin{equation*}
\nabla_{\xi} v:=\nabla_{\rho(\xi)} v, \quad \forall \xi \in \Gamma(E), v \in \Gamma(V) \tag{5.8}
\end{equation*}
$$

Let $\nabla^{E}$ be an $E$-connection on $E$. According to [1],

$$
\begin{equation*}
\mathrm{C}_{\boldsymbol{\nabla}}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=\frac{1}{2} \operatorname{cycl}_{123} \mathrm{~g}\left(\frac{1}{3}\left(\llbracket \xi_{1}, \xi_{2} \rrbracket-\llbracket \xi_{2}, \xi_{1} \rrbracket\right)-\left(\boldsymbol{\nabla}_{\xi_{1}}^{E} \xi_{2}-\nabla_{\xi_{2}}^{E} \xi_{1}\right), \xi_{3}\right) \tag{5.9}
\end{equation*}
$$

defines a section in $\Gamma\left(\wedge^{3} E\right)$, where $\operatorname{cycl}_{123}$ denotes the sum over cyclic permutations. This is called the torsion of $E$ with respect to $\nabla^{E}$. If $\nabla^{E}$ is induced from an ordinary connection $\nabla^{E}$ as in Eq. (5.8), then $\mathrm{C}_{\nabla}$ coincides with the torsion $\mathrm{C}_{\nabla}$, introduced previously in (5.4).

Let $\left(\xi^{a}\right)$ be a local frame of $E$. For any $\xi \in \Gamma(E)$, the function $\operatorname{Div} \xi \in \mathrm{C}^{\infty}(M)$, defined by

$$
\begin{equation*}
\operatorname{Div} \xi=\sum_{a} \mathrm{~g}\left(\nabla_{\xi^{a}}^{E} \xi, \xi^{a}\right) \tag{5.10}
\end{equation*}
$$

does not depend on the chosen local frame $\left(\xi^{a}\right)$. We call Div $\xi$ the divergence of $\xi$. Then,

$$
\begin{equation*}
\nabla_{\xi}^{\Lambda} s:=\mathrm{L}_{\rho(\xi)} s-(\operatorname{Div} \xi) s \tag{5.11}
\end{equation*}
$$

defines an $E$-connection on the line bundle $\wedge^{\text {top }} T^{*} M$. Here $\mathrm{L}_{\rho(\zeta)} s$ stands for the Lie derivative of $s \in \Gamma\left(\wedge^{\text {top }} T^{*} M\right)$ along the vector field $\rho(\zeta) \in \mathfrak{X}(M)$.

Lemma 5.11. Let $\nabla^{E}$ be a metric connection on $E$. It induces an $E$-connection on $E$ as in Eq. (5.8) and an E-connection $\nabla^{\Lambda}$ on $\wedge^{t o p} T^{*} M$ via Eq. (5.11). For any linear connection $\nabla^{T M}$ on $T M$, we have the relation

$$
\nabla_{\xi}^{\Lambda}=\nabla_{\rho(\xi)}^{\Lambda}+\mathrm{g}(\xi, \operatorname{Tr} \nabla \rho), \quad \forall \xi \in \Gamma(E)
$$

where $\nabla^{\Lambda}$ and $\nabla$ are induced connections on $\wedge^{\text {top }} T^{*} M$ and $E \otimes T M$ respectively.
Proof. The Lie derivative on $\wedge^{\text {top }} T^{*} M$ satisfies the relation:

$$
\begin{equation*}
\mathrm{L}_{\rho(\xi)}=\nabla_{\rho(\xi)}^{\Lambda}+\operatorname{Tr} \nabla^{T M}(\rho(\xi)) \tag{5.12}
\end{equation*}
$$

Since $\nabla^{E}$ preserves the metric, we have $\nabla^{T M}(\rho(\xi))=\mathrm{g}(\nabla \rho, \xi)+\mathrm{g}\left(\rho, \nabla^{E} \xi\right)$, which is an equality in $\Gamma\left(T M \otimes T^{*} M\right)$. By taking the trace, we obtain

$$
\begin{equation*}
\operatorname{Tr} \nabla^{T M}(\rho(\xi))=\mathrm{g}(\operatorname{Tr} \nabla \rho, \xi)+\operatorname{Div} \xi \tag{5.13}
\end{equation*}
$$

where the divergence of $\xi$ is computed for the $E$-connection on $E$ induced by $\nabla^{E}$ as in (5.10). The conclusion thus follows by combining Eqns (5.11), (5.12) and (5.13).

An $E$-connection on $E$ preserves the metric if

$$
\rho\left(\xi_{1}\right)\left[\mathrm{g}\left(\xi_{2}, \xi_{3}\right)\right]=\mathrm{g}\left(\boldsymbol{\nabla}_{\xi_{1}}^{E} \xi_{2}, \xi_{3}\right)+\mathrm{g}\left(\xi_{2}, \boldsymbol{\nabla}_{\xi_{1}}^{E} \xi_{3}\right)
$$

for all $\xi_{1}, \xi_{2}, \xi_{3} \in \Gamma(E)$. An $E$-connection $\nabla^{E}$ on $E$ and an $E$-connection $\nabla^{S}$ on $S$ are compatible if $\nabla^{E}$ preserves the metric, and their induced connections on the Clifford algebra bundle coincide under the isomorphism $\mathbb{C l}(E) \cong \operatorname{End} S$. By Eq. (5.11), we see that such a pair of compatible $E$-connections induces an $E$-connection on the twisted spinor bundle $\mathbb{S}=S \otimes\left(\operatorname{det} S^{*}\right)^{1 / N} \otimes\left|\wedge^{\text {top }} T^{*} M\right|^{1 / 2}$.

Proposition 5.12. Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ be a Courant algebroid with twisted spinor bundle $\mathbb{S}$. The unique skew-symmetric Dirac generating operator is given by

$$
\begin{equation*}
D=\mathrm{g}_{a b} \gamma\left(\xi^{a}\right) \nabla_{\xi^{b}}^{\mathbb{S}}-\gamma\left(\mathrm{C}_{\boldsymbol{\nabla}}\right) \tag{5.14}
\end{equation*}
$$

where $\boldsymbol{\nabla}^{\mathbb{S}}$ is the $E$-connection on $\mathbb{S}$ induced by compatible $E$-connections on $S$ and $E$. Here, $\left(\xi^{a}\right)$ is a local frame of $E$ and $\mathrm{g}_{a b}$ are the metric components.

Proof. According to [1, Theorem 4.3], the formula (5.14) does not depend on the choice of $E$ connections on $E$ and $S$. Hence, we can start with $E$-connections induced by ordinary linear connections. In this case, we have $\mathrm{C}_{\boldsymbol{\nabla}}=\mathrm{C}_{\nabla}$ and $\nabla_{\xi}^{\mathbb{S}}=\nabla_{\rho(\xi)}^{\mathbb{S}}+\frac{1}{2} \mathrm{~g}(\xi, \operatorname{Tr} \nabla \rho)$ by Lemma 5.11. Substituting both equalities in Eq. (5.14) yields Eq. (5.7). The result thus follows.
5.5. A Courant algebroid invariant. Since a skew-symmetric Dirac generating operator $D$ is unique, then $D^{2} \in \mathrm{C}^{\infty}(M)$ is an invariant of the Courant algebroid. It is natural to ask how this invariant can be described geometrically.

Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ be a Courant algebroid. By abuse of notation, we denote by the same symbol g the metric on $\wedge E$ induced by the metric on $E$ as follows: $\mathrm{g}(\xi, \eta)=\operatorname{det}\left(\mathrm{g}\left(\xi_{i}, \eta_{j}\right)\right)$ for all $\xi=\xi_{1} \wedge \cdots \wedge \xi_{k}$ and $\eta=\eta_{1} \wedge \cdots \wedge \eta_{k}$ in $\Gamma\left(\wedge^{k} E\right)$. Using a metric connection $\nabla^{E}$ on $E$ and a linear connection $\nabla^{T M}$ on $T M$, we define $f_{E} \in \mathrm{C}^{\infty}(M)$ by

$$
\begin{equation*}
f_{E}=\mathrm{g}\left(\mathrm{C}_{\nabla}, \mathrm{C}_{\nabla}\right)-\frac{1}{4} \mathrm{~g}(\operatorname{Tr} \nabla \rho, \operatorname{Tr} \nabla \rho)-\frac{1}{2} \operatorname{Div}(\operatorname{Tr} \nabla \rho), \tag{5.15}
\end{equation*}
$$

where $\mathrm{C}_{\nabla}$ is the torsion defined as in (5.4), $\nabla$ denotes the induced connection on $E \otimes T M$ and $\operatorname{Div}(\operatorname{Tr} \nabla \rho)=\sum_{a} \mathrm{~g}\left(\xi^{a}, \nabla_{\rho\left(\xi^{a}\right)}^{E}(\operatorname{Tr} \nabla \rho)\right)$ is the divergence of $\operatorname{Tr} \nabla \rho \in \Gamma(E)$ (see Eq. (5.10)).

Theorem 5.13. The function $f_{E}$ satisfies the following properties:
(i) $f_{E}$ is independent of the choice of the connections $\nabla^{T M}$ and $\nabla^{E}$;
(ii) $\left\{\Theta, f_{E}\right\}=0$, where $\Theta=\rho-\mathrm{C}_{\nabla}$ is the Hamiltonian generating function of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$;
(iii) if ( $E, \mathrm{~g}$ ) admits a twisted spinor bundle, then $f_{E}=-D^{2}$, where $D$ is the skew-symmetric Dirac generating operator of $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket)$ as in Theorem 5.6.

Proof. We prove the last assertion first. In view of (5.15), it suffices to prove this locally over a contractible open subset $U \subset M$. Being restricted to $U$, a pseudo-Euclidean vector bundle $\left(\left.E\right|_{U}, \mathrm{~g}\right)$ always admits a twisted spinor bundle $\left.\mathbb{S}\right|_{U}$. According to Theorem 5.6, we obtain a skew-symmetric Dirac generating operator $D \in \mathcal{D}\left(U,\left.\mathbb{S}\right|_{U}\right)_{\mathbb{R}}$. By a straightforward computation using Eq. (5.7), we check that $f_{E}=-D^{2}$.

Since the skew-symmetric Dirac generating operator $D$ is independent of the choice of the connections $\nabla$ and $\nabla^{E}$ according to Theorem [5.6] so does $f_{E}=-D^{2}$. This proves the assertion (i).

Finally, we have $\left\{\Theta, f_{E}\right\}=-\sigma_{2}\left(\left[D, D^{2}\right]\right)=0$. Hence the assertion (ii) follows.
As a consequence, the function $f_{E}$ is indeed an invariant of the Courant algebroid. Our next goal is to provide an intrinsic formula for this function. We first recall some standard facts on the structure of a Courant algebroid $(E, \mathrm{~g}, \rho, \llbracket \cdot, \cdot \rrbracket)$. It is well-known that $(\operatorname{ker} \rho)^{\perp} \subset \operatorname{ker} \rho$ (see e.g. [32]). Moreover, $(\operatorname{ker} \rho)^{\perp}$ and ker $\rho$ are two-sided ideals in $E$ for the Dorfman bracket. Hence, the quotient

$$
\mathcal{G}:=\operatorname{ker} \rho /(\operatorname{ker} \rho)^{\perp}
$$

is a bundle of quadratic Lie algebras, whose fiberwise Lie brackets $[\cdot, \cdot]^{\mathcal{G}}$ and non-degenerate adinvariant scalar product $(\cdot, \cdot)^{\mathcal{G}}$ are inherited, respectively, from the Dorfman bracket $\llbracket \cdot, \cdot \rrbracket$ and the metric g (for further details, see [10]). Since fibers of $\mathcal{G}$ are quadratic Lie algebras, we have an analogue of Cartan 3-form:

$$
\mathrm{C}_{\mathcal{G}}\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}\right):=\left.\left(\left[\mathbf{r}_{1}, \mathbf{r}_{2}\right]^{\mathcal{G}}, \mathbf{r}_{3}\right)^{\mathcal{G}} \in \wedge^{3} \mathcal{G}\right|_{x},
$$

for all $\mathbf{r}_{1}, \mathbf{r}_{2},\left.\mathbf{r}_{3} \in \mathcal{G}\right|_{x}, \forall x \in M$. If $\rho$ is of constant rank over a neighborhood of a point $x \in M$, we say that the Courant algebroid is regular at the point $x$. In this case, $\mathrm{C}_{\mathcal{G}}$ is smooth in a neighborhood of $x$. Note that regular points form a dense open subset $M_{\text {reg }}$ of the base manifold $M$.

Theorem 5.14. Let $(E, \mathrm{~g}, \rho, \llbracket \cdot, \rrbracket \rrbracket)$ be a Courant algebroid. On $M_{\text {reg }}$, we have

$$
f_{E}=\mathrm{g}\left(\mathrm{C}_{\mathcal{G}}, \mathrm{C}_{\mathcal{G}}\right)
$$

Proof. As both sides of the above equality depend only on the local structure of the Courant algebroid, we can assume that $E$ is a regular Courant algebroid.

We recall some useful constructions regarding regular Courant algebroids [10]. The vector bundle $E$ admits a splitting:

$$
\begin{equation*}
E \cong F^{*} \oplus \mathcal{G} \oplus F \tag{5.16}
\end{equation*}
$$

where $F=\rho(E) \subseteq T M$ is an integrable distribution. Under the above isomorphism, the anchor map $\rho$ becomes the projection $\rho: E \rightarrow F$ and each section $\xi \in \Gamma(E)$ decomposes as $\xi=\eta+\mathbf{r}+x$, with $\eta \in \Gamma\left(F^{*}\right), \mathbf{r} \in \Gamma(\mathcal{G})$ and $x \in \Gamma(F)$. In the remainder of the proof, we use such a decomposition $\xi_{i}=\eta_{i}+\mathbf{r}_{i}+x_{i}$, for all $\xi_{i} \in \Gamma(E)$. Let $\nabla^{T M}$ be a torsion-free connection on $T M$ and denote by $\nabla^{F}$ and $\nabla^{F^{*}}$ the induced connections on $F$ and $F^{*}$ respectively. According to [10, Proposition 1.1] and 10, Lemma 1.4], there exists a metric $F$-connection $\nabla^{\mathcal{G}}: \Gamma(F) \otimes \Gamma(\mathcal{G}) \rightarrow \Gamma(\mathcal{G})$ and a section $\mathcal{H} \in \Gamma\left(\wedge^{3} F^{*}\right)$ such that the formula

$$
\begin{equation*}
\nabla_{\xi_{1}}^{\prime} \xi_{2}=\left(\nabla_{x_{1}}^{F_{1}^{*}} \eta_{2}-\frac{1}{3} \mathcal{H}\left(x_{1}, x_{2}, \cdot\right)\right)+\left(\nabla_{x_{1}}^{\mathcal{G}} \mathbf{r}_{2}+\frac{2}{3}\left[\mathbf{r}_{1}, \mathbf{r}_{2}\right]^{\mathcal{G}}\right)+\nabla_{x_{1}}^{F} x_{2}, \quad \forall \xi_{1}, \xi_{2} \in \Gamma(E), \tag{5.17}
\end{equation*}
$$

defines an $E$-connection $\nabla^{\prime}$ on $E$. As shown in [10, Formula (3.1)], the torsion of $\boldsymbol{\nabla}^{\prime}$ reads as

$$
\begin{equation*}
\mathrm{C}_{\boldsymbol{\nabla}^{\prime}}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=\mathcal{H}\left(x_{1}, x_{2}, x_{3}\right)+\operatorname{cycl}_{123}\left(R^{\mathcal{G}}\left(x_{1}, x_{2}\right), \mathbf{r}_{3}\right)^{\mathcal{G}}-\mathrm{C}_{\mathcal{G}}\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}\right), \tag{5.18}
\end{equation*}
$$

for all $\xi_{i} \in \Gamma(E), i=1,2,3$, where $R^{\mathcal{G}}: \wedge^{2} F \rightarrow \mathcal{G}$ is a certain bundle map.
To compute $f_{E}$ via Eq. (5.15), we need a metric connection on $E$ and a linear connection on $T M$. For the latter, we choose the above connection $\nabla^{T M}$. As for the metric connection on $E$, we define it as follows. Pick a Riemannian metric on $M$ and denote by $F^{\perp} \subseteq T M$ the distribution orthogonal to $F$.

Choose a metric connection $\tilde{\nabla}^{\mathcal{G}}$ on the vector bundle $\mathcal{G}$ and define a new metric connection on $\mathcal{G}$ by setting

$$
\nabla_{X}^{\mathcal{G}} \mathbf{r}:=\nabla_{X_{F}}^{\mathcal{G}}(\mathbf{r})+\tilde{\nabla}_{X_{F \perp}}^{\mathcal{G}}(\mathbf{r}), \quad \forall X \in \mathfrak{X}(M), \mathbf{r} \in \Gamma(\mathcal{G}),
$$

where the splitting $X=X_{F}+X_{F^{\perp}}$ corresponds to the decomposition $T M \cong F \oplus F^{\perp}$. Then a metric connection on $E$ can be defined by the following formula

$$
\begin{equation*}
\nabla^{E}:=\nabla^{F^{*}} \oplus \nabla^{\mathcal{G}} \oplus \nabla^{F} . \tag{5.19}
\end{equation*}
$$

Since the anchor map $\rho$ is identified with the projection onto $F$ through the decomposition (5.16), it is simple to check that

$$
\nabla \rho=\nabla^{T M} \circ \rho-\rho \circ \nabla^{E}=0 .
$$

Therefore, according to Eq. (5.15), we have

$$
\begin{equation*}
f_{E}=\mathrm{g}\left(\mathrm{C}_{\nabla}, \mathrm{C}_{\nabla}\right), \tag{5.20}
\end{equation*}
$$

with $\mathrm{C}_{\nabla}$ being the torsion of $E$ with respect to $\nabla^{E}$.
Our next goal is to compute $\mathrm{C}_{\nabla}$. The linear connection $\nabla^{E}$ defined in (5.19) induces an $E$ connection on $E$ given explicitly by

$$
\nabla_{\xi_{1}} \xi_{2}=\nabla_{x_{1}}^{F^{*}} \eta_{2}+\nabla_{x_{1}}^{\mathcal{G}} \mathbf{r}_{2}+\nabla_{x_{1}}^{F} x_{2}, \quad \forall \xi_{1}, \xi_{2} \in \Gamma(E) .
$$

In view of Eq. (5.17), the $E$-connections $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}^{\prime}$ are related as follows:

$$
\nabla_{\xi_{1}}^{\prime} \xi_{2}-\nabla_{\xi_{1}} \xi_{2}=-\frac{1}{3} \mathcal{H}\left(x_{1}, x_{2}, \cdot\right)+\frac{2}{3}\left[\mathbf{r}_{1}, \mathbf{r}_{2}\right]^{\mathcal{G}} .
$$

Using the fact that $\mathrm{C}_{\boldsymbol{\nabla}}=\mathrm{C}_{\nabla}$ and Eq. (5.9), we deduce that

$$
\begin{aligned}
\mathrm{C}_{\nabla}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)-\mathrm{C}_{\nabla^{\prime}}\left(\xi_{1}, \xi_{2}, \xi_{3}\right) & =\operatorname{cycl}_{123} \mathrm{~g}\left(-\frac{1}{3} \mathcal{H}\left(x_{1}, x_{2}, \cdot\right)+\frac{2}{3}\left[\mathbf{r}_{1}, \mathbf{r}_{2}\right]^{\mathcal{G}}, \xi_{3}\right) \\
& =-\frac{1}{3} \operatorname{cycl}_{123}\left(\mathcal{H}\left(x_{1}, x_{2}, x_{3}\right)\right)+\frac{2}{3} \operatorname{cycl}_{123} \mathrm{~g}\left(\left[\mathbf{r}_{1}, \mathbf{r}_{2}\right]^{\mathcal{G}}, \mathbf{r}_{3}\right), \\
& =-\mathcal{H}\left(x_{1}, x_{2}, x_{3}\right)+2 \mathrm{C}_{\mathcal{G}}\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}\right)
\end{aligned}
$$

for all $\xi_{i}=\eta_{i}+\mathbf{r}_{i}+x_{i} \in \Gamma(E), i=1,2,3$. Therefore, by Eq. (5.18), we have

$$
\mathrm{C}_{\nabla}\left(\xi_{1}, \xi_{2}, \xi_{3}\right)=\mathrm{C}_{\mathcal{G}}\left(\mathbf{r}_{1}, \mathbf{r}_{2}, \mathbf{r}_{3}\right)+\operatorname{cycl}_{123}\left(R^{\mathcal{G}}\left(x_{1}, x_{2}\right), \mathbf{r}_{3}\right)^{\mathcal{G}}
$$

This means that $\mathrm{C}_{\nabla}-\mathrm{C}_{\mathcal{G}} \in \Gamma\left(\wedge^{2} F^{*} \otimes \mathcal{G}\right)$. Using the fact that $\mathrm{g}\left(F^{*}, F^{*} \oplus \mathcal{G}\right)=0$ and Eq. (5.20), we obtain $f_{E}=\mathrm{g}\left(\mathrm{C}_{\mathcal{G}}, \mathrm{C}_{\mathcal{G}}\right)$. This concludes the proof of the theorem.

Remark 5.15. By definition-see Eq. (5.15), the function $f_{E}$ is always smooth, even for a nonregular Courant algebroid $E$. Therefore, one can consider the function $f_{E}$ as the smooth extension of $\mathrm{g}\left(C_{\mathcal{G}}, C_{\mathcal{G}}\right)$ from $M_{\text {reg }}$ to $M$. There is no reason to expect that the function $\mathrm{g}\left(C_{\mathcal{G}}, C_{\mathcal{G}}\right)$ itself is smooth at non-regular points of $E$. However we were unable to find an example of Courant algebroid such that $\mathrm{g}\left(C_{\mathcal{G}}, C_{\mathcal{G}}\right) \neq f_{E}$.

## 6. Applications to Lie bialgebroids

6.1. Weyl quantization in the splittable case. From now on, we assume that the pseudoEuclidean vector bundle ( $E, \mathrm{~g}$ ) splits as $E:=A \oplus A^{*}$ and the pseudo-metric on $E$ is given by

$$
\begin{equation*}
\mathrm{g}\left(\zeta_{1}+\eta_{1}, \zeta_{2}+\eta_{2}\right)=\left\langle\zeta_{1}, \eta_{2}\right\rangle+\left\langle\zeta_{2}, \eta_{1}\right\rangle, \quad \forall \zeta_{1}, \zeta_{2} \in \Gamma(A), \eta_{1}, \eta_{2} \in \Gamma\left(A^{*}\right), \tag{6.1}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ is the duality pairing between $A$ and $A^{*}$. The previous constructions can then be described more explicitly.

According to Example 3.1, $S_{\mathbb{R}}:=\wedge A^{*}$ is a real spinor bundle of $(E, \mathrm{~g})$. Under the identification

$$
\Gamma\left(S_{\mathbb{R}}\right)=\mathcal{O}(A[1])
$$

the Clifford action of $\Gamma(E)$, given in (3.3), reads as

$$
\begin{equation*}
\gamma(\zeta) \phi=\left\langle\zeta, \eta^{a}\right\rangle\left(\frac{\partial}{\partial \eta^{a}} \phi\right) \quad \text { and } \quad \gamma(\eta) \phi=\eta \phi, \quad \forall \zeta \in \Gamma(A), \eta \in \Gamma\left(A^{*}\right), \phi \in \mathcal{O}(A[1]) \tag{6.2}
\end{equation*}
$$

where $\left(\eta^{a}\right)$ is a local frame of $A^{*}$. We will need the following relations later on:

$$
\begin{equation*}
\gamma(\psi) \phi=\psi \phi \quad \text { and } \quad \gamma(\psi \zeta) \phi=\left(\left\langle\zeta, \eta^{a}\right\rangle \psi \frac{\partial}{\partial \eta^{a}}+\frac{(-1)^{\kappa}}{2}\langle\zeta, \psi\rangle\right) \phi \tag{6.3}
\end{equation*}
$$

for all $\zeta \in \Gamma(A), \psi \in \mathcal{O}_{\kappa}(A[1]), \phi \in \mathcal{O}(A[1])$. Assume that the line bundle $\mathcal{L}:=\wedge^{\text {top }} A \otimes \wedge^{\text {top }} T^{*} M$ admits a square root, denoted $\mathcal{L}^{1 / 2}$, and set $\mathbb{S}_{\mathbb{R}}:=\wedge A^{*} \otimes \mathcal{L}^{1 / 2}$. Since $\operatorname{det} S_{\mathbb{R}} \cong\left(\wedge^{\text {top }} A^{*}\right)^{\frac{N}{2}}$, with $N$ being the rank of $\wedge A^{*}$, the $\mathbb{C}$-vector bundle $\mathbb{S}:=\mathbb{S}_{\mathbb{R}} \otimes \mathbb{C}$ defines a twisted spinor bundle (see Section 3.3). Accordingly, the $\mathbb{R}$-vector bundle $\mathbb{S}_{\mathbb{R}}$ is called a real twisted spinor bundle. By pull-back along $\pi: A[1] \rightarrow M$, the line bundle $\pi^{*} \mathcal{L}$ can be identified with the Berezinian line bundle $\operatorname{Ber}_{A[1]} \rightarrow A[1]$ (see e.g. [7, 16, 21]). Therefore, we obtain an isomorphism of $\mathcal{O}(A[1])$-modules

$$
\begin{equation*}
v: \Gamma\left(\mathbb{S}_{\mathbb{R}}\right) \xrightarrow{\sim} \Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right) \tag{6.4}
\end{equation*}
$$

A linear connection on the vector bundle $A$ induces a metric connection $\nabla^{E}$ on $E$ and a compatible spinor connection $\nabla^{S}$ on $S_{\mathbb{R}}$. It in turn, together with a connection on $T M$, induces a spinor connection $\nabla^{\mathbb{S}}$ on $\mathbb{S}_{\mathbb{R}}$, compatible with $\nabla^{E}$, and a connection $\nabla$ on the Berezinian line bundle $\operatorname{Ber}_{A[1]}^{1 / 2} \rightarrow A[1]$. The latter can be defined as the pull back connection, via the canonical projection $\pi: A[1] \rightarrow M$, of the induced connection on the line bundle $\mathcal{L}^{1 / 2} \rightarrow M$. It can be expressed explicitly as follows. First, note that the space of vector fields $\mathfrak{X}(A[1])$ is linearly spanned by the vector fields $X$ which, as derivations of $\mathcal{O}(A[1])=\Gamma\left(S_{\mathbb{R}}\right)$, take the form

$$
\begin{equation*}
X=f \cdot \nabla_{X_{0}}^{S}+g \cdot \gamma(\zeta) \tag{6.5}
\end{equation*}
$$

with $f, g \in \mathcal{O}(A[1]), X_{0} \in \mathfrak{X}(M)$ and $\zeta \in \Gamma(A)$. By setting

$$
\begin{equation*}
\nabla_{X}:=v \circ\left(f \cdot \nabla_{X_{0}}^{\mathbb{S}}+g \cdot \gamma(\zeta)\right) \circ v^{-1} \tag{6.6}
\end{equation*}
$$

one obtains a well-defined linear connection $\nabla$ on $\operatorname{Ber}_{A[1]}^{1 / 2} \rightarrow A[1]$.
The algebra $\mathcal{D}\left(A[1], \operatorname{Ber}_{A[1]}^{1 / 2}\right)$, of differential operators on $\operatorname{Ber}_{A[1]}^{1 / 2}$, is a subalgebra of $\operatorname{End}\left(\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)\right)$ generated by multiplication operators by functions on $A[1]$ and covariant derivatives $\nabla_{X}, \forall X \in$ $\mathfrak{X}(A[1])$-see (6.6).

Lemma 6.1. The map defined by $\Upsilon_{A}(D):=v \circ D \circ v^{-1}, \forall D \in \mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}}\right)$, provides an isomorphism of algebras

$$
\Upsilon_{A}: \mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}}\right) \xrightarrow{\sim} \mathcal{D}\left(A[1], \operatorname{Ber}_{A[1]}^{1 / 2}\right)
$$

Proof. By (6.4), the map $\Upsilon_{A}(\cdot)=v \circ \cdot \circ v^{-1}$ is an algebra isomorphism between $\operatorname{End}\left(\Gamma\left(\mathbb{S}_{\mathbb{R}}\right)\right)$ and $\operatorname{End}\left(\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)\right)$. Since generators of $\mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}}\right)$ and $\mathcal{D}\left(A[1], \operatorname{Ber}_{A[1]}^{1 / 2}\right)$ are clearly in bijection via $\Upsilon_{A}$, the result follows.

Since the metric $g$ in Eq. (6.1) is the duality pairing, by Lemma 2.3, we have a symplectic diffeomorphism $\widetilde{\Xi}_{\nabla}$ between $\left(T^{*}[2](A[1]), \omega_{\text {can }}\right)$ and $\left(T^{*}[2] M \oplus E[1], \omega_{\mathrm{g}, \nabla^{E}}\right)$. Therefore, the map $\Phi_{A}:=\left(\widetilde{\Xi}_{\nabla}\right)^{-1}$ defines a symplectic diffeomorphism:

$$
\begin{equation*}
\Phi_{A}: T^{*}[2] M \oplus E[1] \xrightarrow{\sim} T^{*}[2](A[1]) \tag{6.7}
\end{equation*}
$$

Consider the Weyl quantization map $\mathcal{W} \mathcal{Q}: \mathcal{O}\left(T^{*}[2] M \oplus E[1]\right) \rightarrow \mathcal{D}(M, \mathbb{S})_{\mathbb{R}}$ (see Proposition 4.12). According to Remark 3.19, the restriction from $\mathbb{S}$ to $\mathbb{S}_{\mathbb{R}}$ induces an algebra isomorphism

$$
\mathcal{R}: \mathcal{D}(M, \mathbb{S})_{\mathbb{R}} \xrightarrow{\sim} \mathcal{D}\left(M, \mathbb{S}_{\mathbb{R}}\right)
$$

Definition 6.2. The Weyl quantization on $T^{*}[2](A[1])$ is the map

$$
\mathcal{W} \mathcal{Q}^{A}: \mathcal{O}\left(T^{*}[2](A[1])\right) \longrightarrow \mathcal{D}\left(A[1], \operatorname{Ber}_{A[1]}^{1 / 2}\right)
$$

given by

$$
\mathcal{W} \mathcal{Q}^{A}:=\Upsilon_{A} \circ \mathcal{R} \circ \mathcal{W} \mathcal{Q} \circ\left(\Phi_{A}\right)^{*}
$$

Note that $\mathcal{W} \mathcal{Q}^{A}$ is a linear isomorphism.

Remark 6.3. Connections on $A$ and $T M$ give rise to a connection $\nabla$ on the graded vector bundle on $T(A[1]) \rightarrow A[1]$. It would be interesting to compare the quantization map $\mathcal{W} \mathcal{Q}^{A}$ with the Weyl quantization constructed directly from the induced connections on $T(A[1]) \rightarrow A[1]$ and $\operatorname{Ber}_{A[1]}^{1 / 2} \rightarrow A[1]$ via Eq. (4.7).

There is a notion of Lie derivative on the space of sections $\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)$ (see e.g. [21]). We will need its expression in local affine coordinates $\left(x^{i}, \eta^{a}\right)$ on $A[1]$. Assume that $X \in \mathfrak{X}(A[1])$ is a vector field of degree $\kappa$. Using the local expression $X=X^{i} \frac{\partial}{\partial x^{i}}+X^{a} \frac{\partial}{\partial \eta^{a}}$, where $X^{i}$ and $X^{a}$ are local functions on $A[1]$, the Lie derivative on $\operatorname{Ber}_{A[1]}^{1 / 2}$ reads as

$$
\begin{equation*}
\mathrm{L}_{X}=X^{i} \frac{\partial}{\partial x^{i}}+X^{a} \frac{\partial}{\partial \eta^{a}}+\frac{1}{2}\left(\frac{\partial}{\partial x^{i}} X^{i}+(-1)^{\kappa+1} \frac{\partial}{\partial \eta^{a}} X^{a}\right) \tag{6.8}
\end{equation*}
$$

in the trivialization of $\operatorname{Ber}_{A[1]}^{1 / 2}$ provided by $\left(\prod \zeta_{a} \otimes \wedge d x^{i}\right)^{1 / 2}$, with $\left(\zeta_{a}\right)$ being the dual frame of $\left(\eta^{a}\right)$.
Proposition 6.4. For any vector field $X \in \mathfrak{X}(A[1])$, we have

$$
\begin{equation*}
\mathcal{W} \mathcal{Q}^{A}\left(F_{X}\right)=\mathrm{L}_{X} \tag{6.9}
\end{equation*}
$$

where $F_{X}$ is the fiberwise linear function on $T^{*}[2](A[1])$ corresponding to $X$.
Proof. Let $X \in \mathfrak{X}(A[1])$ be any vector field of degree $\kappa$. It suffices to prove Eq. (6.9) at each point $x \in M$. Consider local affine coordinates $\left(x^{i}, \eta^{a}\right)$ on $A[1]$ obtained by pull-back of a Cartesian coordinate system on $T_{x} M \times A_{x}$, via the map $\mathcal{T}_{x}$ defined in (4.3). This system of coordinates induces fiberwise coordinates $\left(\zeta_{a}\right)$ of $A$ and $\left(p_{i}\right)$ of $T^{*} M$. Then, $\left(x^{i}, \eta^{a}, p_{i}, \zeta_{a}\right)$ is a local coordinate system of both $T^{*}[2](A[1])$ and $T^{*}[2] M \oplus E[1]$. By Eq. (2.6), the map $\Phi_{A}: T^{*}[2] M \oplus E[1] \rightarrow T^{*}[2](A[1])$ in (6.7) satisfies

$$
\left(\Phi_{A}\right)^{*} x^{i}=x^{i}, \quad\left(\Phi_{A}\right)^{*} p_{i}^{\nabla}=p_{i}, \quad\left(\Phi_{A}\right)^{*} \eta^{a}=\eta^{a}, \quad\left(\Phi_{A}\right)^{*} \zeta_{a}=\zeta_{a}
$$

where $p_{i}^{\nabla} \in C^{\infty}\left(T^{*}[2](A[1])\right)$ is the fiberwise linear function on $T^{*}[2](A[1])$ corresponding to the vector field $\nabla_{i}^{A}$ on $A[1]$. Note that $\nabla_{i}^{A}=\frac{\partial}{\partial x^{i}}+\Gamma_{i b}^{a} \eta^{b} \frac{\partial}{\partial \eta^{a}}$, where the real function $\Gamma_{i b}^{a}$ vanishes at $x$ by definition of the coordinates $\left(x^{i}, \eta^{a}\right)$. Using the local expression $X=X^{i} \frac{\partial}{\partial x^{i}}+X^{a} \frac{\partial}{\partial \eta^{a}}$, we obtain

$$
\left(\Phi_{A}\right)^{*} F_{X}=X^{i} \cdot\left(p_{i}+\Gamma_{i b}^{a} \eta^{b} \zeta_{a}\right)+X^{a} \zeta_{a}
$$

From Eqns (4.15)-(4.17) and $\Gamma_{i b}^{a}(x)=0$, we deduce that

$$
\left(\mathcal{W} \mathcal{Q}^{A}\left(F_{X}\right) \psi\right)(x, \eta)=\left(\gamma\left(X^{i}(x, \eta)\right) \frac{\partial}{\partial x^{i}}+\frac{1}{2} \gamma\left(\frac{\partial}{\partial x^{i}} X^{i}(x, \eta)\right)+\gamma\left(X^{a}(x, \eta) \zeta_{a}\right)\right) \psi(x, \eta)
$$

in the trivialization of $\operatorname{Ber}_{A[1]}^{1 / 2}$ provided by $\left(\prod \zeta_{a} \otimes \wedge d x^{i}\right)^{1 / 2}$. The result thus follows from Eqns (6.3) and (6.8).

Remark 6.5. Let $X \in \mathfrak{X}(A[1])$ be a vector field as in Eq. (6.5). According to Eqns (4.18) and (6.9), the Lie derivative on $\operatorname{Ber}_{A[1]}^{1 / 2}$ is given by

$$
\mathrm{L}_{X}=\nabla_{X}+\frac{1}{2}\left(\operatorname{Tr} \nabla\left(f X_{0}\right)+\langle\zeta, g\rangle\right)
$$

Remark 6.6. Let $\alpha^{\prime}: \mathcal{O}^{\mathbb{C}}(A[1]) \rightarrow \mathcal{O}^{\mathbb{C}}(A[1])$ be the $\mathrm{C}^{\infty}(M)$-linear antiautomorphism satisfying $\alpha^{\prime}\left(\phi_{0}\right)=\phi_{0}$ if $\phi_{0} \in \mathcal{O}_{1}^{\mathbb{C}}(A[1])$. The map $\alpha^{\prime}$ extends naturally to $\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right) \otimes \mathbb{C} \cong \mathcal{O}^{\mathbb{C}}(A[1]) \otimes_{R} \Gamma\left(\mathcal{L}^{1 / 2}\right)$ by setting $\alpha:=\alpha^{\prime} \otimes \mathrm{id}$. Let

$$
\varepsilon= \begin{cases}1 & \text { if } \operatorname{rk} A \equiv 0,1 \quad \bmod 4, \\ i & \text { if } \operatorname{rk} A \equiv 2,3 \quad \bmod 4 .\end{cases}
$$

Via the Berezin integration over the supermanifold $A[1]$, we define a pseudo-Hermitian scalar product

$$
(\phi, \psi)=\varepsilon \int_{A[1]} \overline{\alpha(\phi)} \psi,
$$

on the space of compactly supported sections $\Gamma_{c}\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right) \otimes \mathbb{C}$. Under the isomorphism $v \otimes \operatorname{id}_{\mathbb{C}}$ : $\Gamma(\mathbb{S}) \xrightarrow{\sim} \Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right) \otimes \mathbb{C}$, the above scalar product on $\operatorname{Ber}_{A[1]}^{1 / 2} \otimes \mathbb{C}$ furnishes a globalization of the local spinor scalar product on $\mathbb{S}$ given in Eq. (3.22). Accordingly, the adjoint operation on $\mathcal{D}(M, \mathbb{S})$, defined in Proposition 3.16, coincides with the one defined on $\mathcal{D}\left(A[1], \operatorname{Ber}_{A[1]}^{1 / 2} \otimes \mathbb{C}\right)$ by the above scalar product.
6.2. Preliminaries on Lie algebroids. Now let $(A,[\cdot, \cdot], \rho)$ be a Lie algebroid. According to [12], the line bundle $\mathcal{L}=\wedge^{\text {top }} A \otimes \wedge^{\text {top }} T^{*} M$ is an $A$-module with the $A$-action being given by

$$
D_{\zeta}(v \otimes s):=[\zeta, v] \otimes s+v \otimes \mathrm{~L}_{\rho(\zeta)} s,
$$

for any $\zeta \in \Gamma(A), v \in \Gamma\left(\wedge^{\text {top }} A\right)$ and $s \in \Gamma\left(\wedge^{\text {top }} T^{*} M\right)$. Here the bracket stands for the Schouten bracket on $\Gamma(\wedge A)$, and $\mathrm{L}_{\rho(\zeta)} s$ stands for the Lie derivative of $s \in \Gamma\left(\wedge^{\text {top }} T^{*} M\right)$ along the vector field $\rho(\zeta) \in \mathfrak{X}(M)$. Since the map $\zeta \mapsto D_{\zeta}$ is $\mathrm{C}^{\infty}(M)$-linear, for each section $v \otimes s \in \Gamma(\mathcal{L})$, there exists a unique $\eta_{0} \in \Gamma\left(A^{*}\right)$ such that

$$
\begin{equation*}
D_{\zeta}(v \otimes s)=\left\langle\eta_{0}, \zeta\right\rangle v \otimes s, \quad \forall \zeta \in \Gamma(A) . \tag{6.10}
\end{equation*}
$$

The element $\eta_{0}$ is called the modular 1-cocycle of the Lie algebroid ( $A,[\cdot, \cdot], \rho$ ) associated to the section $v \otimes s \in \Gamma(\mathcal{L})$. Assume the square root of $\mathcal{L}$ exists. Then, $\mathcal{L}^{1 / 2}$ admits an action of the Lie algebroid $A$, defined by

$$
\begin{equation*}
\tilde{D}_{\zeta}\left((v \otimes s)^{1 / 2}\right):=\frac{1}{2}\left\langle\eta_{0}, \zeta\right\rangle(v \otimes s)^{1 / 2} . \tag{6.11}
\end{equation*}
$$

The Chevalley-Eilenberg differential $Q: \Gamma\left(\wedge^{k} A^{*}\right) \rightarrow \Gamma\left(\wedge^{k+1} A^{*}\right)$ of the Lie algebroid $A$ is given by

$$
\begin{aligned}
Q\left(\phi_{0}\right)\left(\zeta_{0}, \ldots, \zeta_{k}\right)= & \sum_{a=0}^{\mathrm{rk} A}(-1)^{a} \rho\left(\zeta_{a}\right)\left[\phi_{0}\left(\zeta_{0}, \ldots, \hat{\zeta}_{a}, \ldots, \zeta_{k}\right)\right] \\
& +\sum_{a<b}(-1)^{a+b} \phi_{0}\left(\left[\zeta_{a}, \zeta_{b}\right], \zeta_{0}, \ldots, \hat{\zeta}_{a}, \ldots, \hat{\zeta}_{b}, \ldots, \zeta_{k}\right),
\end{aligned}
$$

where $\phi_{0} \in \Gamma\left(\wedge^{k} A^{*}\right)$ and $\zeta_{0}, \ldots, \zeta_{k} \in \Gamma(A)$. Equivalently, $Q$ can be considered as a homological vector field on $A[1]$ [38]. In local coordinates $\left(x^{i}, \eta^{a}\right), Q$ is expressed as follows:

$$
\begin{equation*}
Q=\rho_{a}^{i} \eta^{a} \frac{\partial}{\partial x^{i}}+\frac{1}{2} C_{b c}^{a} \eta^{c} \eta^{b} \frac{\partial}{\partial \eta^{a}}, \tag{6.12}
\end{equation*}
$$

where, by definition,

$$
\rho_{a}^{i}=\left\langle\rho\left(\zeta_{a}\right), d x^{i}\right\rangle \quad \text { and } \quad C_{b c}^{a}=\left\langle\left[\zeta_{b}, \zeta_{c}\right], \eta^{a}\right\rangle
$$

are smooth functions on the base manifold and $\left(\zeta_{a}\right)$ is the dual frame to $\left(\eta^{a}\right)$. From Eq. (6.11), we obtain another differential $\widetilde{Q}$, defined on the complex $\bigoplus_{k} \Gamma\left(\wedge^{k} A^{*} \otimes \mathcal{L}^{1 / 2}\right) \cong \Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)$ :

$$
\begin{aligned}
\widetilde{Q}(\phi)\left(\zeta_{0}, \ldots, \zeta_{k}\right)= & \sum_{a=0}^{\mathrm{rk} A}(-1)^{a} \tilde{D}_{\zeta_{a}}\left(\phi\left(\zeta_{0}, \ldots, \hat{\zeta}_{a}, \ldots, \zeta_{k}\right)\right) \\
& +\sum_{a<b}(-1)^{a+b} \phi\left(\left[\zeta_{a}, \zeta_{b}\right], \zeta_{0}, \ldots, \hat{\zeta}_{a}, \ldots, \hat{\zeta}_{b}, \ldots, \zeta_{k}\right),
\end{aligned}
$$

where $\phi \in \Gamma\left(\wedge^{k} A^{*} \otimes \mathcal{L}^{1 / 2}\right)$ and $\zeta_{0}, \ldots, \zeta_{k} \in \Gamma(A)$.
Proposition 6.7. Let $Q \in \mathfrak{X}(A[1])$ be the homological vector field of the Lie algebroid $A$. As an operator on $\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)$, the differential $\widetilde{Q}$ satisfies

$$
\begin{equation*}
\widetilde{Q}=\mathrm{L}_{Q} . \tag{6.13}
\end{equation*}
$$

If $\phi=\phi_{0} \otimes(v \otimes s)^{1 / 2} \in \Gamma\left(\wedge A^{*} \otimes \mathcal{L}^{1 / 2}\right)$, we have

$$
\begin{equation*}
\mathrm{L}_{Q}(\phi)=\left(Q\left(\phi_{0}\right)+\frac{1}{2} \eta_{0} \phi_{0}\right) \otimes(v \otimes s)^{1 / 2} \tag{6.14}
\end{equation*}
$$

where $\eta_{0} \in \Gamma\left(A^{*}\right)$ is the modular 1-cocycle of the Lie algebroid $A$ associated to the section $v \otimes s$.
Proof. Let $\phi=\phi_{0} \otimes(v \otimes s)^{1 / 2} \in \Gamma\left(\wedge A^{*} \otimes \mathcal{L}^{1 / 2}\right)$. A direct computation shows that

$$
\widetilde{Q} \phi=\left(Q+\frac{1}{2} \eta_{0}\right) \phi_{0} \otimes(v \otimes s)^{1 / 2} .
$$

It suffices to prove that $\mathrm{L}_{Q}=\widetilde{Q}$ in a local coordinate system $\left(x^{i}, \eta^{a}\right)$ on $A[1]$. We work in the trivialization of $\mathcal{L}^{1 / 2}$ provided by the local section $\left(\Pi \zeta_{a} \otimes \wedge d x^{i}\right)^{1 / 2}$, with $\left(\zeta_{a}\right)$ being the dual frame to $\left(\eta^{a}\right)$. By Eqns (6.2), (6.8) and (6.12), we have

$$
\mathrm{L}_{Q}=Q+\frac{1}{2}\left(\frac{\partial}{\partial x^{i}} \rho_{b}^{i}+C_{b a}^{a}\right) \eta^{b}
$$

By Eq. (6.10), the modular 1-cocycle with respect to the local section $\left(\prod \zeta_{a} \otimes \wedge d x^{i}\right)$ satisfies $\eta_{0}=$ $\frac{\partial}{\partial x^{2}} \rho_{b}^{i}+C_{b a}^{a}$. The result thus follows.

Remark 6.8. Each nowhere vanishing section $v \otimes s \in \Gamma(\mathcal{L})$ defines a nowhere vanishing section of the Berezinian bundle $1 \otimes(v \otimes s) \in \Gamma\left(\operatorname{Ber}_{A[1]}\right)$ and then a divergence by the formula: Div $X=\frac{\mathrm{L}_{X}(1 \otimes v \otimes s)}{1 \otimes v \otimes s}$ (see e.g. [21]). The above proposition shows that $\frac{1}{2} \operatorname{Div} Q=\eta_{0}$. This result was proved in the more general situation of skew-algebroids in (16].
6.3. Dirac generating operators for Lie bialgebroids. Assume that both $(A,[\cdot, \cdot], \rho)$ and $\left(A^{*},[\cdot, \cdot]_{*}, \rho_{*}\right)$ are Lie algebroids, with homological vector fields $Q \in \mathfrak{X}(A[1])$ and $Q_{*} \in \mathfrak{X}\left(A^{*}[1]\right)$ respectively. The Lie algebroid brackets, extended by the Leibniz rule, turn $\left(\mathcal{O}\left(A^{*}[1]\right),[\cdot, \cdot]\right)$ and $\left(\mathcal{O}(A[1]),[\cdot, \cdot]_{*}\right)$ into Gerstenhaber algebras [20, 41]. The pair $\left(A, A^{*}\right)$ is a Lie bialgebroid if $Q_{*}$ is a derivation of the Gerstenhaber algebra $\left(\mathcal{O}\left(A^{*}[1]\right),[\cdot, \cdot]\right)$, or equivalently if $Q$ is a derivation of the Gerstenhaber algebra $\left(\mathcal{O}(A[1]),[\cdot, \cdot]_{*}\right)$.

The duality pairing between $A$ and $A^{*}$ extends to their exterior powers and leads to an isomorphism 9:

$$
\beta_{k}: \wedge^{k} A \otimes\left(\wedge^{n} A^{*} \otimes \wedge^{\mathrm{top}} T^{*} M\right)^{1 / 2} \longrightarrow \wedge^{n-k} A^{*} \otimes \mathcal{L}^{1 / 2}
$$

where $n$ is the rank of $A$ and $0 \leq k \leq n$. In turn, the maps $\left(\beta_{k}\right)$ induce an isomorphism

$$
\beta: \Gamma\left(\operatorname{Ber}_{A^{*}[1]}^{1 / 2}\right) \xrightarrow{\sim} \Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)
$$

By symmetry in $A$ and $A^{*}$, the maps $\Phi_{A}, \Upsilon_{A}$ and $\mathcal{W} \mathcal{Q}^{A}$ introduced in Section 6.1 have counterparts $\Phi_{A^{*}}, \Upsilon_{A^{*}}$ and $\mathcal{W} \mathcal{Q}^{A^{*}}$, respectively. We also denote by $\mathrm{L}_{X}$ the Lie derivative on $\Gamma\left(\operatorname{Ber}_{A^{*}[1]}^{1 / 2}\right)$ along a vector field $X \in \mathfrak{X}\left(A^{*}[1]\right)$.

As a consequence of Theorem [5.6, we recover the following theorem in (9].
Theorem $6.9(9])$. Let $\left(A, A^{*}\right)$ be a pair of Lie algebroids with homological vector fields $Q \in \mathfrak{X}(A[1])$ and $Q_{*} \in \mathfrak{X}\left(A^{*}[1]\right)$, respectively. Let $F_{Q} \in \mathcal{O}\left(T^{*}[2](A[1])\right)$ and $F_{Q_{*}} \in \mathcal{O}\left(T^{*}[2]\left(A^{*}[1]\right)\right)$ be their corresponding Hamiltonian functions. Set $\Theta=F_{Q}+\left(\Phi_{A^{*}} \circ \Phi_{A}^{-1}\right)^{*} F_{Q_{*}}$. The following statements are equivalent:
(i) $\left(A, A^{*}\right)$ is a Lie bialgebroid;
(ii) $\{\Theta, \Theta\}=0$;
(iii) $\left(\mathrm{L}_{Q}+\beta \circ \mathrm{L}_{Q_{*}} \circ \beta^{-1}\right)^{2} \in \mathrm{C}^{\infty}(M)$.

Proof. The equivalence between the first two assertions was proved in 31.
We prove the equivalence between the last two assertions. According to Theorem5.6, $\mathcal{W} \mathcal{Q}\left(\left(\Phi_{A}\right)^{*} \Theta\right)$ is a Dirac generating operator if and only if $\left\{\left(\Phi_{A}\right)^{*} \Theta,\left(\Phi_{A}\right)^{*} \Theta\right\}=0$, as a function on $T^{*}[2] M \oplus E[1]$. Since $\Upsilon_{A}$ is an algebra isomorphism and $\Phi_{A}$ is a symplectic diffeomorphism, we deduce that $\{\Theta, \Theta\}=0$ if and only if $\left(\mathcal{W} \mathcal{Q}^{A}(\Theta)\right)^{2} \in \mathrm{C}^{\infty}(M)$, where $\mathcal{W} \mathcal{Q}^{A}$ is the quantization map as in Definition 6.2, The conclusion follows from the lemma below.

Lemma 6.10. Under the same hypothesis of Theorem 6.9, we have $\mathcal{W} \mathcal{Q}^{A}(\Theta)=\mathrm{L}_{Q}+\beta \circ \mathrm{L}_{Q_{*}} \circ \beta^{-1}$.
Proof. According to Proposition 6.4, we have $\mathcal{W} \mathcal{Q}^{A}\left(F_{Q}\right)=\mathrm{L}_{Q}$. Similarly, we also have $\mathcal{W} \mathcal{Q}^{A^{*}}\left(F_{Q_{*}}\right)=$ $\mathrm{L}_{Q_{*}}$. Since $\Upsilon_{A}(\cdot)=\beta \circ \Upsilon_{A^{*}}(\cdot) \circ \beta^{-1}$, the conclusion thus follows.

According to Proposition 6.7, the above Dirac generating operator is also equal to

$$
\mathcal{W} \mathcal{Q}^{A}(\Theta)=\widetilde{Q}+\beta \circ \widetilde{Q_{*}} \circ \beta^{-1}
$$

where $\widetilde{Q}$ is the differential operator on $\Gamma\left(\operatorname{Ber}_{A[1]}^{1 / 2}\right)$ and $\widetilde{Q_{*}}$ is the differential operator on $\Gamma\left(\operatorname{Ber}_{A^{*}[1]}^{1 / 2}\right)$. This expression is the one obtained in [9].

Choosing $s \in \Gamma\left(\wedge^{\text {top }} T^{*} M\right), v \in \Gamma\left(\wedge^{\text {top }} A\right)$ and $w \in \Gamma\left(\wedge^{\text {top }} A^{*}\right)$, we obtain modular 1-cocycles $\eta_{0} \in \Gamma\left(A^{*}\right)$ and $\zeta_{0} \in \Gamma(A)$ of the Lie algebroids $A$ and $A^{*}$, respectively. We work locally and assume that $\langle v, w\rangle=1$. Then, $v$ provides a local isomorphism $v^{\sharp}: \Gamma\left(\wedge^{k} A^{*}\right) \rightarrow \Gamma\left(\wedge^{n-k} A\right)$, and we define $\widehat{Q}_{*}:=(-1)^{k}\left(v^{\sharp}\right)^{-1} \circ Q_{*} \circ v^{\sharp}$ on $\Gamma\left(\wedge^{k} A^{*}\right)$, for all $k \in \mathbb{N}$. According to [9, we have

$$
\mathcal{W} \mathcal{Q}^{A}(\Theta)=\left(Q-\widehat{Q}_{*}+\frac{1}{2} \eta_{0}+\frac{1}{2} \gamma\left(\zeta_{0}\right)\right) \otimes \operatorname{id}_{\mathcal{L}^{1 / 2}}
$$

under the local trivialization of $\mathcal{L}^{1 / 2}$ provided by the section $(v \otimes s)^{1 / 2}$. As a consequence of Proposition 5.13, we obtain

Corollary 6.11. The function $2\left(\mathcal{W} \mathcal{Q}^{A}(\Theta)\right)^{2}=\frac{1}{2}\left\langle\zeta_{0}, \eta_{0}\right\rangle-\widehat{Q}_{*} \eta_{0}$ is an invariant of the Lie bialgebroid $\left(A, A^{*}\right)$.

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Department of Physics, University of Jena, Germany
Email address: melchiorG@gmail.com
Department of Mathematics, University of Liège, Belgium
Email address: jpmichel82@gmail.com
Department of Mathematics, Penn State University, United States
Email address: ping@math.psu.edu


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[^1]:    ${ }^{1}$ To the best of our knowledge, Cabras-Vinogradov's contributions to Courant brackets seem to have been overlooked by the community so far.

[^2]:    ${ }^{2}$ In this paper, we use the notation $\hbar$ to denote a nonzero variable in $\mathbb{C}$. In most situation, $\hbar$ can be interpreted as the Planck constant. However, we sometimes let $\hbar=\mathrm{i}$.

