Asymptotic analysis of multiple mode structures equipped with multiple tuned mass dampers

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Abstract

The design of a mitigation dissipative system composed of several tuned mass dampers (TMDs) is not a simple task. As soon as several structural modes or several additional degrees-of-freedom are considered, the analytical response of the coupled system is not tractable anymore. The generic problem involving any number of structural modes and any number of attached tuned mass dampers is considered in this paper. The appropriate re-scaling of the problem is formulated in order to highlight the governing scales in the problem. An asymptotic analysis in the frequency domain, similar to the multiple timescale spectral analysis, is developed. The resulting expression for the frequency response function of the coupled system is expressed in a lower dimensional space, which offers the possibility to derive simple and accurate analytical solutions in other cases than the well-known single-mode single-TMD case. Although the major contribution of this paper consists in the derivation of this analytical formulation, two illustrations are also provided. The first one concerns the well-known single-mode single-TMD case, which illustrates the benefits of the proposed formulation and validates the resulting approximations against established literature. The second example concerns a single-mode double-TMD, where the proposed approximation is used to discuss the accuracy with which the two damping units need to be tuned in order to provide reasonable optimality with respect to mitigation performances.

Keywords: vibration mitigation, perturbation, multiple timescale, TMD

1. Introduction

Slender structures such as modern footbridges suffer from vibration serviceability problems, mainly because of their low stiffness and low damping ratio. To deal with such unwanted behavior, passive linear vibration absorbers are generally installed. Tuned Mass Dampers (TMD) are the most popular devices because of their simplicity, efficiency, and affordable cost. They consist of a mass, a damping component and a spring (stiffness component), and are installed at critical locations on a structure, such that the added components interact with the structural characteristics to fulfill an anti-resonant role [26, 7]. The technology is involved in several applications in civil engineering, as it contributes, inter alia, to the reduction of oscillations due to wind and wave loading in offshore installations [22], due to vortex shedding in bridge decks [20], and to seismic excitation in multi-storey buildings [10, 21].

The effectiveness of a TMD relies on its accurate tuning, the parameters (damping ratio and especially natural frequency) must be adjusted with high precision to meet the expected performance, which constitutes a manufacturing challenge. This sensitivity to tuning opened investigations around the robustness in its design, see e.g. [3] and [15]. For such a technology, the mass ratio can reach 3% [2] or more in order to offer sufficient robustness against mistuning. Higher values can be virtually reached with inerter-based absorbers, a technology borrowed from the automotive industry [24, 23] and used to suppress cable vibrations [17, 18, 13].

Some simple criteria exist for the design of a tuned mass damper for Single-Degree-Of-Freedom (SDOF) structures. Both [4] and [25] provide optimal tuning parameters for the effective attenuation of the structural response. For Multi-Degree-Of-Freedom (MDOF) structures with sufficiently different natural frequencies, a simple strategy is to associate one TMD with each structural mode. The modes are studied separately and their interaction is neglected [1]. In case of close natural frequencies and mode shapes, these interactions can no longer be neglected and the (favorable or not) influence of one TMD on one or several modes at a time needs to be carefully studied. To the authors' knowledge there is no analytical solution available today as soon as several degrees-of-freedom are involved. This is explained by the fact that a formal derivation of the response of, say, a 2-DOF structure with one TMD, or the other way around a 1-DOF structure with two TMDs, involves in total a 3-DOF dynamical model. The characteristic governing equation of this low-order model is a 6^{th}-degree polynomial whose general solution cannot be obtained explicitly. More complicated models follow the same observations. Current trends today in the design of multiple TMDs for multiple mode vibration mitigation is to recourse to optimal control [27] or repeated numerical simulation either in full parameter space [11, 6] or with the help of response surface methods [12].

In order to circumvent the need for numerical simulation, this work develops an asymptotic approach of the problem, by considering that several parameters are represented by small quantities : the structural damping ratio, the mass ratio which represents the mass of TMD(s) over structural modal mass(es), as well as the mistuning ratio (ratio of frequencies of structural modes and TMD modes). A similar approach has already been used for a SDOF structure equipped with a single TMD which allowed to derive very simple design formulas [8]. This work extends those solutions to MDOF structures and/or multiples TMDs. The same problem has been addressed by Abe and Igusa [1] who derived design criteria for TMDs on structures with closely spaced natural frequencies. They follow a similar procedure based on the comparison of the orders of magnitudes of the various terms entering the solution of the problem; they derive an analytical solution which admits closed-form solutions for one or two natural frequencies, and under harmonic loading. The developments presented in this paper continue along the same line, while providing a complete asymptotic analysis of the coupled Frequency Response Function (FRF) of a structure with any number of mode shapes and any number of tuned mass dampers. The game of stretching and squeezing coordinate allow to focus on the important timescales of the problem which finally results in a simple formulation for the coupled FRF. Explicit inversion of this matrix remains impossible in many cases but the proposed formulation proves helpful in understanding the majors trends in more complex problems. Beside alleviating the complexity of the mathematical solution, the proposed solution has the merit of retaining only the necessary pieces of information of the problem, in the neighborhood of resonance, where interaction takes places between the dynamics of the structural modes and of the TMDs. .

The method is illustrated through two examples. The elementary case of a single TMD on an SDOF structure is first presented; this shows that well-known design criteria are retrieved. A second example deals with the configuration of two TMDs to mitigate the vibrations of a single structural mode. The motivation for this second example stems from the fact that TMDs are sometimes split into several units and it is necessary to answer the question of how similar need to be two nominally identical units to not prevent the optimality of the technical solution. While a formal solution of this problem is mathematically cumbersome, the proposed approach is able to provide affordable solutions to this question.

2. Problem statement

The dynamics of a structure in its modal basis is governed by the equation of motion

$$\mathbf{M}_{\mathrm{s}}\ddot{\mathbf{q}}_s + \mathbf{C}_{\mathrm{s}}\dot{\mathbf{q}}_s + \mathbf{K}_{\mathrm{s}}\mathbf{q}_s = \mathbf{p}_{\mathrm{s}} \tag{1}$$

where $\mathbf{q}_s(t)$ collects the n_{m} modal coordinates (the dot denotes time derivatives), $\mathbf{M}_s$, $\mathbf{C}_s$ and $\mathbf{K}_s$ are the structural modal matrices ($n_{\mathrm{m}} \times n_{\mathrm{m}}$), and $\mathbf{p}_{\mathrm{s}}$ is the vector of modal loads. The considered modal basis is the basis of undamped normal modes of vibration so that $\mathbf{M}_s$ and $\mathbf{K}_s$ are diagonal matrices. As customary the inherent damping matrix $\mathbf{C}_s$ is assumed to be diagonal too; it is for instance constructed by means of a chosen modal damping ratio or with a Rayleigh damping model [16]. The number of modes n_{m} is much lower than the number of DOFs.

One or several tuned mass dampers with small mass ratios are added to this structure. A number n_{d} of such devices is installed at locations in the structure where the modal coordinates are gathered in the coordinate matrix $\boldsymbol{\varphi}$ ($n_{\mathrm{m}} \times n_{\mathrm{d}}$), where φ_{mk} represents the modal coordinates of damper $k \in [1, n_{\mathrm{d}}]$ in mode $m \in [1, n_{\mathrm{m}}]$. Introducing the coordinates $\mathbf{q}_{\mathrm{TMD}}(t)$ corresponding to the absolute displacements of the tuned masses, the system is governed by the augmented state equations

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p} \tag{2}$$

where the augmented coordinate vector $\mathbf{q}(t)$ is defined by

$$\mathbf{q} = \begin{pmatrix} \mathbf{q}_s \\ \mathbf{q}_{\mathrm{TMD}} \end{pmatrix} \tag{3}$$

and the augmented structural matrices are given by

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix} \mathbf{M}_{\mathrm{s}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{M}_{\mathrm{d}} \end{pmatrix} \\
\mathbf{C} &= \begin{pmatrix} \mathbf{C}_{\mathrm{s}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varphi}\mathbf{C}_{\mathrm{d}}\boldsymbol{\varphi}^T & -\boldsymbol{\varphi}\mathbf{C}_{\mathrm{d}} \\ -\mathbf{C}_{\mathrm{d}}\boldsymbol{\varphi}^T & \mathbf{C}_{\mathrm{d}} \end{pmatrix} \\
\mathbf{K} &= \begin{pmatrix} \mathbf{K}_{\mathrm{s}} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varphi}\mathbf{K}_{\mathrm{d}}\boldsymbol{\varphi}^T & -\boldsymbol{\varphi}\mathbf{K}_{\mathrm{d}} \\ -\mathbf{K}_{\mathrm{d}}\boldsymbol{\varphi}^T & \mathbf{K}_{\mathrm{d}} \end{pmatrix}
\end{aligned} \tag{4}$$

Matrices $\mathbf{M}_{\mathrm{d}}$, $\mathbf{C}_{\mathrm{d}}$ and $\mathbf{K}_{\mathrm{d}}$ are diagonal matrices collecting, on their diagonal elements, the mass, viscosity and stiffness associated with each tuned mass damper. Since these matrices are diagonal, each tuned mass damper interacts with the structural degrees-of-freedom only; direct interactions between tuned mass dampers, such as the double tuned mass damper [14] are therefore not covered by the following developments. The augmented load vector $\mathbf{p}$ is composed of the loads that are physically applied to the structure $\mathbf{p}_\mathrm{s}$ and a zero loading on the TMDs,

$$\mathbf{p} = \left(\begin{array}{c} \mathbf{p}_s \\ \mathbf{0} \end{array} \right). \tag{5}$$

Equation (4) shows that the addition of TMDs in a structure couples modal responses, i.e. a load in a given mode no longer induces a response in that mode only.

In practical applications, tuned mass dampers are devices that are added to the bare structure in order to improve its dynamic behavior, i.e. to limit the amplitudes of vibration for given loading $\mathbf{p}(t)$. The mechanical properties of the damped system, which comes in the model through $\mathbf{M}_\mathrm{d}$, $\mathbf{C}_\mathrm{d}$ and $\mathbf{K}_\mathrm{d}$, therefore need to be determined in such a way that a cost function related to the structural response is minimized. In most cases, they involve the steady-state response of the structure, which is obtained by the frequency response function $\mathbf{H}(\omega)$, as the Fourier transform of (2), $\mathbf{Q}(\omega) = \mathbf{H}(\omega)\,\mathbf{P}(\omega)$, where $\mathbf{Q}(\omega)$ and $\mathbf{P}(\omega)$ are the Fourier transforms of $\mathbf{q}(t)$ and $\mathbf{p}(t)$ respectively and

$$\mathbf{H} = \left(-\mathbf{M}\omega^2 + \mathrm{i}\omega\mathbf{C} + \mathbf{K}\right)^{-1}. \tag{6}$$

Apart from the very simple case of a single degree-of-freedom structure ($n_\mathrm{m} = 1$) damped with a single damper ($n_\mathrm{d} = 1$), which admits explicit solutions, the design process of tuned mass dampers requires numerical techniques, which do not ease the design process. The main bottleneck is that, as soon as there are two structural modes or two tuned mass dampers, the frequency response function $\mathbf{H}$ being expressed as the inverse of a matrix, takes very long expressions, which makes it intractable. Due to the mechanical coupling evidenced by the non-diagonal nature of $\mathbf{C}$ and $\mathbf{K}$, this is indeed a full matrix inversion.

It this paper, we provide a simple but approximate expression for the frequency response function $\mathbf{H}(\omega)$. As shown next, it takes advantage of the small numbers present in this problem, mainly the small mass ratios. Then an asymptotic solution of the problem is developed. The dimensionless formulation is developed in Sections 3 and 4, the perturbation approach is described in Section 5, while examples of application are given in Section 7.

3. Dimensionless formulation

The adimensionalization of the governing equation requires a characteristic time $t^\star$, a characteristic displacement for the structure $q_{\mathrm{s}}^\star$ and a characteristic displacement for the tuned mass dampers $q_{\mathrm{TMD}}^\star$. With these, a dimensionless time and dimensionless displacements can be determined

$$\tau = \frac{t}{t^\star} \qquad ; \qquad \mathbf{x}_{\mathrm{s}} = \frac{\mathbf{q}_{\mathrm{s}}}{q_{\mathrm{s}}^\star} \qquad ; \qquad \mathbf{x}_{\mathrm{TMD}} = \frac{\mathbf{q}_{\mathrm{TMD}}}{q_{\mathrm{TMD}}^\star} \tag{7}$$

so that the augmented coordinate vector $\mathbf{q}(t)$ is rewritten

$$\mathbf{q} = \begin{pmatrix} q_{\mathrm{s}}^\star \mathbf{x}_{\mathrm{s}} \\ q_{\mathrm{TMD}}^\star \mathbf{x}_{\mathrm{TMD}} \end{pmatrix} = q_{\mathrm{s}}^\star \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \frac{q_{\mathrm{TMD}}^\star}{q_{\mathrm{s}}^\star} \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{\mathrm{s}} \\ \mathbf{x}_{\mathrm{TMD}} \end{pmatrix} := \mathbf{q}^\star \mathbf{x} \tag{8}$$

where $\mathbf{q}^\star$ is the diagonal scaling matrix and $\mathbf{x}\left[\tau\left(t\right)\right]$ is a dimensionless augmented coordinate vector. The two characteristic displacements in the scaling matrix $\mathbf{q}^\star$ will be determined later in order to have dimensionless coordinates of order 1.

The characteristic time $t^\star$ is chosen with respect to the dynamics (oscillations) of the primary system which serves as a reference; it is anyway expected that the tuned mass dampers have similar timescales, otherwise the coupling would be limited and they would be inefficient. It is therefore required to define a characteristic structural mass $m_{\mathrm{s}}^\star$ (which could be the generalized mass of the first structural undamped mode, $M_{\mathrm{s},11}$) and a characteristic structural stiffness $k_{\mathrm{s}}^\star$ (which could be the generalized stiffness of the first structural undamped mode, $K_{\mathrm{s},11}$). The characteristic timescale is then chosen as

$$t^\star = \sqrt{\frac{m_{\mathrm{s}}^\star}{k_{\mathrm{s}}^\star}} := \frac{1}{\omega^\star}. \tag{9}$$

The mass and stiffness matrices are written $\mathbf{M} = m_{\mathrm{s}}^\star \tilde{\mathbf{M}}$ and $\mathbf{K} = k_{\mathrm{s}}^\star \tilde{\mathbf{K}}$ where $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ are dimensionless augmented mass and stiffness matrices. The elements in the upper part of these matrices (those associated with the structural degrees-of-freedom) are assumed to have the same order of magnitude, while the elements in the lower part of these matrices (those associated with the coordinates of the tuned mass dampers) are much smaller. This smallness is exploited in the perturbation analysis in Section 5 to derive simple asymptotic solutions.

Combining (7) and (9) into (2) and using the chain rule differentiation, the governing equation is rewritten

$$m_{\mathrm{s}}^\star {\omega^\star}^2 q_{\mathrm{s}}^\star \tilde{\mathbf{M}} \mathbf{q}^\star \mathbf{x}'' + \omega^\star q_{\mathrm{s}}^\star \mathbf{C} \mathbf{q}^\star \mathbf{x}' + k_{\mathrm{s}}^\star q_{\mathrm{s}}^\star \tilde{\mathbf{K}} \mathbf{q}^\star \mathbf{x} = \mathbf{p} \tag{10}$$

where the apex in $\mathbf{x}(\tau)$ denotes differentiation with respect to the dimensionless time τ. Defining a characteristic critical viscosity $c_{\mathrm{s}}^{\star} = 2m_{\mathrm{s}}^{\star}\omega^{\star}\xi_{s}^{\star} = 2\xi_{s}^{\star}\sqrt{k_{\mathrm{s}}^{\star}m_{\mathrm{s}}^{\star}}$, a dimensionless augmented damping matrix $\mathbf{C} := c_{\mathrm{s}}^{\star}\tilde{\mathbf{C}} = 2\xi_{\mathrm{s}}^{\star}m_{\mathrm{s}}^{\star}\omega^{\star}\tilde{\mathbf{C}}$ is also introduced. After substitution in (10) and simplification, the governing equation reads

$$\mathcal{M}\mathbf{x}'' + \mathcal{C}\mathbf{x}' + \mathcal{K}\mathbf{x} = \mathcal{P} \tag{11}$$

where $\mathcal{M} = \mathbf{q}^{\star}\tilde{\mathbf{M}}\mathbf{q}^{\star}$, $\mathcal{C} = 2\xi_{s}^{\star}\mathbf{q}^{\star}\tilde{\mathbf{C}}\mathbf{q}^{\star}$, $\mathcal{K} = \mathbf{q}^{\star}\tilde{\mathbf{K}}\mathbf{q}^{\star}$, and where we have also defined $\mathcal{P} = \frac{1}{k_{\mathrm{s}}^{\star}q_{\mathrm{s}}^{\star}}\mathbf{q}^{\star}\mathbf{p}$, using therefore $k_{\mathrm{s}}^{\star}q_{\mathrm{s}}^{\star}$ as a characteristic modal load.

4. Appropriate Scaling

Now the augmented coordinate system has been rewritten in dimensionless coordinates, relying exclusively on the macroscopic mechanical properties of the structural modes, a closer look can be paid to the equation in order to highlight the relative smallness of the mechanical properties related to the tuned mass dampers. The dimensionless augmented mass matrix reads

$$\mathcal{M} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \frac{q_{\mathrm{TMD}}^{\star}}{q_{\mathrm{s}}^{\star}}\mathbf{I} \end{pmatrix} \begin{pmatrix} \frac{1}{m_{\mathrm{s}}^{\star}}\mathbf{M}_{\mathrm{s}} & 0 \\ 0 & \frac{1}{m_{\mathrm{s}}^{\star}}\mathbf{M}_{\mathrm{d}} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \frac{q_{\mathrm{TMD}}^{\star}}{q_{\mathrm{s}}^{\star}}\mathbf{I} \end{pmatrix} = \begin{pmatrix} \frac{1}{m_{\mathrm{s}}^{\star}}\mathbf{M}_{\mathrm{s}} & 0 \\ 0 & \frac{1}{m_{\mathrm{s}}^{\star}}\left(\frac{q_{\mathrm{TMD}}^{\star}}{q_{\mathrm{s}}^{\star}}\right)^{2}\mathbf{M}_{\mathrm{d}} \end{pmatrix}. \tag{12}$$

Let us also introduce the notations

$$\mathbf{M}_{\mathrm{s}} = m_{\mathrm{s}}^{\star}\tilde{\mathbf{M}}_{\mathrm{s}} \qquad ; \qquad \mathbf{M}_{\mathrm{d}} = m_{\mathrm{TMD}}^{\star}\tilde{\mathbf{M}}_{\mathrm{d}} = \varepsilon^{2}\, m_{\mathrm{s}}^{\star}\tilde{\mathbf{M}}_{\mathrm{d}} \tag{13}$$

where $\tilde{\mathbf{M}}_{\mathrm{s}}$ and $\tilde{\mathbf{M}}_{\mathrm{d}}$ are the (diagonal) dimensionless mass matrices. They represent the relative magnitude of the modal masses of the bare structure on the one hand, and of the tuned mass dampers on the other hand. If $m_{\mathrm{s}}^{\star}$ and $m_{\mathrm{TMD}}^{\star} \ll m_{\mathrm{s}}^{\star}$ are chosen as the largest modal mass and the largest tuned mass, both $\tilde{\mathbf{M}}_{\mathrm{s}}$ and $\tilde{\mathbf{M}}_{\mathrm{d}}$ are composed of elements smaller than one. These two matrices are therefore of order 1 at most. We have also introduced the small number, $0 < \varepsilon \ll 1$, defined by the ratio of the characteristic TMD mass over the characteristic structural modal mass,

$$\varepsilon := \sqrt{\frac{m_{\mathrm{TMD}}^{\star}}{m_{\mathrm{s}}^{\star}}}. \tag{14}$$

In case there is only one structural mode and one tuned mass damper (see Illustration 1, Section 7), it corresponds to the square root of the mass ratio, $\varepsilon = \sqrt{\mu}$. This ratio is typically of the order of 1%

to 3% in practical applications [9], and a little more for inerters [17, 18, 13]. Substitution of (13) into (12) yields

$$\mathcal{M} = \begin{pmatrix} \tilde{\mathbf{M}}_\mathrm{s} & 0 \\ 0 & \left(\varepsilon \frac{q^\star_\mathrm{TMD}}{q^\star_\mathrm{s}}\right)^2 \tilde{\mathbf{M}}_\mathrm{d} \end{pmatrix}. \tag{15}$$

In order to obtain a problem with all masses of order 1, it is now clear that the characteristic displacements to be used to scale the response of the structural modes $q^\star_\mathrm{s}$ and those to be used to scale the tuned mass damper displacements $q^\star_\mathrm{TMD}$, need to satisfy $\varepsilon \frac{q^\star_\mathrm{TMD}}{q^\star_\mathrm{s}} \sim 1$, so that we choose

$$\frac{q^\star_\mathrm{TMD}}{q^\star_\mathrm{s}} = \frac{1}{\varepsilon}. \tag{16}$$

This conclusion supports the well-accepted fact that the motion of a tuned mass damper is larger than the motion of the structure [8]. This equation shows that it actually scales with the square-root of the masses. This scaling naturally comes from the governing equations.

Similarly, the relative smallness in the dimensionless stiffness matrix can be made explicit by introducing

$$\mathbf{K}_\mathrm{s} = k^\star_\mathrm{s}\tilde{\mathbf{K}}_\mathrm{s} \qquad \text{and} \qquad \mathbf{K}_\mathrm{d} = k^\star_\mathrm{TMD}\tilde{\mathbf{K}}_\mathrm{d} \tag{17}$$

which defines, $\tilde{\mathbf{K}}_\mathrm{s}$ and $\tilde{\mathbf{K}}_\mathrm{d}$, the dimensionless stiffness matrices which have elements of order 1. Similarly to (14), the small ratio of the characteristic stiffnesses $k^\star_\mathrm{TMD}$ and $k^\star_\mathrm{s}$ is made explicit by introducing the parameter $\alpha \sim 1$ as

$$\sqrt{\frac{k^\star_\mathrm{TMD}}{k^\star_\mathrm{s}}} := \alpha\varepsilon \qquad \Leftrightarrow \qquad \alpha^2 = \frac{k^\star_\mathrm{TMD}/m^\star_\mathrm{TMD}}{k^\star_\mathrm{s}/m^\star_\mathrm{s}} = \frac{k^\star_\mathrm{TMD}}{\varepsilon^2 k^\star_\mathrm{s}}. \tag{18}$$

This ratio is close to unity when the dynamics of the structure and of the tuned mass dampers interact.

With these notations, the dimensionless augmented stiffness matrix defined in (11) reads

$$\mathcal{K} = \begin{pmatrix} \tilde{\mathbf{K}}_\mathrm{s} & 0 \\ 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} \varepsilon^2 \boldsymbol{\varphi}\tilde{\mathbf{K}}_\mathrm{d}\boldsymbol{\varphi}^T & -\varepsilon\boldsymbol{\varphi}\tilde{\mathbf{K}}_\mathrm{d} \\ -\varepsilon\tilde{\mathbf{K}}_\mathrm{d}\boldsymbol{\varphi}^T & \tilde{\mathbf{K}}_\mathrm{d} \end{pmatrix}. \tag{19}$$

At leading order, as $\varepsilon \to 0$, we notice that this matrix is diagonal, composed of $\tilde{\mathbf{K}}_\mathrm{s}$ and $\alpha^2\tilde{\mathbf{K}}_\mathrm{d}$, with all elements of order 1. This highlights the appropriateness of the scaling.

Finally, the same operation is performed with the damping matrices, by introducing

$$\mathbf{C}_\mathrm{s} = c^\star_\mathrm{s}\tilde{\mathbf{C}}_\mathrm{s} = 2\xi^\star_\mathrm{s}\sqrt{k^\star_\mathrm{s} m^\star_\mathrm{s}}\tilde{\mathbf{C}}_\mathrm{s} \qquad \text{and} \qquad \mathbf{C}_\mathrm{d} = c^\star_\mathrm{TMD}\tilde{\mathbf{C}}_\mathrm{d} = 2\xi^\star_\mathrm{TMD}\sqrt{k^\star_\mathrm{TMD} m^\star_\mathrm{TMD}}\tilde{\mathbf{C}}_\mathrm{d} \tag{20}$$

which defines, $\tilde{\mathbf{C}}_{\mathrm{s}}$ and $\tilde{\mathbf{C}}_{\mathrm{d}}$, the dimensionless damping matrices with elements of order 1 at most. The characteristic viscosity $c^{\star}_{\mathrm{TMD}}$ shall not be expressed by reference to the characteristic viscosity of the undamped structure, since it is possible that it is several orders of magnitude larger. Instead, it is defined with regards to the characteristic mass and stiffness of the tuned mass dampers. With these notations, the dimensionless augmented damping matrix $\mathcal{C} = 2\xi^{\star}_{s}\mathbf{q}^{\star}\tilde{\mathbf{C}}\mathbf{q}^{\star}$, as defined in (11), can be written,

$$\mathcal{C} = 2\xi^{\star}_{s}\begin{pmatrix} \tilde{\mathbf{C}}_{\mathrm{s}} & 0 \\ 0 & 0 \end{pmatrix} + 2\xi^{\star}_{s}\frac{c^{\star}_{\mathrm{TMD}}}{\varepsilon^2 c^{\star}_{\mathrm{s}}}\begin{pmatrix} \varepsilon^2\boldsymbol{\varphi}\tilde{\mathbf{C}}_{\mathrm{d}}\boldsymbol{\varphi}^T & -\varepsilon\boldsymbol{\varphi}\tilde{\mathbf{C}}_{\mathrm{d}} \\ -\varepsilon\tilde{\mathbf{C}}_{\mathrm{d}}\boldsymbol{\varphi}^T & \tilde{\mathbf{C}}_{\mathrm{d}} \end{pmatrix}. \tag{21}$$

By introducing the characteristic damping ratios in the structure and in the tuned mass damper, whose definitions are recalled in (20), we have

$$\frac{c^{\star}_{\mathrm{TMD}}}{\varepsilon^2 c^{\star}_{\mathrm{s}}} = \frac{1}{\varepsilon^2}\frac{\xi^{\star}_{\mathrm{TMD}}}{\xi^{\star}_{\mathrm{s}}}\left(\alpha\varepsilon\right)\varepsilon = \alpha\frac{\xi^{\star}_{\mathrm{TMD}}}{\xi^{\star}_{\mathrm{s}}}, \tag{22}$$

so that finally

$$\mathcal{C} = 2\xi^{\star}_{\mathrm{s}}\begin{pmatrix} \tilde{\mathbf{C}}_{\mathrm{s}} & 0 \\ 0 & 0 \end{pmatrix} + 2\alpha\xi^{\star}_{\mathrm{TMD}}\begin{pmatrix} \varepsilon^2\boldsymbol{\varphi}\tilde{\mathbf{C}}_{\mathrm{d}}\boldsymbol{\varphi}^T & -\varepsilon\boldsymbol{\varphi}\tilde{\mathbf{C}}_{\mathrm{d}} \\ -\varepsilon\tilde{\mathbf{C}}_{\mathrm{d}}\boldsymbol{\varphi}^T & \tilde{\mathbf{C}}_{\mathrm{d}} \end{pmatrix}. \tag{23}$$

In summary, having introduced the small parameter ε, defined in (14), we have been able to write the dimensionless augmented structural matrices by highlighting explicitly the orders of magnitude of the different terms. They are given in (15), (19) and (23). In these expressions, all symbols, except ε and the small damping ratios $\xi^{\star}_{\mathrm{s}}$ and $\xi^{\star}_{\mathrm{TMD}}$, represent quantities or order 1.

We are now in a position to derive an asymptotic analysis of the problem. A distinguished limit is obtained by combining the following three ingredients : (i) small TMD mass ratio, (ii) small damping ratios, (iii) small mistuning. The first having been already formalized through (14), the last two statements are now detailed.

The smallness of the damping ratio is formalized by writing

$$\xi^{\star}_{\mathrm{s}} = \varepsilon^2\,\tilde{\xi}_{\mathrm{s}} \qquad \text{and} \qquad \xi^{\star}_{\mathrm{TMD}} = \varepsilon\,\tilde{\xi}_{\mathrm{TMD}} \tag{24}$$

where $\tilde{\xi}_{\mathrm{s}}$ and $\tilde{\xi}_{\mathrm{TMD}}$ are re-scaled damping ratios of order 1 at most.

Furthermore, we also want to make explicit that the natural frequencies of the structural modes and of the tuned mass dampers included in the analysis are in the same range, otherwise there would be no benefit to exploit the coupling between either the mode shapes together, either the structural mode(s) and the tuned mass damper(s). This is formally expressed by writing

$$\alpha = 1 + \varepsilon\,\delta \tag{25}$$

where $\delta \sim 1$ is a detuning parameter of order 1. This translates the fact that the characteristic frequency of the structural modes and of the tuned mass dampers are close to each other. Besides, it is also important to state that all structural modes have close natural frequencies and so do all tuned mass dampers. This is stated by writing

$$\tilde{\mathbf{K}}_{\mathrm{s}} = \tilde{\mathbf{M}}_{\mathrm{s}}\left(\mathbf{I} + 2\varepsilon\,\boldsymbol{\Delta\Omega}_{\mathrm{s}}\right) \qquad ; \qquad \tilde{\mathbf{K}}_{\mathrm{d}} = \tilde{\mathbf{M}}_{\mathrm{d}}\left(\mathbf{I} + 2\varepsilon\,\boldsymbol{\Delta\Omega}_{\mathrm{d}}\right) \tag{26}$$

where $\mathbf{I}$ is the identity matrix, with appropriate size (the same symbol is used to refer to identity matrices of size $(n_{\mathrm{m}} \times n_{\mathrm{m}})$ and $(n_{\mathrm{d}} \times n_{\mathrm{d}})$) and where $\boldsymbol{\Delta\Omega}_{\mathrm{s}}$ and $\boldsymbol{\Delta\Omega}_{\mathrm{d}}$ are diagonal matrices quantifying the mistuning of the natural frequencies of structural modes on one hand, and of tuned mass dampers on the other hand.

After substitution of these expressions in (15), (19) and (23), and after some simplifications, the asymptotic expansions of the augmented structural matrices are obtained. They read

$$\mathcal{M} = \mathcal{M}_0; \quad \mathcal{C} = \varepsilon\,\mathcal{C}_1 + \mathrm{ord}\left(\varepsilon^2\right); \quad \mathcal{K} = \mathcal{K}_0 + \varepsilon\,\mathcal{K}_1 + \mathrm{ord}\left(\varepsilon^2\right) \tag{27}$$

with

$$\mathcal{M}_0 = \begin{pmatrix} \tilde{\mathbf{M}}_{\mathrm{s}} & 0 \\ 0 & \tilde{\mathbf{M}}_{\mathrm{d}} \end{pmatrix} \quad ; \quad \mathcal{K}_0 = \begin{pmatrix} \tilde{\mathbf{M}}_{\mathrm{s}} & 0 \\ 0 & \tilde{\mathbf{M}}_{\mathrm{d}} \end{pmatrix}$$
$$\mathcal{C}_1 = 2\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\xi}_{\mathrm{TMD}}\tilde{\mathbf{C}}_{\mathrm{d}} \end{pmatrix}; \quad \mathcal{K}_1 = \begin{pmatrix} 2\tilde{\mathbf{M}}_{\mathrm{s}}\boldsymbol{\Delta\Omega}_{\mathrm{s}} & -\boldsymbol{\varphi}\tilde{\mathbf{M}}_{\mathrm{d}} \\ -\tilde{\mathbf{M}}_{\mathrm{d}}\boldsymbol{\varphi}^T & 2\tilde{\mathbf{M}}_{\mathrm{d}}\left(\delta\mathbf{I} + \boldsymbol{\Delta\Omega}_{\mathrm{d}}\right) \end{pmatrix} \tag{28}$$

It is important to notice that $\mathcal{K}_0 = \mathcal{M}_0$ because of (26). Although seemingly unimportant, this properly plays a central role in the subsequent developments. It has been achieved thanks to the careful and appropriate scaling. It indicates that all natural frequencies involved in the problem are in the range $\Omega \sim 1$, after having accounted for the small mistuning(s).

5. Asymptotic analysis of the FRF

The dimensionless frequency response function associated with (11) reads

$$\mathcal{H}\left(\Omega\right) = \left(-\mathcal{M}\Omega^2 + \mathrm{i}\Omega\mathcal{C} + \mathcal{K}\right)^{-1} \tag{29}$$

where Ω is the dimensionless frequency associated with the dimensionless time τ. Following the same procedure as that described in the Multiple Timescale Spectral Analysis of stochastically excited structures [5], a stretched frequency

$$\Omega = 1 + \varepsilon\eta + \mathrm{ord}(\varepsilon^2) \tag{30}$$

with $\eta \sim 1$ is introduced in order to focus on the frequency range of interest. This stretching can be seen as the formal introduction of several timescales in classical multiple timescale methods [5]. The ansatz

$$\mathcal{H} := \frac{1}{\varepsilon}\mathcal{H}_0 + \mathcal{H}_1 + \mathrm{ord}(\varepsilon^2) \tag{31}$$

is chosen for the frequency response function. It stems from the fact that $\mathcal{M} = \mathcal{K} \sim 1$ and $\mathcal{C} \sim \varepsilon$ at leading order, see (27). The terms $\mathcal{H}_0$ and $\mathcal{H}_1$ in the series expansion can be determined by noticing that $\mathcal{H}^{-1}\mathcal{H} = \left(-\mathcal{M}\Omega^2 + \mathrm{i}\Omega\mathcal{C} + \mathcal{K}\right)\frac{1}{\varepsilon}\left(\mathcal{H}_0 + \varepsilon\mathcal{H}_1\right) = \mathbf{I}$, i.e.

$$\left(-\mathcal{M}_0\left(1+2\varepsilon\eta\right) + \mathrm{i}\left(1+\varepsilon\eta\right)\varepsilon\mathcal{C}_1 + \mathcal{K}_0 + \varepsilon\,\mathcal{K}_1\right)\left(\mathcal{H}_0 + \varepsilon\mathcal{H}_1\right) = \varepsilon\mathbf{I}. \tag{32}$$

Substituting (28) and balancing the similar powers of ε, the leading order solution is

$$\mathcal{H}_0 = \left(-2\eta\mathcal{M}_0 + \mathrm{i}\mathcal{C}_1 + \mathcal{K}_1\right)^{-1}, \tag{33}$$

or, after substitution of the full expression for $\mathcal{M}_0$, $\mathcal{C}_1$ and $\mathcal{K}_1$,

$$\mathcal{H}_0 = \frac{1}{2}\begin{pmatrix} \tilde{\mathbf{M}}_\mathrm{s}\left(-\eta\mathbf{I} + \boldsymbol{\Delta\Omega}_\mathrm{s}\right) & -\frac{1}{2}\boldsymbol{\varphi}\tilde{\mathbf{M}}_\mathrm{d} \\ -\frac{1}{2}\tilde{\mathbf{M}}_\mathrm{d}\boldsymbol{\varphi}^T & \tilde{\mathbf{M}}_\mathrm{d}\left(\left(-\eta+\delta\right)\mathbf{I} + \boldsymbol{\Delta\Omega}_\mathrm{d}\right) + \mathrm{i}\tilde{\xi}_\mathrm{TMD}\tilde{\mathbf{C}}_\mathrm{d} \end{pmatrix}^{-1}. \tag{34}$$

6. Discussion

This equation expresses, in its most simple form, the leading order dynamics of the structure and of the tuned mass dampers for small mistuning and in the neighborhood of resonance. After truncation at first order, it exclusively depends on $\mathcal{M}_0$, $\mathcal{C}_1$ and $\mathcal{K}_1$ which capture the main information about the structural dynamics in near-resonance conditions : the small added mass, the damping ratio and the tuning frequency.

In this formulation, we have transformed the formal expression (29) of the frequency response function which features $2n$ poles into a n-pole expression. Indeed, while (29) involves the inverse of a quadratic in Ω (with all matrices are $n \times n$), (33) just involves the inverse of η. This is a consequence of the use of the stretched coordinate (30) which allows focusing on the resonance peak(s) located on the positive side of the frequency axis only, see [5] for more details. As a consequence, the analytical expressions are more tractable than in the formal case. This makes it possible to determine simple analytical solutions in case where a formal derivation (without explicitly exploiting the smallness of the numerous small numbers in this problem) would yield very long and cumbersome expressions.

Last but not least, (34) degenerates into simple known cases. Some of them are shown in the illustrations of the following section. We can also observe that if the tuned mass dampers were placed at antinodes of vibration modes, $\boldsymbol{\varphi} = 0$, or if the mass of the additional devices was tending to zero $\tilde{\mathbf{M}}_\mathrm{d} \to 0$, there would be no benefit for the structure. This is indeed what our expression translates since the coupling terms $-\frac{1}{2}\boldsymbol{\varphi}\tilde{\mathbf{M}}_\mathrm{d}$ vanish as soon as one of these two conditions is met. This shows that the structure and TMDs would live independently, as expected. It that case, the dynamics of the structure would be ruled out by the upper left term in $\mathcal{H}_0$ (it yields unbounded response because structural inherent damping is assumed of order ε^2) and those of the additional devices by the bottom right part of $\mathcal{H}_0$. Since these latter degrees-of-freedom are not excited by any external loading, this yields the trivial solution $\mathbf{q}_\mathrm{TMD} = 0$ in the uncoupled case.

So the coupling term $-\frac{1}{2}\boldsymbol{\varphi}\tilde{\mathbf{M}}_\mathrm{d}$ is responsible for the intricate mixing of the dynamics of the structure, and of the tuned mass damper(s). It actually hinders to further pursue the analytical derivation. In fact the Frobenius-Schur block matrix inversion formula can be used to extract the upper left partition of the inverse but the exact expression remains rather long. As such, (34) is the most compact form that we could obtain and that translates the complete dynamics of the problem.

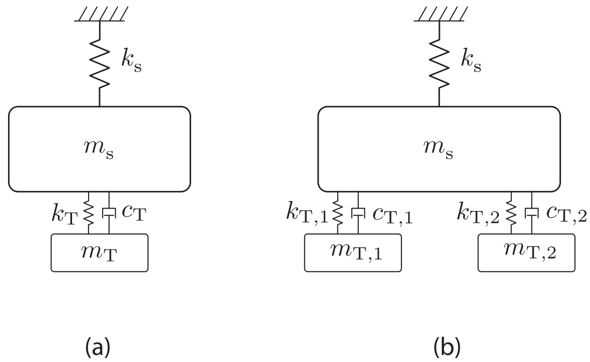


Figure 1: Sketch of the practical applications

	Illustration 1	Illustration 2
$m_s^\star$	M_1	M_1
$k_s^\star$	K_1	K_1
$m_{TMD}^\star$	m_{TMD}	$m_{TMD,1} + m_{TMD,2}$
$k_{TMD}^\star$	k_{TMD}	$k_{TMD,1} + k_{TMD,2}$
$\xi_{TMD}^\star$	ξ_{TMD}	$\xi_{TMD,1} + \xi_{TMD,2}$

Table 1: Reference characteristics used in the following illustrations

7. Illustrations

The very general equation (34) is now specialized to the two cases of practical application that are sketched in Figure (1). The first illustration corresponds to the well-known case where only one tuned mass damper is used to mitigate vibrations in a single structural mode, see Figure (1)-a. In the second illustration, a single structural mode is damped with two tuned mass dampers. Practically this configuration might occur when a single tuned mass damper is split into two (for available space or size reasons) or when two slightly different tuned mass dampers are installed in order to provide more robustness against mistuning. In that case, see Figure (1)-b, the two tuned mass dampers are nominally identical or slightly different on purpose.

7.1. One mode, one TMD

The problem drastically simplifies if $n_d = 1$ and $n_m = 1$. In this first example, there is only one structural mode (degree-of-freedom) and one tuned mass damper. It is therefore natural to choose the characteristic values $m_{\mathrm{s}}^{\star}$ and $k_{\mathrm{s}}^{\star}$ as the modal mass and stiffness, and $m_{\mathrm{TMD}}^{\star}$ and $k_{\mathrm{TMD}}^{\star}$ as the mass and stiffness of the small added mass. The characteristic damping ratios $\xi_{\mathrm{s}}^{\star}$ and $\xi_{\mathrm{TMD}}^{\star}$ are also naturally chosen as the damping ratios of the structural mode and of the tuned mass damper, respectively ($\xi_{\mathrm{s}}^{\star} = \xi_{\mathrm{s}}$; $\xi_{\mathrm{TMD}}^{\star} = \xi_{\mathrm{TMD}}$), see Table 1. This results in $\tilde{\mathbf{M}}_{\mathrm{s}} = \tilde{\mathbf{M}}_{\mathrm{d}} = \tilde{\mathbf{K}}_{\mathrm{s}} = \tilde{\mathbf{K}}_{\mathrm{d}} = \tilde{\mathbf{C}}_{\mathrm{s}} = \tilde{\mathbf{C}}_{\mathrm{d}} = 1$.

Matrix $\mathcal{H}_0$ can be obtained by substituting this into (34), but it is interesting to develop intermediate results for this simple and well-known problem. Furthermore, the mass ratio $\mu = m_{\mathrm{TMD}}^{\star}/m_{\mathrm{s}}^{\star}$ is equal to ε^2. Seen differently, ε is defined by $\sqrt{\mu}$. This gives the physical meaning to the small number. The magnitude φ of the mode shape at the position of the tuned mass damper becomes a scalar. Finally, since there is only one mode shape, $\Delta\Omega_{\mathrm{s}} = 0$ becomes useless; same for $\Delta\Omega_{\mathrm{d}} = 0$.

The formal expressions (27) of the augmented structural matrices boil down to

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mathcal{K} = \begin{pmatrix} 1 + \mu\alpha^2\varphi^2 & -\sqrt{\mu}\alpha^2\varphi \\ -\sqrt{\mu}\alpha^2\varphi & \alpha^2 \end{pmatrix}; \quad \mathcal{C} = 2\xi_{\mathrm{s}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2\alpha\xi_{\mathrm{TMD}} \begin{pmatrix} \mu\varphi^2 & -\sqrt{\mu}\varphi \\ -\sqrt{\mu}\varphi & 1 \end{pmatrix}. \tag{35}$$

They are expressed as a function of the mass ratio μ, the frequency ratio $\alpha = \omega_{\mathrm{TMD}}/\omega_{\mathrm{s}}$, the damping ratios ξ_{s} and ξ_{TMD} and the mode shape amplitude φ at the location of the installed damper. The exact expression of the frequency response function (29), obtained with these matrices will serve as a reference solution. Alternatively the leading order expressions of the structural matrices are

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \mathcal{K}_1 = \begin{pmatrix} 0 & -\varphi \\ -\varphi & 2\delta \end{pmatrix} \quad \mathcal{C}_1 = 2 \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\xi}_{\mathrm{TMD}} \end{pmatrix} \tag{36}$$

where $\tilde{\xi}_{\mathrm{TMD}} = \xi_{\mathrm{TMD}}/\sqrt{\mu}$ and $\delta = (\alpha - 1)/\sqrt{\mu}$ is the mistuning defined in (25). They are obtained by specializing the general solution (28) to the current case or by introducing the small number $\varepsilon = \sqrt{\mu}$ in (35). What the multiple timescale analysis indicates is that the coupled dynamics of this structure (damped with a tuned mass damper) is represented, at first order, by these three simple and short matrices, instead of the more complex expressions corresponding to the exact solution. Quite interestingly, thanks to this, the FRF of the structural DOF takes a very simple expression.

$$\mathcal{H}_0(1,1) = -\frac{2\left(\delta - \eta + i\tilde{\xi}_{\mathrm{TMD}}\right)}{4\eta\left(\delta - \eta + i\tilde{\xi}_{\mathrm{TMD}}\right) + \varphi^2}. \tag{37}$$

The module of this function is depicted by dashed lines in Figure 2-a (analytic), where it is also compared to the exact solution obtained by numerical inversion of the full matrix, which is represented by solid lines (numeric). The double peak pattern is well reproduced with one peak being slightly overestimated while the other is slightly underestimated, but with a well balanced compensation. A better estimate could be obtained by including the second-order term $\mathcal{H}_1$ in the response but this goes beyond the scope of this communication.

The variance of the displacement under a unit white noise (broadband) loading corresponds to the 2-norm of the frequency response function

$$\sigma_x^2 = \|H\|_2 = \int_{-\infty}^{+\infty} |H|^2 \, d\omega. \tag{38}$$

This integral is difficult to estimate in the general case. On the contrary, its computation is straightforward with the proposed simplifications. Using Cauchy's residue theorem we obtain

$$\sigma_x^2 = \frac{\pi}{4\varphi^2\sqrt{\mu}} \frac{4\tilde{\xi}_{\mathrm{TMD}}^2 + 4\delta^2 + \varphi^2}{\tilde{\xi}_{\mathrm{TMD}}} = \frac{\pi}{\mu\varphi^2} \frac{\xi_{\mathrm{TMD}}^2 + \left(\frac{\omega_{\mathrm{TMD}}}{\omega_{\mathrm{s}}} - 1\right)^2 + \frac{\mu}{4}\varphi^2}{\xi_{\mathrm{TMD}}}. \tag{39}$$

This simple expression captures the well-known features of the influence of a TMD on the response of a single degree-of-freedom structure. Specifically, it reveals the existence of optimal values of the damping ratio ξ_{TMD} and the mistuning $\alpha = \frac{\omega_{\mathrm{TMD}}}{\omega_{\mathrm{s}}}$. Since α appears in the numerator only, the response is minimized by minimizing $(\alpha - 1)^2$, i.e. adjusting ω_{TMD} and ω_{s}. A trade-off is however required for the damping ratio which appears in both the numerator and denominator of fraction (39). The variance σ_x^2 is seen to be minimum when the damping ratio $\xi_{\mathrm{TMD}} = \sqrt{(\alpha-1)^2 + \mu\varphi^2/4}$. Combining both conditions, the proposed model provides the optimal tuning parameter $(\alpha, \xi_{\mathrm{TMD}})_{\mathrm{opt}} = \left(1, \sqrt{\mu}\varphi/2\right)$. Furthermore, in the proposed model (39), the modal amplitude φ at the location of the TMD is always combined with the mass ratio μ. This translates the well-known fact that the placement of a device with mass ratio μ away from the antinode of the structural mode shape corresponds to placing a smaller mass ratio $\mu\varphi^2$ exactly at the antinode.

Figure (2)-b shows the variance of the response to unit broadband excitation as a function of the damping ratio of the tuned mass damper and for two values of the mistuning. Again, the exact (numerical) solution and the proposed (analytical) solution are compared. The agreement is very good, especially for $\alpha \simeq 1$. The contours in Figure (2)-(c,d) illustrate the variance of the response as a function of the TMD parameters. They also illustrate the accuracy of the proposed solution.

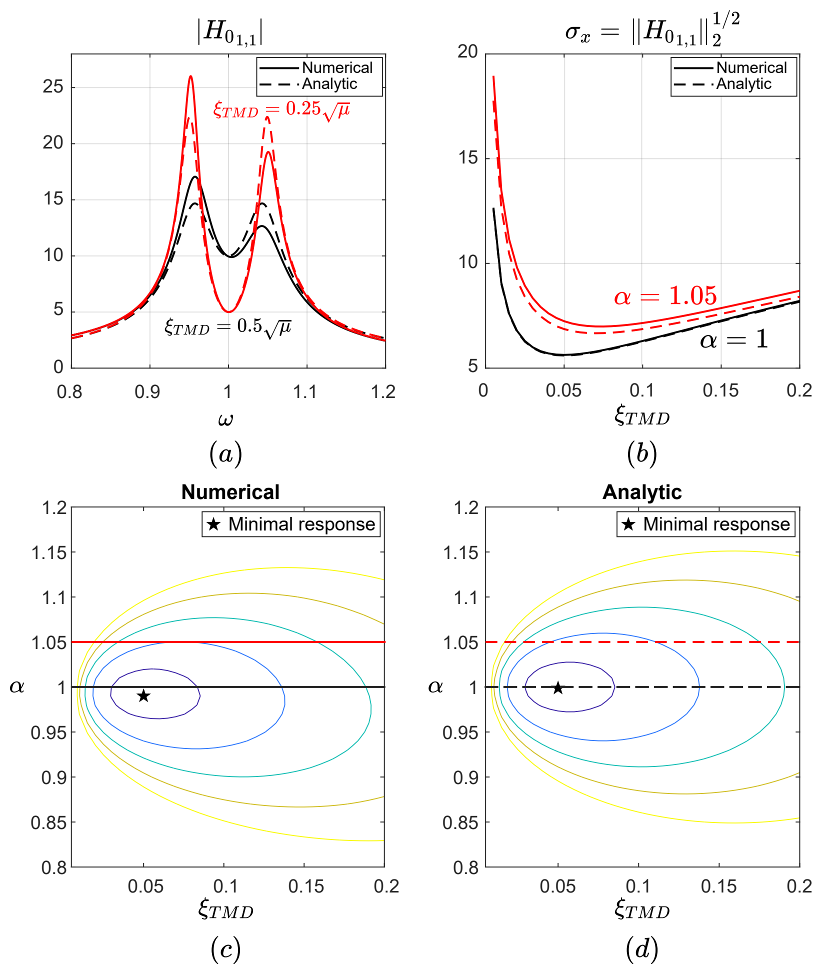


Figure 2: Illustration 1: Comparison of the exact (numerical) and approached (analytic) transfer function ($\mu = 0.01$, $\varphi = 1$). (a) Frequency Response Function (FRF), (b) 2-norm of the FRF as a function of the damping ratio ξ_{TMD}, (c-d) contours of the 2-norm of the FRF as a function of the damping ratio ξ_{TMD} and mistuning parameter α.

Tuning parameters	α	ξ_{TMD}
Den Hartog	$\frac{1}{1+\mu} \sim 1-\mu$	$\sqrt{\frac{3}{8}\frac{\mu}{1+\mu}} \sim \sqrt{\frac{3}{8}}\sqrt{\mu}$
Warburton	$\frac{1}{1+\mu}\sqrt{\frac{2+\mu}{2}} \sim 1-\mu$	$\sqrt{\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}} \sim \frac{1}{2}\sqrt{\mu}$
Fujino	$\frac{\sqrt{1+\mu/2}}{1+\mu} \sim 1-\mu$	$\frac{1}{2}\sqrt{\frac{\mu(1+3\mu/4)}{(1+\mu)(1+\mu/2)}} \sim \frac{1}{2}\sqrt{\mu}$
Proposed	1	$\frac{1}{2}\sqrt{\mu}$

Table 2: Comparison with other criteria available in the literature.

The location of the optimum is slightly off since the proposed method results in $\alpha = 1$ instead of an optimal value slightly smaller than 1 in the exact formulation. This is because only the leading order solution has been kept in the proposed derivation. The location of the optimum in the $(\alpha, \xi_{\mathrm{TMD}})$ space can be compared to available solutions in the literature, see Table (2). The series expansions, for small mass ratio, of all these expressions also yield $\alpha \sim 1$ at leading order, which is fully consistent with the proposed formulation which is truncated after leading order. Concerning the damping ratio, the optimal value suggested by our model is consistent with the results of Warburton [25] and Fujino [8], but also close to the criteria of Den Hartog $\xi_{\mathrm{TMD}} \simeq 0.61\sqrt{\mu}$. This difference is explained by the fact that Den Hartog's strategy of equalizing the height of both peaks in the FRF corresponds to minimizing the $\infty-$norm of $\mathcal{H}(\omega)$ instead of the 2-norm as considered in this illustration and in the works of Warburton and Fujino.

To close the discussion, it is interesting to point out that the location of the TMD φ comes in the denominator of (39) which means that the response grows unbounded as $\varphi \to 0$. As already introduced, in that case, the coupling vanishes in (34) and the TMD-structure system behaves as a 2-DOF uncoupled system. Since the structural damping ratio does not enter the governing equations at leading order, the response is unbounded.

This first illustration aimed at recovering known results from the literature and provide a general framework that can be followed for higher dimensional systems. The second illustration is the first step along that line.

7.2. One mode, two TMDs

In the second illustration, there is one structural degree-of-freedom and two additional masses. It is therefore natural to choose the structural characteristic values as those of the physical system, in particular $m_{\mathrm{s}}^{\star} = m_s$, so that $\tilde{\mathbf{M}}_{\mathrm{s}} = \tilde{\mathbf{K}}_{\mathrm{s}} = \tilde{\mathbf{C}}_{\mathrm{s}} = 1$ and $\Delta\Omega_{\mathrm{s}} = 0$. The characteristic mass $m_{\mathrm{TMD}}^{\star}$ can be chosen as the sum of the two masses, $m^\star_{\mathrm{TMD}} := m_{\mathrm{TMD},1} + m_{\mathrm{TMD},2}$. Once the characteristic masses are defined, the small number ε is obtained by (14), i.e. $\varepsilon = \sqrt{m^\star_{\mathrm{TMD}}/m^\star_{\mathrm{s}}}$. The characteristic stiffness used for the TMDs is consistently chosen as $k^\star_{\mathrm{TMD}} := k_{\mathrm{TMD},1} + k_{\mathrm{TMD},2}$, and the dimensionless mass and stiffness associated with TMDs are

$$\tilde{\mathbf{M}}_{\mathrm{d}} = \begin{pmatrix} m_{\mathrm{TMD},1}/m^\star_{\mathrm{TMD}} & 0 \\ 0 & m_{\mathrm{TMD},2}/m^\star_{\mathrm{TMD}} \end{pmatrix} \quad ; \quad \tilde{\mathbf{K}}_{\mathrm{d}} = \begin{pmatrix} k_{\mathrm{TMD},1}/k^\star_{\mathrm{TMD}} & 0 \\ 0 & k_{\mathrm{TMD},2}/k^\star_{\mathrm{TMD}} \end{pmatrix}. \tag{40}$$

In the rest of this illustration, it is assumed that, for practical reasons, a single TMD was split into two smaller units of equal mass ($m_{\mathrm{TMD11}} = m_{\mathrm{TMD21}}$ so that the diagonal elements of $\tilde{\mathbf{M}}_{\mathrm{d}}$ are equal to 1/2). An objective of this study is to determine whether it is possible to provide better performances with two slightly different units rather than two identical units.

Small mistuning is invoked so that (26) gives

$$\frac{k_{\mathrm{TMD},i}}{k^\star_{\mathrm{TMD}}} = \frac{m_{\mathrm{TMD},i}}{m^\star_{\mathrm{TMD}}} \left(1 + 2\varepsilon\, \Delta\Omega_{\mathrm{d,i}}\right) \tag{41}$$

for $i = 1, 2$. The sum of these two equations yields, term by term, $1 = 1 + 2\varepsilon \frac{m_{\mathrm{TMD},1}\Delta\Omega_{\mathrm{d},1} + m_{\mathrm{TMD},2}\Delta\Omega_{\mathrm{d},2}}{m^\star_{\mathrm{TMD}}}$, or

$$\Delta\Omega_{\mathrm{d},1} + \Delta\Omega_{\mathrm{d},2} = 0 \tag{42}$$

for equal masses. Furthermore, the natural frequency of each TMD taken separately is asymptotically given by

$$\omega_{\mathrm{TMD},i} = \sqrt{\frac{k_{\mathrm{TMD},i}}{m_{\mathrm{TMD},i}}} \sim \sqrt{\frac{k^\star_{\mathrm{TMD}}}{m^\star_{\mathrm{TMD}}}} \left(1 - \varepsilon\, \Delta\Omega_{\mathrm{d,i}}\right) \tag{43}$$

so that the relative distance between the natural frequencies reads

$$\Delta := \frac{\omega_{\mathrm{TMD},1} - \omega_{\mathrm{TMD},2}}{\omega_{\mathrm{TMD},1} + \omega_{\mathrm{TMD},2}} = \frac{\varepsilon}{2} \left(\Delta\Omega_{\mathrm{d},1} - \Delta\Omega_{\mathrm{d},2}\right) = \varepsilon\, \Delta\Omega_{\mathrm{d},1}. \tag{44}$$

This indicates that Δ is a small number, which is expected in view of the small mistuning (41). Equation (44) is used in the sequel in order to get rid of $\Delta\Omega_{\mathrm{d},1}$ and use the more meaningful quantity Δ instead. In order to simplify the following notation, we also define $\tilde{\Delta} = \Delta\Omega_{\mathrm{d},1} = \mathrm{ord}(1)$, which also means $\Delta\Omega_{\mathrm{d},2} = -\tilde{\Delta}$.

Assuming that the dimensionless damping ratio $\widetilde{\xi}_{\mathrm{TMD}}$ is equal for the two dampers and that the two modal amplitudes φ_1 and φ_2 are equal at first order (because the two units are located close to each other), $\varphi_1 = \varphi_2 := \varphi$, the asymptotic and stretched versions of the structural matrices take the simple form

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}; \mathcal{K}_1 = \begin{pmatrix} 0 & -\varphi/2 & -\varphi/2 \\ -\varphi/2 & \delta + \tilde{\Delta} & 0 \\ -\varphi/2 & 0 & \delta - \tilde{\Delta} \end{pmatrix}; \mathcal{C}_1 = \widetilde{\xi}_{\text{TMD}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which are characterized by the two mistuning parameters: δ (structure vs. TMDs) and $\tilde{\Delta}$ (TMD1 vs. TMD2). These expressions are doubtlessly simpler than the formal expressions.

The general expression (34) of the frequency response function of the coupled system can be specialized with these matrices. In particular, the first element, related to the structural response, reads

$$\mathcal{H}_{0_{1,1}}(\eta) = -\frac{2\left(\left(\delta - \eta + \mathrm{i}\widetilde{\xi}_{\text{TMD}}\right)^2 - \tilde{\Delta}^2\right)}{4\eta\left(\left(\delta - \eta + \mathrm{i}\widetilde{\xi}_{\text{TMD}}\right)^2 - \tilde{\Delta}^2\right) + \varphi^2\left(\delta - \eta + \mathrm{i}\widetilde{\xi}_{\text{TMD}}\right)}. \tag{45}$$

Again, it is highlighted that this expression is drastically more compact than the exact solution (involving the inverse of a 3×3 matrix). It provides however very accurate solutions as long as $\varepsilon \ll 1$. For instance, Figure 3-a shows $\left|\mathcal{H}_{0_{1,1}}\right|$ for two different values of $\widetilde{\xi}_{\text{TMD}}$. The proposed method (analytic, dashed lines) is compared to a full numerical estimate (inverse of 3×3 matrix) of the response. Numerical values chosen for the illustration are a mass ratio $\mu = 0.01$, i.e. $\varepsilon = 0.1$, and dampers located at the antinode of the mode, $\varphi = 1$, while the mistuning parameters are $\alpha = 1$, i.e. $\delta = 0$ and $\Delta = \varepsilon\,\tilde{\Delta} = 0.05$. It shows that the proposed method is able to accurately capture the resonant peaks as well as the monotonic decrease in the far field, away from resonances; this is all the more appreciated that the proposed method is based on a stretched coordinate in the neighborhood of resonances.

The 2-norm of the frequency response functions, defined as in (38), do not accept simple closed form expressions, neither in the exact case, neither in the proposed approach. The 2-norm of the frequency response function obtained by numerical integration is shown as a function of the damping ratio ξ_{TMD} in Figure 3-(b,c,d). The proposed formulation is again shown to be accurate, and most importantly the existence of an optimum damping ratio is well captured by the simplified model, which is not necessary a trivial feature when dealing with multiple scales methods [19]. The largest relative errors are 0.6% and 2.9% for $\alpha = 1$ and $\alpha = 1.05$, respectively.

Although the 2-norm of the frequency response function does not accept a closed form expression in the general case of the proposed formulation, some more insightful conclusions based on analytical derivation can be obtained in specific cases. In particular, the influence of the mistuning of the two

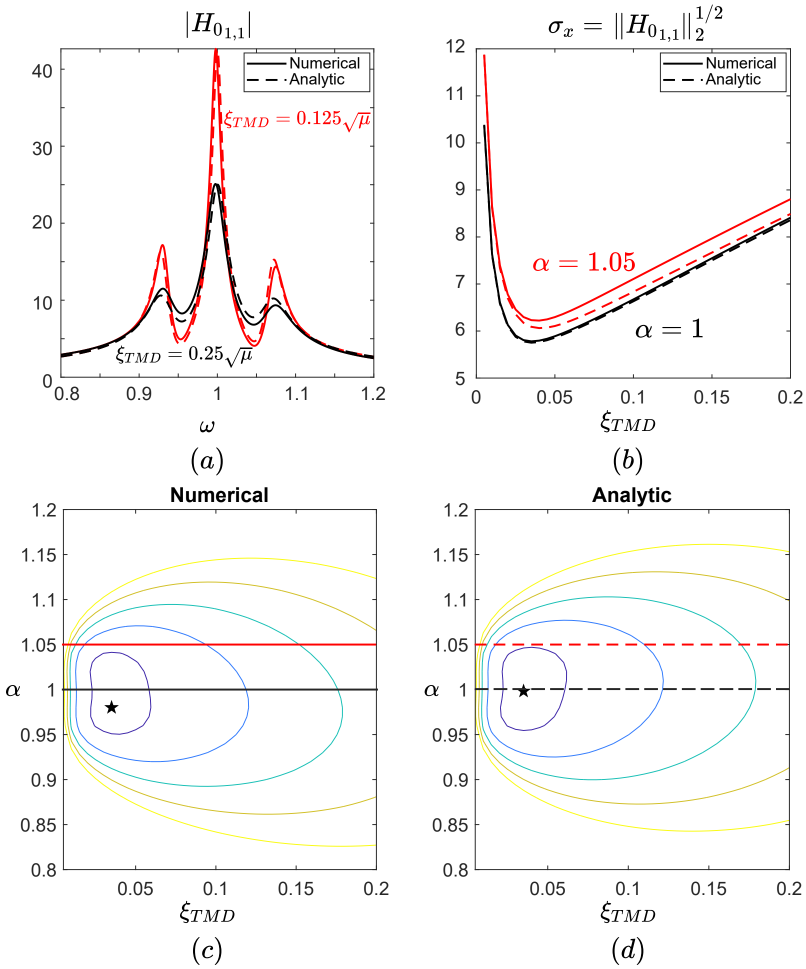


Figure 3: (a) Comparison of the proposed approximation of $\mathcal{H}_{0_{1,1}}$ (analytical) and the exact value numerically obtained by inversion of the 3×3 matrix, (b,c,d) 2-norm of the frequency response function $\mathcal{H}_{0_{1,1}}$ represented for $\alpha = 1$ ($\delta = 0$) and $\alpha = 1.05$ ($\delta = 0.5$) or as a function of the tuning parameters $(\alpha, \xi_{\mathrm{TMD}})$ of the TMDs. Numerical values: $\varepsilon = 0.1$, $\mu = 0.01$, $\varphi = 1$, $\Delta = 0.05$, $\tilde{\Delta} = 0.5$.

tuned mass dampers can be studied. It is expected that the two damping units do not work as a whole when their natural frequencies are too different from each other. With the proposed formulation, it is possible to determine the maximum relative difference (in terms of frequencies for instance) of the two units so that they behave as a single tuned mass damper. To do so, it is noticed first that, in the case $\Delta = 0$, the two TMD units are identical (same mass, same frequency, same damping ratio), and (45) interestingly degenerates into (37) after simplifying numerator and denominator by $\left(\delta - \eta + \mathrm{i}\widetilde{\xi}_{\mathrm{TMD}}\right)$. This division by a function of the stretched coordinate η consists in dropping one pole (forget one dimension in the dynamics) and it is indeed observed that we recover the model considered in the first illustration with one mode and one TMD. For small values of the mistuning $\tilde{\Delta}$, the leading order expression of the variance of the response is therefore similar to (39) and given by

$$\sigma^2_{x,\tilde{\Delta}\ll 1} = \frac{\pi}{4\varphi^2\sqrt{\mu}} \frac{4\tilde{\xi}^2_{\mathrm{TMD}} + 4\delta^2 + \varphi^2}{\tilde{\xi}_{\mathrm{TMD}}} + \mathrm{ord}\left(\tilde{\Delta}\right). \tag{46}$$

This interesting conclusion was already hinted by the fact that the frequency response function (45) tends towards the expression (37) for small values of $\tilde{\Delta}$. The Figure 4-a represents the structural response as a function of $\Delta := \varepsilon\tilde{\Delta}$. The configuration that provides the minimal variance is obtained when Δ is minimum, the numerical and analytic curves are in accordance and match the limit value given in (46). For large mistuning, the FRF is approximated by

$$\lim_{\tilde{\Delta}\to+\infty} \mathcal{H}_{0_{1,1}}(\eta) = -\frac{2\left(-\tilde{\Delta}^2\right)}{4\eta\left(-\tilde{\Delta}^2\right) + \varphi^2\left(\delta - \eta + \mathrm{i}\widetilde{\xi}_{\mathrm{TMD}}\right)} \tag{47}$$

and the variance of the response to broadband loading is

$$\sigma^2_{x,\tilde{\Delta}\gg 1} = \frac{1}{\varphi^2\sqrt{\mu}\tilde{\xi}_{\mathrm{TMD}}} \frac{4\pi\tilde{\Delta}^4}{4\tilde{\Delta}^2 + \varphi^2} = \frac{\pi\tilde{\Delta}^2}{\varphi^2\sqrt{\mu}\tilde{\xi}_{\mathrm{TMD}}} + \mathrm{ord}\left(\frac{1}{\tilde{\Delta}^2}\right). \tag{48}$$

The standard deviation of the response ultimately grows linearly with the mistuning.

Numerical integration of (45) (labeled "Semi-analytic") is compared to the numerical integral of the original problem in Figure 4-a for $\delta = 0.1$ and as a function of Δ. The two asymptotic solutions are also represented and show very good agreement. The proposed method provides an acceptable estimation of the numerical computation. The relative error with respect to the numerical model is plotted in order to validate the asymptotic proposition for different tuning configurations (α, Δ). According to Figure 4-b, the approximation offers a good representation of the exact behavior (error lower than 10%) despite the existence of zones of non-optimal functioning that appear when the natural frequencies of the TMDs are far away from each other and so wide of the mark.

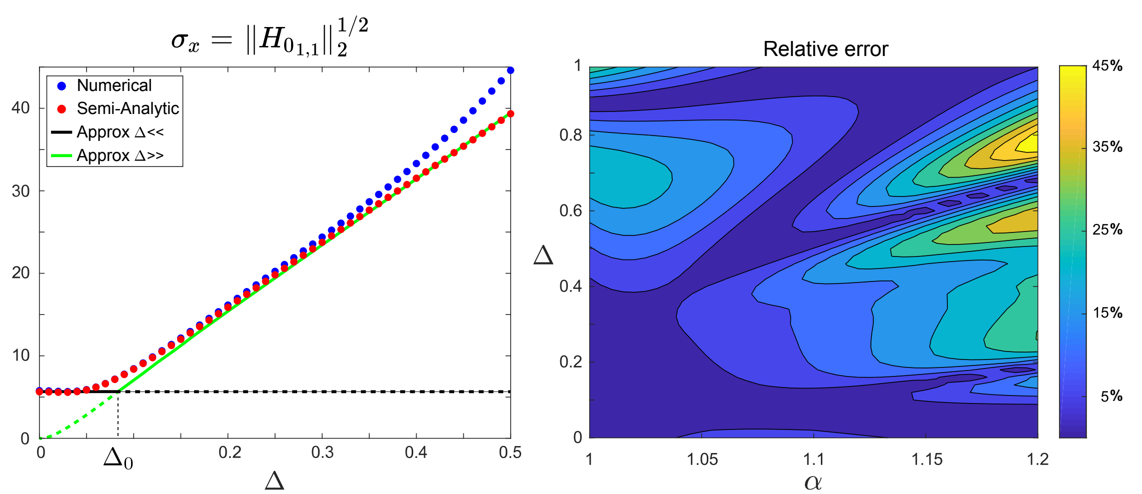
Although the problem could not be fully solved with closed-form expressions, the proposed formulation allowed to derive two simple asymptotic cases for small and respectively large $\tilde{\Delta}$. The response of the structure equipped with two tuned mass dampers can be idealized by considering that the efficiency of the two units is optimal and unchanged for $\tilde{\Delta} \in [0, \tilde{\Delta}_0]$ where $\tilde{\Delta}_0$ corresponds to the intersection of the two asymptotes, then increases monotonically for larger mistuning. Solving $\sigma^2_{x,\tilde{\Delta}\ll 1} = \sigma^2_{x,\tilde{\Delta}\gg 1}$ for $\tilde{\Delta}$ yields

$$\tilde{\Delta}_0^2 = \tilde{\xi}^2_{\mathrm{TMD}} + \delta^2 + \frac{1}{4}\varphi^2 \quad \rightarrow \Delta_0^2 = \varepsilon^2 \tilde{\Delta}_0^2 = \xi^2_{\mathrm{TMD}} + (1-\alpha)^2 + \frac{1}{4}\mu\varphi^2$$

Provided the two units are tuned for optimal conditions, $\alpha = 1$ and $\tilde{\xi}_{\mathrm{TMD}} = 1/2$, the former derivation shows that two slightly mistuned units tend to work together as long as the mistuning of their frequencies is lower than

$$\Delta_0 = \frac{1}{2}\sqrt{1+\mu\varphi^2}$$

after what the performance of the 2-unit damping system worsens.



(a) (b)

Figure 4: (a) Comparison of the semi-analytic approximation of the standard deviation for different values of Δ, (b) Relative error of the std for different tuning configurations (numerical integration of the exact and approached FRFs)

8. Conclusions

The proposed method developed in this paper suggests a simple mathematical model for the analysis of multiple degree-of-freedom structural system equipped with multiple tuned mass dampers. A final aim is to predict the optimal adjustment of the dampers associated to the minimal structural response under wide-band excitation. This strategy stands out from the classical ones as it begins by putting all the assumptions on the order of magnitude of the different parameters, then concentrates all the computational efforts on the areas of the problem that are crucial, and finally tackles the calculation of the response.

The analysis is frequential and implemented in the modal basis. The structural matrices and thus the FRF appear to take a heavy and non-explicit form. In order to lighten up the computations and make them applicable for design, a proper scaling of the different parameters involved is performed. This opens the gates to exploit perturbation methods, and allows to gather with thoroughness the essence of the information contained in the first order. The expressions of the transfer function and thus the variance of the response are condensed, simplified and more meaningful. The asymptotical developments are validated through two illustrations, as it predicts in a solid manner the structural behavior.

A low-order mathematical model has been derived for the analysis of multiple degree-of-freedom structural systems equipped with multiple tuned mass dampers. By carefully expressing the orders of magnitude of the different quantities involved in this generic problem, it was shown that the frequency response function (FRF) of the coupled system can be expressed with a simple expression. It has been obtained with the multiple timescale spectral approach and by invoking the smallnesses of all possible problem parameters. It is the simplest form that can be obtained for the general problem without formulating additional assumptions. The corresponding solution is accurate as long as the mass ratio remains small and the response to the considered loading is mostly contributed by structural resonance.

Once an approximate expression for the system FRF is obtained, any subsequent information can be estimated. While the infinite norm of the FRF would inform about the response to harmonic loading, the 2-norm corresponds to the response under broadband excitation. This latter case has been investigated for two particular examples of application of the proposed formulation. In particular, it has been demonstrated that it is possible to derive approximate solutions to problems that are more complex than those enjoying today's available closed-form solutions. In the second example, we could derive a (first-order accurate) condition on the maximum frequency mistuning between two separate

TMDs so that they can offer the same mitigation properties as a perfectly tuned device. This example shows the perspectives of the simple model. The same methodology could be used to investigate the response of a single mode under TMDs split into more than 2 mistuned units, the possible usage of a single TMD to damp two or more structural modes at a time. Other scopes of application concern the double TMD which do require, however, the derivation of another family of governing equations, which includes interactions between the small masses of the problem.

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