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Dissertation originale présentée en vue de l'obtention du grade académique de Docteur en Sciences — Juin 2022

Expansions in Cantor real bases and alternate bases:

Combinatorial, algebraic and ergodic properties

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"*The recent development of combinatorics is somewhat of a cinderella story. It used to be looked down on by mainstream mathematicians as being somehow less respectable than other areas, in spite of many services rendered to both pure and applied mathematics. Then along came the prince of computer science with its many mathematical problems and needs and it was combinatorics that best fitted the glass slipper held out.*"

Björner and Stanley

REMERCIEMENTS

Je remercie de tout cœur Émilie Charlier pour une multitude de raisons. Je ne peux les citer toutes, mais les suivantes permettront de comprendre à quel point Émilie a été sans conteste le pilier central de cet accomplissement. Tout d'abord, je la remercie d'avoir accepté de me superviser pendant ces quatre années de doctorat. Je la remercie également pour les merveilleuses questions de recherche qu'elle m'a proposées et pour le savoir qu'elle m'a transmis afin de les résoudre. Sa disponibilité, son enthousiasme sans faille et ses encouragements m'ont permis de mener à bien cette folle aventure doctorale mais surtout de la vivre sans crainte et sans doute. Je la remercie ensuite et surtout pour son rire communicatif pendant nos longues heures de recherche et pour l'amitié qui s'est finalement créée entre nous. Durant ce parcours, je me suis d'abord rendue compte que j'avais une super promotrice. Ensuite, la super promotrice est devenue une "maman de thèse" que je n'aurais voulu échanger pour rien au monde.

Après la "maman de thèse" que m'a offert ce doctorat, j'ai eu l'immense chance de trouver ma "sœur de thèse" Manon Stipulanti. Ce petit bout de femme pleine de surprises a été à la fois une collaboratrice exceptionnelle et une amie incroyable. Son soutien professionnel m'a permis d'avancer dans mes recherches avec l'impression que le monde s'ouvrait à nous. Son soutien personnel m'a permis de traverser les bons et mauvais moments de ces dernières années. Je profite de ces quelques lignes pour lui exprimer mon amitié la plus sincère.

Le bon déroulement d'un projet de thèse repose aussi sur le partage de connaissances. C'est donc tout naturellement que je tiens à remercier mes co-auteurs. Plus précisément, en plus des deux femmes incroyables décrites ci-dessus, j'ai eu la chance de collaborer avec d'autres superbes personnes : Karma Dajani, Savinien Kreczman, Zuzana Masáková, Adeline Massuir et Edita Pelantová. Chacune de ces collaborations fût agréable et enrichissante.

J'exprime aussi ma gratitude auprès de Jean-Paul Allouche, Charlène Kalle, Cor Kraaikamp, Julien Leroy, Zuzana Masáková et Michel Rigo de me faire l'honneur de faire partie de mon jury de thèse. Je me sens privilégiée de pouvoir compter sur leur expertise.

Je remercie le Fonds de la Recherche Scientifique (FNRS) de m'avoir octroyé cette bourse doctorale. Cela m'a donné la chance de vivre cette fabuleuse expérience, et ce, notamment en voyageant. Grâce à ces voyages, j'ai pu rencontrer de nombreux chercheurs internationaux et j'ai pu réaliser des séjours de recherche mémorables. Par ailleurs, d'un point de vue personnel, ces aventures m'ont énormément apporté et j'ai pu voir le monde sous un autre angle.

Enfin, j'adresse mes remerciements les plus sincères à ma famille et mes amis. Je pense plus particulièrement à mes parents, ma soeur, mon beau-frère et mon compagnon qui m'ont soutenue et encouragée tout au long de ces années. Cette tâche n'a pas été simple mais ils l'ont accomplie d'une façon remarquable et avec un amour sans limite. Cette aventure n'aurait jamais été la même sans le soutien et la bonne humeur de mon entourage. Ils ont, eux aussi, contribué à la réalisation de ce travail en me permettant de déconnecter quand j'en avais besoin. J'en profite pour leur rappeler à quel point je les aime.

Pour finir, j'adresse les quelques lignes qu'il me reste au Covid-19. Je me dois de ne pas remercier du tout ce virus qui a décidé de faire sa triste apparition au milieu de ma deuxième année de thèse. Il a été la cause de toutes ces conférences annulées (ou simplement non-organisées) et de ce télétravail qui empêchait des séances de recherche agrémentées de rires autour d'un café. L'expérience doctorale était déjà géniale et intense ainsi, je n'ose imaginer ce qu'elle aurait pu être sans cette pandémie.

CONTENTS

INTRODUCTION

Mathematics is notably concerned with the study of numbers and the arithmetic properties of these numbers in relation with the syntactical properties of their representations by sequences of symbols (usually called *digits*). In order to approach such questions, we first need to know how to represent numbers since there are many ways to write them. Usually, numbers are represented by words over an alphabet of digits with respect to a base. In everyday life, the decimal representation is used, that is, the base elements are the powers of 10. In computer science, the binary base is preferred for some practical aspects. More generally, any integer $b \geq 2$ can be considered as a base. We then obtain words written over the alphabet $\{0, 1, \dots, b-1\}$ called the base-b representations. Towards a general study, mathematicians are interested in other various ways to represent numbers.

Two well-known generalizations of integer base representations are Cantor and real base representations. The former was introduced by Cantor in 1869 [Can69]. A Cantor representation of a real number x via a base sequence $(b_n)_{n\in\mathbb{N}}$ of integers greater than or equal to 2 is an infinite sequence $a_0 a_1 a_2 \cdots$ of non-negative integers such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} b_i}.$$

If for each $n \in \mathbb{N}$, the digit a_n belongs to the alphabet $\{0, 1 \dots, b_n - 1\}$ and if infinitely many digits a_n are non-zero, then the series is called the Cantor series of x. Many studies are devoted to Cantor series, a large amount of which are concerned with the digit frequencies; see [ER59, Gal76, KT84, Rén56] to cite just a few. The latter was defined by Rényi in 1957 [Rén57] and well understood since the pioneering work of Parry in 1960 [Par60]. A representation of a real number x via a real base $\beta > 1$ is an infinite sequence $a_0a_1a_2\cdots$ of non-negative integers such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{n+1}}.$$

The digits a_n can be chosen by using several appropriate algorithms. Typically each point in $[0,1)$ has uncountably many representations [Sid03]. The most commonly used algorithms are the greedy and lazy algorithms giving rise respectively to the largest and the smallest representations in the lexicographic order. Representations in a real base are extensively studied and we can only cite a few of the many possible references [BM86, Lot02, Par60, Sch80, IS09, Fro92, Sol94, FS10, KL98, Ped05, DK02b, DdVKL12, DK07].

This doctoral dissertation is dedicated to the investigation of series expansions of real numbers that are based on a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. We call such a base sequence $\boldsymbol{\beta}$ a *Cantor real base*. A representation of a real number x via a Cantor real base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ is an infinite sequence $a_0a_1a_2\cdots$ of non-negative integers such that

$$x = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} \beta_i}.$$

We talk about *$\boldsymbol{\beta}$-representations*. In doing so, we generalize both representations of real numbers through Cantor series and real bases. The digits of a $\boldsymbol{\beta}$-representation can be chosen by using several appropriate algorithms. As in the real base theory, in order to represent non-negative real numbers smaller than or equal to $x_{\boldsymbol{\beta}}$, where

$$x_{\boldsymbol{\beta}} = \sum_{n\in\mathbb{N}} \frac{\lceil \beta_n \rceil - 1}{\prod_{i=0}^{n} \beta_i},$$

we will consider the greedy algorithm and the lazy one. In the greedy algorithm, each digit is chosen as the largest possible among $0,\ldots,\lceil \beta_n \rceil - 1$ at position n. At the other extreme, the lazy algorithm picks the least possible digit at each step. The so-obtained $\boldsymbol{\beta}$-representations are respectively called the greedy and lazy $\boldsymbol{\beta}$-expansions.

The goal of this thesis centered at the study of $\boldsymbol{\beta}$-representations is to figure out if the properties of representations in real bases can be generalized while considering Cantor real bases. The framework of this doctoral dissertation encompasses several related but distinct domains, namely, numeration systems, combinatorics on words, formal language theory, algebra, dynamical systems, ergodic theory and number theory.

Note that these type of representations involving more than one base simultaneously and independently have recently aroused the interest of other mathematicians [CD20, Li21, Neu21, KLZ21]. Each gives a different generalization of representations via real bases and with different global interests. These papers all present a generalization of Parry's theorem [Par60] to their respective frameworks. But so far, all the research was concentrated on the symbolic properties of these representations. In this work, we also give algebraic and dynamical properties which are nowhere else studied.

Throughout this text, in order to provide a clear presentation, we illustrate the concepts under consideration thanks to a number of examples. This doctoral dissertation is articulated as follows.

In the first chapter, without attempting to provide an exhaustive description, we recall the necessary backgrounds for a clear understanding of this work. We start with some algebraic structures and related conventions. Then, we briefly introduce words, languages and automata. Next, we state material about measure preserving dynamical systems. This chapter ends with an entire section devoted to the key notion of representations of real numbers in real bases. This section deals with an overview of the combinatorial, algebraic and dynamical properties of β-representations. The goal is to give the state of the art on β-representations by stating results which will then be generalized (or not) in the subsequent chapters to the Cantor real base framework in general or in the particular case of periodic Cantor real bases, called alternate bases. Therefore, the stated results will not be "used" to prove analogue ones for Cantor real bases but they are stated in order to compare theories of real bases and Cantor real bases.

The second chapter aims at defining Cantor real bases and proving fundamental combinatorial properties of $\boldsymbol{\beta}$-representations. We first give a characterization of those infinite words a over the alphabet $\mathbb{R}_{\geq 0}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. Next, we introduce the greedy algorithm and study the combinatorial properties of greedy $\boldsymbol{\beta}$-expansions, each of which extends existing results on representations in a real base. In particular, we introduce the quasi-greedy $\boldsymbol{\beta}$-expansion $d^*_{\boldsymbol{\beta}}(1)$ of 1 and show that $d^*_{\boldsymbol{\beta}}(1)$ is the lexicographically greatest $\boldsymbol{\beta}$-representation not ending in 0^ω of all real numbers in $[0, 1]$. We then prove a generalization of Parry's theorem [Par60] characterizing sequences of non-negative integers that are the greedy $\boldsymbol{\beta}$-representations of some real number in the interval $[0, 1)$. We end this section by introducing the notion of greedy $\boldsymbol{\beta}$-shift and give a description of it in full generality. In the fourth section, the lazy algorithm in real bases is generalized to the setting of Cantor real bases when $x_{\boldsymbol{\beta}} < +\infty$. It is shown that the lazy $\boldsymbol{\beta}$-expansions are obtained by "flipping" the digits of the greedy $\boldsymbol{\beta}$-expansions. As a consequence, combinatorial properties of the previous section are "flipped" to the lazy framework. Then, we show that the same "flip" permits us to go from the quasi-greedy $\boldsymbol{\beta}$-expansion to a quasi-lazy one. Consequently, a Parry-like criterion characterizing sequences of non-negative integers that are the lazy $\boldsymbol{\beta}$-expansions of some real number in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is proved. Moreover, we define and study a lazy $\boldsymbol{\beta}$-shift. Note that lazy real base expansions have been widely studied in terms of dynamics and, to the best of our knowledge, not really in terms of combinatorics. Hence, since real bases are particular cases of Cantor real bases, this section also gives a new study of lazy β-expansions for real bases $\beta > 1$.

In the third chapter, we focus on the combinatorial properties of periodic Cantor real bases

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots),$$

which we call *alternate bases.* Note the importance of these particular Cantor real bases since they will also be central for the next two chapters. In both the greedy and lazy cases, we are able to give more precise results than in the general framework of Cantor real bases. In particular, generalizing Parry's result [Par60], we obtain a characterization of the greedy $\boldsymbol{\beta}$-expansion of 1 among all $\boldsymbol{\beta}$-representations of 1. Moreover, generalizing a result of Bertrand-Mathis [BM86], we show that for any alternate base $\boldsymbol{\beta}$, the greedy $\boldsymbol{\beta}$-shift is sofic, that is, its factors form a language that is accepted by a finite automaton, if and only if all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic, where

$$\boldsymbol{\beta}^{(i)} = (\beta_i, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$$

is the i^{th} shift of the Cantor real base $\boldsymbol{\beta}$. Since real bases $\beta > 1$ determining sofic β-shifts are called Parry numbers, we call *Parry alternate bases* the alternate bases such that all quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are ultimately periodic. Using the "flip" from greedy to lazy $\boldsymbol{\beta}$-expansions, analogue results are proved for lazy $\boldsymbol{\beta}$-expansions. In particular, we prove that an alternate base $\boldsymbol{\beta}$ is a Parry alternate base if and only if all quasi-lazy $\boldsymbol{\beta}^{(i)}$-expansions of $x_{\boldsymbol{\beta}^{(i)}} - 1$ are ultimately periodic. Moreover, we show that the alternate base $\boldsymbol{\beta}$ is a Parry alternate base if and only if the lazy $\boldsymbol{\beta}$-shift is sofic.

The fourth chapter deals with some algebraic properties of alternate base expansions. In the real base case, an algebraic description of Parry numbers $\beta > 1$ is not obvious. It is known that the set of Parry numbers includes

Pisot numbers, that is, algebraic integers greater than 1 with Galois conjugates inside the unit circle [Ber77], and that this statement cannot be reversed. The first aim of this chapter is to give such algebraic properties of Parry alternate bases. In particular, we show a necessary condition for an alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$ to be a Parry one is that the product $\beta = \prod_{i=0}^{p-1} \beta_i$ is an algebraic integer and all of the bases $\beta_0, \ldots, \beta_{p-1}$ belong to the algebraic field $\mathbb{Q}(\beta)$. On the other hand, we also give a sufficient condition: if β is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\beta)$, then $\boldsymbol{\beta}$ is a Parry alternate base. The importance of the class of Pisot bases in connection to automata was pointed out also by Frougny [Fro92] who showed that normalization in a real base $\beta > 1$ which maps any β-representation of a real number in $[0, 1)$ to its greedy β-expansion is computable by a finite Büchi automaton if β is a Pisot number. The second aim of this chapter is to provide an analogue of Frougny's result concerning greedy and lazy normalizations in alternate bases. We show that given an alternate base $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots)$ such that $\beta = \prod_{i=0}^{p-1} \beta_i$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\beta)$, the greedy and lazy normalization functions are computable by finite Büchi automata, and furthermore, we effectively construct such automata. An important tool in our proofs is the spectrum of numeration systems associated with alternate bases. Its definition shows that one needs to consider the spectrum of $\beta = \prod_{i=0}^{p-1} \beta_i$ with a more general alphabet of non-integer digits. Hence, we first study the spectrum in the general framework of a complex base δ such that $|\delta| > 1$ with an alphabet $A \subset \mathbb{C}$, which is defined as

$$X^A(\delta) = \{\sum_{i=0}^{n} a_i \delta^i : n \in \mathbb{N}, a_i \in A\}.$$

The notion of spectrum was originally introduced by Erdős, Joó and Komornik for a base $\delta \in (1, 2)$ and an alphabet of the form $A = \{0, 1, \ldots, m\}$ [EJK90]. Topological properties of the spectrum determine many of the arithmetical aspects of the numeration system; see [FP18]. One of the main problems in the study of spectra is to describe bases which give spectra without accumulation points in dependence on the alphabet. For the case of real bases and symmetric integer alphabets, a complete characterization was given by Akiyama and Komornik [AK13] and Feng [Fen16]. In this chapter, as an analogy to the results of [FP18], we prove that the set of representations of zero in a complex base δ such that $|\delta| > 1$ and an alphabet A of complex number is accepted by a finite Büchi automaton if and only if the spectrum $X^A(\delta)$ has no accumulation point. Next, we deduce an analogue in the alternate base case. This result makes use of a Büchi automaton called the zero automaton which generalizes that defined by Frougny [Fro92] and which is intimately linked with the Büchi automata computing the greedy and lazy normalization functions in alternate bases.

The fifth chapter is concerned with the dynamical properties of alternate base expansions. We know that, considering a real base $\beta > 1$, an interesting feature of greedy and lazy β-expansions is that they can be dynamically generated by iterating respectively the so-called greedy β-transformation

$$T_\beta \colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor,$$

and lazy β-transformation

$$L_\beta \colon (x_\beta - 1, x_\beta] \to (x_\beta - 1, x_\beta],\ x \mapsto \beta x - \lceil \beta x - x_\beta \rceil,$$

where $x_\beta = \frac{\lceil \beta \rceil - 1}{\beta - 1}$. Thus it is natural to wonder if, given an alternate base $\boldsymbol{\beta}$, one can find an alternate greedy transformation $T_{\boldsymbol{\beta}}$ and an alternate lazy transformation $L_{\boldsymbol{\beta}}$, iterations of which generate the greedy and lazy $\boldsymbol{\beta}$-expansions respectively. This will be the focus of this final chapter. Moreover, in the real base case, the dynamical properties of T_β and L_β are now well understood since the seminal works of Rényi and Parry; for example, see [DK02b]. Hence, the aim of this chapter is to describe the measure theoretical dynamical behaviors of such transformations $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$. We first prove the existence of a unique absolutely continuous $T_{\boldsymbol{\beta}}$-invariant measure (with respect to an extended Lebesgue measure, called the p-Lebesgue measure where p is the period of the alternate base $\boldsymbol{\beta}$). We then show that this unique measure is in fact equivalent to the p-Lebesgue measure and that the corresponding dynamical system is ergodic and has entropy $\frac{1}{p}\log(\beta)$ where $\beta = \prod_{i=0}^{p-1} \beta_i$. Using tools from ergodic theory, we are able to exhibit an explicit expression of the density function of this invariant measure and to compute the frequencies of letters in the greedy $\boldsymbol{\beta}$-expansions. Furthermore, we show that the dynamical system underlying the greedy $\boldsymbol{\beta}$-expansion is measure theoretically isomorphic to the dynamical system underlying the lazy $\boldsymbol{\beta}$-expansion as well as to the dynamical systems underlying natural greedy an lazy generalizations of the so-called β-shift. As a consequence, the four transformations have the same dynamical behavior. Another interesting property of alternate base expansions is that when every p-terms are written as one fraction, then one is able to rewrite the involved series in the form

$$x = \sum_{n \in \mathbb{N}} \frac{d_n}{\beta^{n+1}},$$

with $\beta = \prod_{i=0}^{p-1} \beta_i$ and d_n belonging to the fixed digit set of real numbers

$$\mathrm{Dig}(\boldsymbol{\beta}) = \Big\{ \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} : \forall i \in [\![0, p-1]\!],\ c_i \in \{0, 1, \ldots, \lceil \beta_i \rceil - 1\} \Big\}.$$

This algebraic operation transforms the considered alternate base expansion to a representation over a general digit set in base β. This is a particular case of Pedicini's extension of real base representations while considering general digit sets [Ped05]. We give a sufficient condition for the representations over $\mathrm{Dig}(\boldsymbol{\beta})$ obtained by grouping p by p the terms of the greedy and lazy $\boldsymbol{\beta}$-expansions to be respectively the greedy and lazy β-expansions over the digit set $\mathrm{Dig}(\boldsymbol{\beta})$. Next, by the greedy and lazy generalizations of Parry's theorem given in Chapter 2, not all p-tuples of letters can appear in the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers in $[0, 1)$ and $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ respectively. Hence, in the last section of the chapter, we construct two subsets of the digit set $\mathrm{Dig}(\boldsymbol{\beta})$ by using respectively the greedy and lazy admissible p-tuples. Then, we prove that the β-representations obtained by grouping p by p the terms of the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers in $[0, 1)$ and $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ are respectively the greedy and lazy ones over these particular digit sets.

This study will finish with several perspectives for future research continuing the work accomplished during this doctoral research.

As a final comment to this introduction, I would like to mention that, in order to create a coherent whole, this dissertation present the contents of four of my papers. However, during these four years of doctoral studies, I also considered other problems giving me the opportunity to write five more papers. The interested reader can find my list of publications in the next pages.

2010 Mathematics Subject Classification: 11A63 · 11K16 · 68Q45 · 37B10 · 37E05 · 37A45 · 28D05

Keywords: Expansions of real number · Cantor real base · Alternate base · Greedy algorithm · Lazy algorithm · Parry's theorem · Subshift · Sofic system · Bertrand-Mathis' theorem · Pisot number · Spectrum · Normalization · Büchi automaton · Measure theory · Ergodic theory · Dynamical system

LIST OF PUBLICATIONS

During these four years of doctoral studies, I had the opportunity to work on various research problems. Below, you can find an overall of the papers I accomplished during this time. To create a coherent whole, I chose to present in this work the contents of the papers $1, 2, 3$ and 9.

1. Émilie Charlier and Célia Cisternino. *Expansions in Cantor real bases*, *Monatsh. Math.*, **195**(4):585-610, 2021.

2. Émilie Charlier, Célia Cisternino, and Karma Dajani. *Dynamical behavios of expansions in alternate bases. Ergodic Theory and Dynamical Systems*, 1-34, 2021.

3. Émilie Charlier, Célia Cisternino, Zuzana Masáková, and Edita Pelantová. *Spectrum, algebraicity and normalization in alternate bases*. Submitted in 2022, arXiv:2202.03718.

4. Émilie Charlier, Célia Cisternino, and Adeline Massuir. *State complexity of the multiples of the Thue-Morse set*, *Tenth International Symposium on Games, Automata, Logics, and Formal Verification (GandALF 2019)*, volume 305 of *Electronic Proceedings in Theoretical Computer Science*, pages 34–49. Open Publishing Association, 2019.

5. Émilie Charlier, Célia Cisternino, and Adeline Massuir. *Minimal automaton for multiplying and translating the Thue-Morse set*, *Electronic Journal of Combinatorics*, **28**, Paper No. 3.12, 36 pp., 2021.

6. Émilie Charlier, Célia Cisternino, and Manon Stipulanti. *Robustness of Pisot-regular sequences*, *Adv. in Appl. Math.*, **125**, 102151, 2021.

7. Émilie Charlier, Célia Cisternino, and Manon Stipulanti, *Regular sequences and synchronized sequences in abstract numeration systems*, *European Journal of Combinatorics*, **101**, 103475, 2022.

8. Émilie Charlier, Célia Cisternino, and Manon Stipulanti. *A full characterization of Bertrand numeration systems*. Submitted in 2022, arXiv:2202.04938.

9. Célia Cisternino. *Combinatorial properties of lazy expansions in Cantor real bases*. Submitted in 2021, arXiv:2202.00437.

CHAPTER

1 PRELIMINARIES

This chapter gives the basic notions that are needed for the comprehension of this work.

First, we recall some usual notion of algebraic structures and we introduce the conventions and notation used in the subsequent chapters. Next, we recap the definitions from combinatorics on words and automata theory. Then, in order to study the representations of real numbers in real bases, namely the β-representations, some basics on measure and ergodic theory are recalled.

Forthwith, the β-representations are defined, studied combinatorics-wise with also an overview on the associated normalization function and then studied in terms of dynamics. The goal of this summary is to know which properties we will look at in the other chapters for Cantor real bases and alternate bases.

Notions from Sections 1.1 and 1.2 must be understood before starting reading this work. Then, the reader can consult these preliminaries when studying the other chapters of this book. More precisely, Chapters 2 and 3 need preliminaries from Section 1.4.1. Groundwork from Sections 1.4.2 and 1.4.3 is required for Chapter 4 and Sections 1.3, 1.4.4 and 1.4.5 are related to Chapter 5.

For further readings on the main discussed topics, we refer the interested reader to [BR10, FS10, Lot97, Lot02, Rig14] for more on combinatorics on words and [BG97, DK21, DK02a, Fur81] for more on ergodic theory.

Contents of the Chapter

1.1 Algebraic structures and related conventions

In this text, we let $\mathbb{N}$ be the set of non-negative integers and for any $m \in \mathbb{N}$ and any $\diamond \in \{<, \leq, >, \geq\}$, we let $\mathbb{N}_{\diamond m}$ denote the set $\{n \in \mathbb{N} : n \diamond m\}$. Moreover, for any integers i and j satisfying $i \leq j$, the interval of integers $\{i, i+1, \ldots, j\}$ is denoted $[\![i, j]\!]$. We make the convention that if $i > j$ then $[\![i, j]\!]$ is the empty set. Moreover, for all $i \in \mathbb{N}$, $[\![i, +\infty]\!]$ denotes the set of integers greater than or equal to i, that is, $[\![i, +\infty]\!] = \mathbb{N}_{\geq i}$.

Similarly, we let $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ be the sets of all integer, rational, real and complex numbers respectively. For any $\diamond \in \{<, \leq, >, \geq\}$, analogously to the set $\mathbb{N}_{\diamond m}$, we define $\mathbb{Z}_{\diamond m}$, $\mathbb{Q}_{\diamond m}$, $\mathbb{R}_{\diamond m}$ and $\mathbb{C}_{\diamond m}$ for $m \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ respectively.

Given a non-negative integer n and a positive integer p, n modulo p, denoted $n \bmod p$ is the remainder of the Euclidean division of n by p.

We let $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ and $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ denote the *ceiling function* and *floor function* respectively defined for all $x \in \mathbb{R}$ by $\lceil x \rceil = \inf\{z \in \mathbb{Z} : z \geq x\}$ and $\lfloor x \rfloor = \sup\{z \in \mathbb{Z} : z \leq x\}$. The *fractional function* $\{\cdot\} \colon \mathbb{R} \to [0, 1)$ is defined for all $x \in \mathbb{R}$ by $\{x\} = x - \lfloor x \rfloor$.

We make use of the common notions of *monoid*, *ring*, *field*, *subring* and *subfield*. With the classical addition and multiplication of numbers, the set $\mathbb{Z}$ is a ring and the sets $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are fields.

We briefly recall additional algebraic definitions needed for this work.

Definition 1.1.1. Let $\mathbb{K}$ be a commutative ring. The *ring of polynomials* with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}[x]$. A *monic polynomial* is a polynomial in $\mathbb{K}[x]$ whose leading coefficient is 1. For $n \in \mathbb{N}_{\geq 1}$, the *ring of polynomials in n indeterminates* with coefficients in $\mathbb{K}$ is denoted $\mathbb{K}[x_1, \ldots, x_n]$.

Definition 1.1.2. An *algebraic number* is a complex number that is a zero of a monic polynomial with coefficients in $\mathbb{Q}$. The *minimal polynomial* of an algebraic number β is the monic polynomial of minimal degree having coefficients in $\mathbb{Q}$ and annihilated by β. The minimal polynomial of an algebraic number β is irreducible over $\mathbb{Q}$ and its degree is the *degree* of the algebraic number β. Zeros of the same irreducible polynomial over $\mathbb{Q}$ are distinct and are said to be *Galois conjugates*. An algebraic number is an *algebraic integer* if it is a zero of a monic polynomial in $\mathbb{Z}[x]$.

It can be shown that the minimal polynomial of an algebraic integer also has integer coefficients.

Proposition 1.1.3. *The set of all algebraic integers is a subring of* $\mathbb{C}$.

Definition 1.1.4. The smallest subfield of the field $\mathbb{C}$ containing $\mathbb{Q}$ and a complex number β is denoted by $\mathbb{Q}(\beta)$.

Example 1.1.5. If β is an algebraic number of degree d then the field $\mathbb{Q}(\beta)$ is of the form

$$\mathbb{Q}(\beta) = \Big\{ \sum_{i=0}^{d-1} a_i \beta^i : a_i \in \mathbb{Q} \Big\}.$$

Definition 1.1.6. A *monoid morphism* is a function $f\colon \mathbb{K} \to \mathbb{K}'$ from a monoid $(\mathbb{K}, \cdot_{\mathbb{K}}, 1_{\mathbb{K}})$ into a monoid $(\mathbb{K}', \cdot_{\mathbb{K}'}, 1_{\mathbb{K}'})$ such that $f(1_{\mathbb{K}}) = 1_{\mathbb{K}'}$ and for all $k_1, k_2 \in \mathbb{K}$, $f(k_1 \cdot_{\mathbb{K}} k_2) = f(k_1) \cdot_{\mathbb{K}'} f(k_2)$. A *monoid isomorphism* is a bijective monoid morphism. A *ring morphism* is a function $f\colon \mathbb{K} \to \mathbb{K}'$ from a ring $(\mathbb{K}, \cdot_{\mathbb{K}}, +_{\mathbb{K}}, 0_{\mathbb{K}}, 1_{\mathbb{K}})$ to a ring $(\mathbb{K}', \cdot_{\mathbb{K}'}, +_{\mathbb{K}'}, 0_{\mathbb{K}'}, 1_{\mathbb{K}'})$ such that for all $k_1, k_2 \in \mathbb{K}$, $f(k_1 \cdot_{\mathbb{K}} k_2) = f(k_1) \cdot_{\mathbb{K}'} f(k_2)$ and $f(k_1 +_{\mathbb{K}} k_2) = f(k_1) +_{\mathbb{K}'} f(k_2)$. A *ring isomorphism* is a bijective ring morphism. In the following, when the context is clear, we simply talk about morphism and isomorphism.

Definition 1.1.7. Let β be an algebraic number of degree d and let $\beta_2, \ldots, \beta_d$ be its Galois conjugates (we set $\beta_1 = \beta$). Then for all $k \in [\![1, d]\!]$, the fields $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\beta_k)$ are isomorphic by the *isomorphism*

$$\psi_k\colon \mathbb{Q}(\beta) \to \mathbb{Q}(\beta_k),\ \sum_{n=0}^{d-1} a_n \beta^n \mapsto \sum_{n=0}^{d-1} a_n (\beta_k)^n.$$

Proposition 1.1.8. *Let β be an algebraic number of degree d, $\beta_2, \dots, \beta_d$ be its Galois conjugates (we set $\beta_1 = \beta$) and $\psi_1, \dots, \psi_d$ be the corresponding isomorphisms. For all $x \in \mathbb{Q}(\beta)$, we have $\prod_{k=1}^{d} \psi_k(x) \in \mathbb{Q}$. Moreover, whenever x is an algebraic integer in $\mathbb{Q}(\beta)$, then $\prod_{k=1}^{d} \psi_k(x)$ is an integer.*

Important classes of algebraic integers that we will deal with are Pisot and Perron numbers.

Definition 1.1.9. A *Pisot number* is an algebraic integer $\beta > 1$ whose Galois conjugates all have modulus less than 1. A *Perron number* is an algebraic integer $\beta > 1$ whose Galois conjugates all have modulus less than $|\beta|$.

Obviously, every Pisot number is a Perron number. Moreover, every integer is a Pisot number and a rational number which is not an integer is never an algebraic integer.

Example 1.1.10. Consider the real number $(1 + \sqrt{5})/2$. This real number will widely be used in the examples of this dissertation. It is called the *Golden ratio* and is denoted φ. The Golden ratio φ is a Pisot number since its minimal polynomial is $x^2 - x - 1$ and its Galois conjugate is the real number $(1 - \sqrt{5})/2$ of modulus less than 1.

Example 1.1.11. The smallest Pisot number is given by the positive zero of the polynomial $x^3 - x - 1$, that is $\beta \simeq 1.3247$.

Example 1.1.12. Consider the real number $\beta > 1$ satisfying $\beta^6 = \beta^5 + 1$, that is $\beta \simeq 1.2852$. This number is a Perron number but it is not a Pisot number since two of its Galois conjugates have modulus greater than 1.

1.2 Words, languages and automata

We now define backgrounds related with combinatorics on words that are needed for this dissertation.

Definition 1.2.1. An *alphabet* is a non-empty finite or infinite set, whose elements are called *letters*. A *finite* (resp., *infinite*) *word* over an alphabet A is a finite (resp., infinite) sequence of letters in A. The *empty word*, denoted by ε, is the empty sequence.

The *length* of a finite word w, denoted by $|w|$, is the number of letters contained in w. The length of an infinite word is set to $+\infty$.

If w is a non-empty (finite or infinite) word, then the letters of w are indexed from 0, that is, for any $n \in [\![0, |w|-1]\!]$, we let w_n denote its $(n+1)^{\text{st}}$ letter.

The set of finite (resp., non-empty finite, infinite) words over an alphabet A is denoted by A^* (resp., A^+, $A^{\mathbb{N}}$).

A *language* over an alphabet A is a subset of A^*. An *ω-languages* over an alphabet A is a subset of $A^{\mathbb{N}}$.

Example 1.2.2. Let $A = \{a, b, \cdots, x, y, z\}$ be the Latin (or Roman) alphabet. The finite word $w = \textit{numeration}$ has length $|w| = 10$.

An alphabet A composed of two letters is usually called a *binary alphabet*. In this text, while working with alphabets made of non-negative integers, we usually make no distinction between the symbols $0, 1, 2, \ldots$ and the integers they represent.

Example 1.2.3. Let $A = \{0, 1\}$ be the alphabet composed of the two letters 0 and 1. Consider the finite word $w = 0110$ over A. Its length is $|w| = 4$ and its second letter is $w_1 = 1$.

We introduce two useful operations on words which will be largely used in this text.

Definition 1.2.4. Let A be an alphabet. The *shift operator* on the infinite words over A, denoted σ_A, is defined by

$$\sigma_A \colon A^{\mathbb{N}} \to A^{\mathbb{N}},\ a_0 a_1 a_2 \cdots \mapsto a_1 a_2 a_3 \cdots .$$

Whenever there is no ambiguity on the alphabet, we drop the subscript and write σ.

Definition 1.2.5. Let A be an alphabet, u be a finite word and v be a finite or infinite word. The *concatenation* of u and v, denoted by uv, is the word w defined by $w_n = u_n$ for all $n \in [\![0, |u| - 1]\!]$ and $w_n = v_{n-|u|}$ for all $n \in [\![|u|, |u| + |v| - 1]\!]$. The concatenation of words is associative. In particular, the set A^* equipped with the concatenation restricted on $A^* \times A^*$ is a monoid with ε as neutral element.

For a finite word u over A and a non-negative integer n, we let u^n denote the concatenation of n copies of u, which is inductively defined by $u^0 = \varepsilon$

and $u^{n+1} = u^n u$ for all $n \in \mathbb{N}$.

We let u^ω define the infinite word made of the concatenation of infinitely many copies of u. An infinite word $w \in A^{\mathbb{N}}$ is said to be *ultimately periodic* if there exist finite words u and v over A with $v \neq \varepsilon$ such that $w = uv^\omega$. Moreover, the word w is called *(purely) periodic* if $u = \varepsilon$, that is $w = v^\omega$.

Example 1.2.6. Over the binary alphabet $A = \{0, 1\}$, the concatenation of the words 0110 and 1001 gives the word 01101001. We have $(01)^2 = 0101$ and $(01)^\omega = 010101\cdots$. The words $(01)^\omega$ and $0(01)^\omega$ are respectively periodic and ultimately periodic.

Definition 1.2.7. Let L and M be two languages over the alphabet A. The *concatenation of L and M* is the language $LM = \{uv : u \in L,\ v \in M\}$. For all $n \in \mathbb{N}$, we let L^n denote the concatenation of n copies of L defined by

$$L^n = \{u^{(1)} \cdots u^{(n)} : u^{(i)} \in L \text{ for all } i \in [\![1, n]\!]\}.$$

We let L^ω denote the ω-language made of the concatenation of infinitely many copies of words in L.

Remark 1.2.8. In this work, for all $n \in \mathbb{N}$, we sometimes summarize the concatenation of the n words $u^{(0)}, u^{(1)}, \ldots, u^{(n-1)}$ by $\prod_{k=0}^{n-1} u^{(k)}$. Moreover, considering alphabets $A_0, \ldots, A_{n-1}$, in order to avoid any confusion with the Cartesian product $\prod_{k=0}^{n-1} A_k$ containing n-tuples, we write $\bigotimes_{k=0}^{n-1} A_k$ for the set of words w of length n with $w_k \in A_k$ for all $k \in [\![0, n-1]\!]$. We extend the notation $\bigotimes$ for infinite words.

We now introduce the notions of factors, prefixes and suffixes of words.

Definition 1.2.9. Let w be a word over an alphabet A. A *factor* of w is a finite word u such that there exist i and j in $[\![0, |w| - 1]\!]$ satisfying $i \leq j$ and $u = w_i \cdots w_j$, in which case w is called the *factor of w starting at position i and ending at position j*. We let $\mathrm{Fac}(w)$ be the set of all factors of w.

Definition 1.2.10. A *prefix* of a word w is a factor starting at position 0. The *prefix of length n* of w with $n \leq |w|$, denoted $\mathrm{Pref}_n(w)$, is the factor $w_0 \cdots w_{n-1}$. We let $\mathrm{Pref}(w)$ be the set of all prefixes of w. A *suffix* of a finite word w is a factor ending at position $|w| - 1$. We let $\mathrm{Suff}(w)$ be the set of all suffixes of w. We extend the definition of suffixes to infinite words as follows: a suffix of an infinite word w is an infinite word $v \in A^{\mathbb{N}}$ such that there exists $u \in \mathrm{Pref}(w)$ satisfying $w = uv$.

Example 1.2.11. Consider the word $w = numeration$. The words *num*, *rat* and *on* are respectively a prefix, a factor and a suffix of w.

Notions of factor, prefix and suffix of words can be extended to languages and ω-language as follows.

Definition 1.2.12. Considering a language (resp., an ω-language) L over A, the set of finite factors of elements in L is denoted $\mathrm{Fac}(L)$. Moreover, let $\mathrm{Pref}(L)$ (resp., $\mathrm{Suff}(L)$) denote the set of the prefixes (resp., suffixes) of its words.

We now endow $A^{\mathbb{N}}$ with a distance. This gives rise to the concept of convergence of sequences of words.

Definition 1.2.13. Let $u, v \in A^{\mathbb{N}}$. We let $\Lambda(u, v)$ denote the *longest common prefix* of u and v. Note that $|\Lambda(u, v)|$ is the smallest index where the two words u and v differ, that is

$$|\Lambda(u, v)| = \inf\{i \in \mathbb{N} : u_i \neq v_i\}.$$

The *(prefix) distance between u and v* is defined by $2^{-|\Lambda(u,v)|}$ if $u \neq v$ and 0 otherwise.

A sequence $(u_n)_{n\in\mathbb{N}}$ of infinite words over the alphabet A *converges* to the infinite word $v \in A^{\mathbb{N}}$ if the distance between u_n and v tends to 0 whenever n tends to $+\infty$. We write $\lim_{n\to+\infty} u_n = v$.

Example 1.2.14. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of infinite words over $\{0, 1\}$ defined by $u_n = 0(01)^n0^\omega$. The distance between $u_1 = 0010^\omega$ and $u_2 = 001010^\omega$ equals 2^{-4}. Moreover, we have $\lim_{n\to+\infty} u_n = 0(01)^\omega$.

We are now able to define the well-known Thue-Morse word [Thu12, Mor21] as the limit of a sequence of finite words. Note that the Thue-Morse word has many other equivalent definitions. This one is chosen based on its use in the subsequent chapters.

Definition 1.2.15. Consider the monoid morphism $\overline{\cdot}\colon \{a, b\}^* \to \{a, b\}^*$ defined by $\overline{a} = b$ and $\overline{b} = a$. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of finite words over the binary alphabet $\{a, b\}$ defined as follows:

$$\begin{cases} u_0 = a \\ u_n = u_{n-1}\overline{u_{n-1}}, \quad \forall n \geq 1. \end{cases} \tag{1.1}$$

The *Thue-Morse word* over $\{a, b\}$ is the infinite word

$$\lim_{n \to +\infty} u_n = abbabaab \cdots .$$

If an alphabet A is endowed with a total order, the sets of words A^* and $A^{\mathbb{N}}$ can be ordered as follows[1].

Definition 1.2.16. Let $(A, <)$ be a totally ordered alphabet. If u and v are two finite words over A, the word u is *lexicographically less* than v, which is denoted $u <_{\text{lex}} v$, either if u is a strict prefix of v or if there exists $\ell \in [\![0, \min\{|u|, |v|\} - 1]\!]$ such that $u_n = v_n$ for all $n \in [\![0, \ell - 1]\!]$ and $u_\ell < v_\ell$. We write $u \leq_{\text{lex}} v$ if either $u <_{\text{lex}} v$ or $u = v$. The lexicographic order is extended to the set of infinite words over A as follows: if u and v are two infinite words over A, $u <_{\text{lex}} v$ if there exists $n \in \mathbb{N}_{\geq 1}$ such that $\text{Pref}_n(u) <_{\text{lex}} \text{Pref}_n(v)$.

Example 1.2.17. Over the binary alphabet, we have $(10)^\omega <_{\text{lex}} 110^\omega$. Over the Latin alphabet $A = \{a, b, \cdots, x, y, z\}$, the lexicographic order is the order used in the dictionary[2]. The word *numeration* comes before *system* in the lexicographical order (and so in the dictionary), that is we have $numeration <_{\text{lex}} system$.

Automata are in some way the simplest model of computation. In the remaining of this section, we recall the definitions and properties needed all along this work.

Definition 1.2.18. A *deterministic automaton* is a 5-tuple

$$\mathcal{A} = (Q, i, F, A, E)$$

where Q is a non-empty set, called the set of *states*, i is a distinguished element of Q, called the *initial state*, $F \subseteq Q$ is the set of *final states*, A is an alphabet and $E \colon Q \times A \to Q$ is the (partial) *transition function*.

A deterministic automaton is *finite* if its set of states is finite and the alphabet is finite.

A *path* in $\mathcal{A}$ is a sequence of states $q_0, \ldots, q_n$ with $n \in \mathbb{N}_{\geq 1}$ and a *label* $a_0 a_1 \cdots a_{n-1}$ such that for all $k \in [\![1, n]\!]$, we have $E(q_{k-1}, a_{k-1}) = q_k$. The path is *initial* (resp., *final*) if $q_0 = i$ (resp., $q_n \in F$). If a path is both initial

[1] An other order on A^* (not used in this work) is also widely studied in combinatorics on words, namely the *radix* or *genealogic order*.

[2] Actually, the lexicographic order is the order used in the dictionary if special symbols like accents and dashes are omitted.

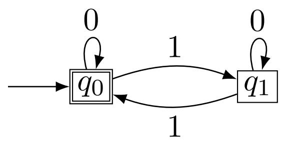

Figure 1.1: A deterministic finite automaton accepting the binary words having an even number of 1.

and final, it is called an *accepting path* in $\mathcal{A}$. A state q is *accessible* if there exists an initial path ending in q. Similarly, a state q is *co-accessible* if there exists a final path starting in q.

A finite word w over A is *accepted* by $\mathcal{A}$ if there exists in $\mathcal{A}$ an accepting path labeled by w. The set of words accepted by $\mathcal{A}$ is the *language accepted by* $\mathcal{A}$.

Deterministic automata can be represented by oriented labeled graphs as follows: nodes are states, the initial state is designated by an incoming arrow, the final states are designated by doubly-framed nodes and for all states p and q and all letters a such that $E(p, a) = q$, there exists an arrow from p to q labeled by a.

Example 1.2.19. Consider the deterministic finite automaton

$$\mathcal{A} = (\{q_0, q_1\}, q_0, \{q_0\}, \{0, 1\}, E)$$

where the transition function E is given by $E(q_0, 0) = q_0$, $E(q_0, 1) = q_1$, $E(q_1, 0) = q_1$ and $E(q_1, 1) = q_0$. The automaton $\mathcal{A}$ is depicted in Figure 1.1. This automaton accepts the set of binary words having an even number of 1.

A generalization of deterministic automata are the non-deterministic ones.

Definition 1.2.20. A *non-deterministic automaton* is a 5-tuple $\mathcal{A} = (Q, I, F, A, E)$ where Q, F and A are defined as in a deterministic automaton, $I \subseteq Q$ is a non-empty set, called the *set of initial states*, $E \subseteq Q \times A \times Q$ is a non-empty set, called the *transition relation*.

The differences between deterministic and non-deterministic automata are the following ones: in a non-deterministic automaton, there may exist several initial states and there may exist several transitions with the same label outgoing from a state.

The notions previously defined for deterministic automata (such as paths, accepted words, accepted language, representations by oriented graphs,...) can be generalized to non-deterministic automata.

Remark 1.2.21. Since the Cartesian product $A \times B$ of two alphabets A and B is still an alphabet, a deterministic automaton can be defined over the alphabet $A \times B$. In that case, a transition is labeled by a pair of letters $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in A \times B$. Such an automaton is called a 2-*tape automaton*.

Since a deterministic automaton is a particular non-deterministic automaton, one could think that there are more languages accepted by non-deterministic automata than are by deterministic automata, but the following proposition shows that this is actually not the case.

Proposition 1.2.22. *A language is accepted by a finite non-deterministic automaton if and only if it is accepted by a finite deterministic automaton.*

We now introduce the central notion of regular languages.

Definition 1.2.23. A language is *regular* if it is accepted by a (deterministic or non-deterministic) finite automaton.

Büchi [Büc60] in 1960, and Muller [Mul63] not much later in 1963, extended the notion of automata in order to accept sets of infinite words. Büchi automata are thoroughly studied in [PP04].

Definition 1.2.24. *Büchi automata* are defined as non-deterministic automata except for the acceptance criterion which has to be adapted in order to deal with infinite words: an infinite word is accepted if it labels a path going infinitely many times through final states. A Büchi automaton is *finite* if its set of states is finite and the alphabet is finite.

Note that the main difference between the theory of classical automata and that of Büchi automata is that an analogue of Proposition 1.2.22 does not hold.

Example 1.2.25. The Büchi automaton depicted in Figure 1.2 accepts the ω-language of infinite binary words over the alphabet $\{a, b\}$ contain-

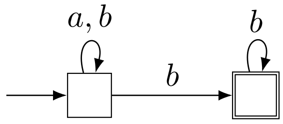

Figure 1.2: A Büchi automaton.

ing finitely many a's. No deterministic Büchi automaton accepts this ω-language.

Regular languages and ω-languages accepted by finite Büchi automata can be characterized in terms of an equivalence relation.

Definition 1.2.26. If L is a language or an ω-language over A and $u \in A^*$, we set

$$u^{-1}L = \{v \in \mathrm{Suff}(L) : uv \in L\}.$$

Two finite words $u, v \in A^*$ are *right congruent with respect to* L, written $u \sim_L v$, if $u^{-1}L = v^{-1}L$. Right congruent words $u, v \in A^*$ are also said *equivalent* for the equivalence relation $\sim_L$.

Theorem 1.2.27. *A language (resp., an ω-language) L is regular (resp., accepted by a finite Büchi automaton) if and only if the equivalence relation $\sim_L$ has only finitely many equivalence classes.*

We end this section by introducing the product of automata in order to state that the intersection of regular languages is a regular language, and similarly, that the intersection of ω-languages accepted by finite Büchi automata is an ω-language accepted by a finite Büchi automaton.

Definition 1.2.28. Consider two automata (resp., Büchi automata) $\mathcal{A}_1 = (Q_1, I_1, F_1, A_1, E_1)$ and $\mathcal{A}_2 = (Q_2, I_2, F_2, A_2, E_2)$. The *product* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is the automaton (resp., Büchi automaton) $\mathcal{A}_1 \times \mathcal{A}_2 = (Q, I, F, A, E)$ where $Q = Q_1 \times Q_2$, $I = I_1 \times I_2$, $F = F_1 \times F_2$, $A = A_1 \times A_2$ and the transition relation $E \subseteq Q \times A \times Q$ is defined by

$$\big((q_1, q_2), (a_1, a_2), (q_1', q_2')\big) \in E \iff (q_1, a_1, q_1') \in E_1 \text{ and } (q_2, a_2, q_2') \in E_2.$$

Proposition 1.2.29. *Let L_1 and L_2 denote respectively the languages (resp., ω-languages) accepted by the finite automata (resp., Büchi automata) $\mathcal{A}_1$*

and $\mathcal{A}_2$. The language (resp., ω-language) $L_1 \cap L_2$ is accepted by the finite automaton (resp., Büchi automaton) $\mathcal{A}_1 \times \mathcal{A}_2$.

1.3 Measure preserving dynamical systems

In this section, we introduce some basics on measure theory. In the next section, the real base expansions will be studied combinatorics-wise and then dynamics-wise. Hence, all the subsequent definitions will be illustrated in Section 1.4.4. Moreover, Chapter 5 is devoted to the study of the dynamical properties of one of the main object of this dissertation, namely the alternate bases.

Definition 1.3.1. Let X be a set. The set made of all subsets of X is denoted $\mathcal{P}(X)$. A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *σ-algebra* over X such that $X \in \mathcal{F}$ and that is closed under complementation and countable unions. The pair $(X, \mathcal{F})$ is called a *measurable space*. The members of $\mathcal{F}$ are called *measurable sets*.

Example 1.3.2. Let X be a set. The collections $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are σ-algebras over X.

Definition 1.3.3. Let X be a set and S be a collection of subsets of X. The smallest σ-algebra containing all sets of S is called the *σ-algebra generated by S* and is denoted $\sigma(S)$.

Example 1.3.4. Let X be a set and A be a subset of X, the collection $\{\emptyset, A, A^c, X\}$ is the σ-algebra generated by $\{A\}$.

An important σ-algebra for this work is the Borel σ-algebra.

Definition 1.3.5. A *topological space* is a set X together with a collection $\mathcal{C}$ of subsets of X such that $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$ and closed under countable unions and finite intersections. The elements of $\mathcal{C}$ are called *open sets* and the collection $\mathcal{C}$ is called a *topology* on X.

Definition 1.3.6. Let X be a topological space. The σ-algebra generated by all open sets is the *Borel σ-algebra over X* and is denoted $\mathcal{B}(X)$. An element $B \in \mathcal{B}(X)$ is called a *Borel set*.

In this text, we will mostly deal with the Euclidean topology on real numbers and Borel σ-algebras over intervals of real numbers.

We now define a measure over a set and an associated σ-algebra. Roughly, a measure on a set is a number intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume.

Definition 1.3.7. Let X be a set and $\mathcal{F}$ be a σ-algebra over X. A map $\mu\colon \mathcal{F} \to [0, +\infty]$ is a *measure over* $\mathcal{F}$ if $\mu(\emptyset) = 0$ and for any sequence $(B_k)_{k\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$, we have

$$\mu\left(\bigcup_{k\in\mathbb{N}} B_k\right) = \sum_{k\in\mathbb{N}} \mu(B_k).$$

If moreover $\mu(X) = 1$ then μ is called a *probability measure over* $\mathcal{F}$.

Definition 1.3.8. A *measure space* is a triplet $(X, \mathcal{F}, \mu)$ where X is a set, $\mathcal{F}$ is a σ-algebra over X and μ is a measure on $\mathcal{F}$. If moreover $\mu(X) = 1$ then the triplet $(X, \mathcal{F}, \mu)$ is called a *probability space*.

Definition 1.3.9. Let $(X, \mathcal{F}, \mu)$ be a measure space. A subset A of X is *μ-negligible* if there exists a set $B \in \mathcal{F}$ such that $A \subseteq B$ and $\mu(B) = 0$. The measure μ is called *complete* if every negligible set is an element of the σ-algebra $\mathcal{F}$. A property over X holds *μ-almost-everywhere*, shortened by *μ-a.e.*, if the set of elements for which the property does not hold is μ-negligible.

A particularly important example of measure is the *Lebesgue measure* on $\mathbb{R}^d$, which assigns the usual volume to subsets of $\mathbb{R}^d$. For instance, the Lebesgue measure of an interval $[a, b)$ of real numbers is its usual length $b-a$. In order to define this measure, we need to introduce outer measures.

Definition 1.3.10. Let X be a set. A map $\mu^*\colon \mathcal{P}(X) \to [0, +\infty) \cup \{+\infty\}$ is an *outer measure* over X if $\mu^*(\emptyset) = 0$, $\mu^*(A) \leq \mu^*(B)$ for all sets $A \subseteq B$ and

$$\mu^*\left(\bigcup_{k\in\mathbb{N}} A_k\right) \leq \sum_{k\in\mathbb{N}} \mu(A_k)$$

for all sequence $(A_k)_{k\in\mathbb{N}}$ of subsets of X.

Definition 1.3.11. Let X be a set and μ^* be an outer measure over X. A set B is *measurable for the outer measure* μ^* if for each subset A of X, we have

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

Proposition 1.3.12. *Let X be a set and μ^* be an outer measure over X. The set of measurable sets for the outer measure μ^* form a σ-algebra. Moreover, the restriction of the outer measure μ^* to the σ-algebra of measurable sets for μ^* defines a complete measure.*

We are now ready to construct the Lebesgue measure.

Definition 1.3.13. Let d be a positive integer. A d-dimensional interval of $\mathbb{R}^d$ is a set of the form $I = \prod_{i=1}^{d} [a_i, b_i]$ where $a_i \leq b_i$ for all $i \in [\![1, d]\!]$ and its *volume* is defined by

$$\mathrm{Vol}(I) = \prod_{i=1}^{d} (b_i - a_i).$$

Let A be a subset of $\mathbb{R}^d$ and let $\mathcal{C}_A$ be the collection of all sequences $(A_k)_{k \in \mathbb{N}}$ of d-dimensional intervals such that $A \subseteq \bigcup_{k \in \mathbb{N}} A_k$. The *Lebesgue outer measure* λ^* is defined by

$$\lambda^* \colon \mathcal{P}(\mathbb{R}^d) \to [0, +\infty],\ A \mapsto \inf \Big\{ \sum_{k \in \mathbb{N}} \mathrm{Vol}(A_k) \colon (A_k)_{k \in \mathbb{N}} \in \mathcal{C}_A \Big\}.$$

The restriction of the outer Lebesgue measure λ^* to the σ-algebra of measurable sets for λ^* is called the *Lebesgue measure* and is denoted λ.

In Chapter 5, the Lebesgue measure will play a considerable role. As said in the following result, every Borel set is λ^*-measurable.

Proposition 1.3.14. *The Lebesgue measure over $\mathbb{R}^d$ is a complete measure defined over the Borel sets in $\mathcal{B}(\mathbb{R}^d)$.*

Two measures over the same measurable space can be compared.

Definition 1.3.15. Let μ and ν be two measures over the same measurable space $(X, \mathcal{F})$. The measure μ is *absolutely continuous with respect to* ν if for all $B \in \mathcal{F}$, $\nu(B) = 0$ implies $\mu(B) = 0$. The measures μ and ν are *equivalent* if they are absolutely continuous with respect to each other. In particular, a measure on $\mathcal{B}(X)$ with $X \subseteq \mathbb{R}$ is *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure λ restricted to $\mathcal{B}(X)$. On the contrary, the measures μ and ν are *mutually singular* if there exist two sets $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, $A \cup B = X$ and $\mu(A) = 0 = \nu(B)$.

Let us now define measurable and integrable maps.

Definition 1.3.16. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be measurable spaces. A map $T\colon X \to Y$ is *measurable* if for all $B \in \mathcal{F}_Y$, then

$$T^{-1}(B) = \{x \in X : T(x) \in B\}$$

belongs to $\mathcal{F}_X$.

Definition 1.3.17. Let $(X, \mathcal{F}, \mu)$ be a measure space. A *simple non-negative function* is a function of the form $\sum_{j=1}^{n} a_j \chi_{B_j}$ where $n \in \mathbb{N}$, $a_1, \ldots, a_n \geq 0$, $B_1, \ldots, B_n \in \mathcal{F}$ and $\chi_{B_1}, \ldots, \chi_{B_n}$ are the characteristic functions of the sets $B_1, \ldots, B_n$ respectively. The set of simple non-negative functions is denoted $S^+(X, \mathcal{F})$. The *integral* of $f = \sum_{j=1}^{n} a_j \chi_{B_j} \in S^+(X, \mathcal{F})$ with respect to μ, denoted $\int f d\mu$, is defined by

$$\int f \, d\mu = \sum_{j=1}^{n} a_j \mu(B_j).$$

The *integral* of a measurable map $f\colon X \to [0, +\infty]$ with respect to μ is defined by

$$\int f \, d\mu = \sup\left\{ \int g \, d\mu : g \in S^+(X, \mathcal{F}) \quad \text{and} \quad g \leq f \right\}.$$

Let $f\colon X \to [-\infty, +\infty]$ be a measurable map. If the *positive part of* f defined by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

and the *negative part of* f defined by

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0, \\ 0 & \text{otherwise} \end{cases}$$

are such that $\int f^+ \, d\mu < +\infty$ and $\int f^- \, d\mu < +\infty$, then f is called μ*-integrable* (or simply *integrable*) and its *integral* with respect to μ is defined by

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

Let $f\colon X \to [-\infty, +\infty]$ be a measurable map and $B \in \mathcal{F}$. Then f is said *integrable over* B if $f\chi_B$ is integrable and in this case, the *integral of* f *over* B is denoted $\int_B f d\mu$ and is defined by

$$\int_B f \, d\mu = \int f \chi_B \, d\mu.$$

Theorem 1.3.18 (Radon-Nikodym)**.** *Let μ and ν be two probability measures over the same measurable space $(X, \mathcal{F})$ such that μ is absolutely continuous with respect to ν. Then there exists a ν-integrable map $f \colon X \mapsto [0, +\infty)$ such that for all $B \in \mathcal{F}$, $\mu(B) = \int_B f \, d\nu$. Moreover, the map f is ν-a.e. unique.*

Definition 1.3.19. The unique map f designated by Radon-Nikodym's theorem is called the *density function* of the measure μ with respect to ν and is usually denoted $\frac{d\mu}{d\nu}$.

Let us define some properties of measurable maps.

Definition 1.3.20. For a measurable space $(X, \mathcal{F})$, a measurable transformation $T \colon X \to X$ and a measure μ on $\mathcal{F}$, the map T is *non-singular with respect to* μ if for all $B \in \mathcal{F}$, $\mu(B) = 0$ if and only if $\mu(T^{-1}(B)) = 0$.

We can now define dynamical systems in terms of a stronger characterization of measurable maps than being non-singular and then study their properties.

Definition 1.3.21. For a measurable space $(X, \mathcal{F})$, a measurable transformation $T \colon X \to X$ and a measure μ on $\mathcal{F}$, the measure μ is *T-invariant*, or equivalently, the transformation $T \colon X \to X$ is *measure preserving with respect to* μ, if for all $B \in \mathcal{F}$, we have $\mu(T^{-1}(B)) = \mu(B)$.

Definition 1.3.22. A *(measure preserving) dynamical system* is a quadruple $(X, \mathcal{F}, \mu, T)$ where $(X, \mathcal{F}, \mu)$ is a probability space and $T \colon X \to X$ is a measure-preserving transformation with respect to μ.

Theorem 1.3.23 (Poincaré's Recurrence Theorem)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system and B be a set in $\mathcal{F}$. If $\mu(B) > 0$ then for μ-almost every point $x \in B$, there exists $k \geq 1$ such that $T^k(x) \in B$.*

Remark 1.3.24. Throughout the text, for a subset A of X, the notation $\mathcal{F} \cap A$ where $\mathcal{F}$ is a σ-algebra designates the σ-algebra $\{B \cap A : B \in \mathcal{F}\}$ over A.

Definition 1.3.25. Two dynamical systems $(X, \mathcal{F}_X, \mu_X, T_X)$ and $(Y, \mathcal{F}_Y, \mu_Y, T_Y)$ are *(measure preservingly) isomorphic* if there exist $M \in \mathcal{F}_X$ and

$N \in \mathcal{F}_Y$ with

$$\mu_X(M) = \mu_Y(N) = 0$$

and

$$T_X(X \setminus M) \subseteq X \setminus M \quad \text{and} \quad T_Y(Y \setminus N) \subseteq Y \setminus N,$$

and if there exists a bijective map

$$\psi\colon X \setminus M \to Y \setminus N$$

which is bimeasurable with respect to the σ-algebras $\mathcal{F}_X \cap (X \setminus M)$ and $\mathcal{F}_Y \cap (Y \setminus N)$ and such that for all $B \in \mathcal{F}_Y \cap (Y \setminus N)$,

$$\mu_Y(B) = \mu_X(\psi^{-1}(B)), \tag{1.2}$$

and finally, such that for all $x \in X \setminus M$,

$$\psi(T_X(x)) = T_Y(\psi(x)).$$

When (1.2) is satisfied for all $B \in \mathcal{F}_Y$, we write $\mu_Y = \mu_X \circ \psi^{-1}$. Figures 1.3 and 1.4 symbolically depict a measure preservingly isomorphism.

We now introduce the notion of ergodicity.

Definition 1.3.26. A dynamical system $(X, \mathcal{F}, \mu, T)$ is *ergodic* if for all $B \in \mathcal{F}$, $T^{-1}(B) = B$ implies $\mu(B) \in \{0, 1\}$.

Roughly speaking we call a dynamical system $(X, \mathcal{F}, \mu, T)$ ergodic if it is impossible to divide X into two pieces A and B (each with positive probability) such that T acts on each piece separately. A non-ergodic map is symbolically depicted in Figure 1.5.

Theorem 1.3.27 (The ergodic theorem)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. For any μ-integrable map $f\colon X \to \mathbb{R}$, the limit*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = f^*(x)$$

exists μ-a.e. and we have $f^* \circ T = f^*$ μ-a.e. and $\int_X f\, d\mu = \int_X f^*\, d\mu$. If moreover the dynamical system $(X, \mathcal{F}, \mu, T)$ is ergodic, then f^* is a constant μ-a.e. and $f^* = \int_X f\, d\mu$.

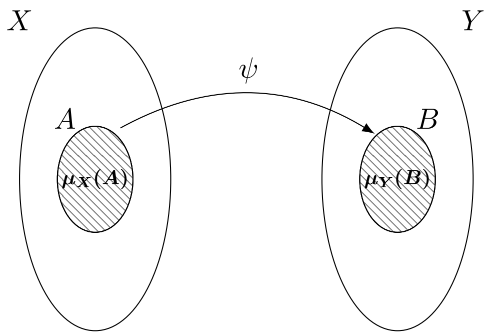

Figure 1.3: The isomorphism from $(X, \mathcal{F}_X, \mu_X, T_X)$ to $(Y, \mathcal{F}_Y, \mu_Y, T_Y)$ gives $\mu_X(A) = \mu_Y(B)$ where $B \in \mathcal{F}_Y$ and $A = \psi^{-1}(B)$.

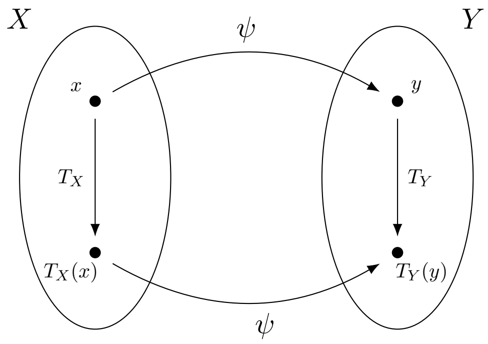

Figure 1.4: A commutative diagram given by an isomorphism from $(X, \mathcal{F}_X, \mu_X, T_X)$ to $(Y, \mathcal{F}_Y, \mu_Y, T_Y)$.

Theorem 1.3.28. *Suppose that μ_1, μ_2 are probability measures on $(X, \mathcal{F})$, and $T\colon X \to X$ is a measure-preserving ergodic transformation with respect to both μ_1 and μ_2. Then either $\mu_1 = \mu_2$ or μ_1 and μ_2 are mutually singular with respect to each other.*

Definition 1.3.29. A dynamical system $(X, \mathcal{F}, \mu, T)$ is *exact* if $\bigcap_{n\in\mathbb{N}}\{T^{-n}(B) : B \in \mathcal{F}\}$ only contains sets of measure 0 or 1.

Clearly, any exact dynamical system is ergodic. Moreover, if a dynamical system $(X, \mathcal{F}, \mu, T)$ is exact, then for all $n \in \mathbb{N}_{\geq 1}$, the dynamical system $(X, \mathcal{F}, \mu, T^n)$ is ergodic.

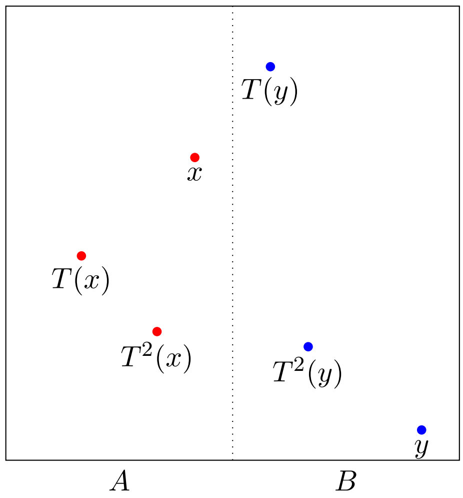

Figure 1.5: A non-ergodic map.

The subsequent definition introduces a notion, called the measure theoretical entropy of a transformation, reflecting the average amount of information gained by a transformation in a dynamical system. Randomness of information in a system was first studied by Shannon in 1948 [Sha48].

Definition 1.3.30. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. Let I be a finite or countable index set. A *partition for* X is a collection $P = \{P_i : i \in I\}$ of measurable sets of X such that $\mu(P_i) > 0$ for all $i \in I$, $\mu(P_i \cap P_j) = 0$ for all $i \neq j$ and $\mu(\bigcup_{i \in I} P_i) = \mu(X)$. When I is finite, the *entropy of the partition* P is given by

$$H(P) = -\sum_{i \in I} \mu(P_i) \log(\mu(P_i)).$$

Given such a partition $P = \{P_i : i \in I\}$ of X, for all $n \in \mathbb{N}$, the partition defined by

$$\{P_{i_0} \cap T^{-1} P_{i_1} \cap \cdots \cap T^{-(n-1)} P_{i_{n-1}} : i_0, \dots i_{n-1} \in I\}$$

is denoted $\bigvee_{i=0}^{n-1} T^{-i} P$. The *entropy of the transformation* T *with respect to* μ *and the partition* P is given by

$$h_\mu(P, T) = \lim_{n \to +\infty} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} T^{-i} P \right).$$

Finally, the *(measure theoretic) entropy* of the transformation T is given by

$$h_\mu(T) = \sup_P h_\mu(P, T)$$

where the supremum is taken over all finite partitions P of X.

Proposition 1.3.31. *Entropy is an isomorphism invariant.*

To end this section, we define the induced systems and give a result about their entropy.

Definition 1.3.32. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system and B be a set of $\mathcal{F}$ such that $\mu(B) > 0$. For $x \in B$, define the *first return time of x to B*, denoted $r(x)$, by

$$r(x) = \inf\{n \geq 1 : T^n(x) \in B\}.$$

Consider the σ-algebra $\mathcal{F} \cap B$ on B and define the measure μ_B and the *induced transformation* T_B by

$$\mu_B(A) = \frac{\mu(A)}{\mu(B)}, \quad \text{for } A \in \mathcal{F} \cap B,$$

and

$$T_B \colon B \to B,\ x \mapsto T^{r(x)}(x), \quad \text{for } x \in B$$

respectively. Then $(B, \mathcal{F} \cap B, \mu_B, T_B)$ is a dynamical system, which is called the *dynamical system induced by B*.

Note that in the previous definition, we have that $r(x)$ is finite μ-a.e. on B by Poincaré's Recurrence Theorem 1.3.23. The induced dynamical system inherits many nice properties of the original system. For example T_B is measure preserving with respect to μ_B. If the original system is ergodic, then the induced system is also ergodic. The converse holds true if $\mu(\bigcup_{n\in\mathbb{N}} T^{-n}(B)) = 1$. Moreover, a famous result of Abramov [Abr59] relates the entropy of the original system with the entropy of the induced system.

Theorem 1.3.33 (Abramov's formula)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system, B be a set of $\mathcal{F}$ such that $\mu(B) > 0$ and $(B, \mathcal{F} \cap B, \mu_B, T_B)$ the corresponding induced system. We have*

$$h_\mu(T) = \mu(B) h_{\mu_B}(T_B).$$

1.4 An overview of β-representations

Generalizing integer base representations, and more particularly the decimal and binary ones, the β-expansions are introduced in the next section and then studied all along the remaining part of this chapter.

1.4.1 Combinatorics of β-representations

Representations of real numbers in real bases were introduced by Rényi in 1957 [Rén57] and well understood since the pioneering work of Parry in 1960 [Par60].

Definition 1.4.1. A *real base* is a real number β greater than 1. We define the *value map* $\mathrm{val}_\beta \colon (\mathbb{R}_{\geq 0})^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{val}_\beta(a) = \sum_{i\in\mathbb{N}} \frac{a_i}{\beta^{i+1}}$$

for any infinite sequence a over $\mathbb{R}_{\geq 0}$, provided that the series converges. A *β-representation* of a non-negative real number x is an infinite sequence a over $\mathbb{N}$ such that $\mathrm{val}_\beta(a) = x$.

There may exist more than one β-representation of the same real number. Between all of them, one plays a crucial role, called the greedy one.

Definition 1.4.2. For $x \in [0,1]$, define a β-representation of x thanks to the *greedy algorithm*: set $r_{-1} = x$ and let, for all $n \in \mathbb{N}$,

$$a_n = \lfloor \beta\, r_{n-1} \rfloor \quad \text{and} \quad r_n = \{\beta\, r_{n-1}\}.$$

The obtained β-representation is called the *greedy β-expansion of x*, or simply the *β-expansion of x*, and is denoted $d_\beta(x)$. For all $x \in [0,1]$, the β-expansion of x is an infinite word over the alphabet $[\![0, \lfloor\beta\rfloor]\!]$. This algorithm is called greedy since at each step it takes the largest possible digit. Indeed, if the first N digits of the β-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest integer such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} \leq x.$$

Example 1.4.3. We have $d_3(1) = 30^\omega$, $d_\varphi(1) = 110^\omega$, $d_\varphi(3-\sqrt{5}) = 10010^\omega$ and $d_{\varphi^2}(1) = 21^\omega$.

In the following, the combinatorial properties of the β-expansions are recalled. We refer the reader to [Lot02, Chapter 7] for a survey. From now on, let $\beta > 1$ be a base.

Proposition 1.4.4. *For each infinite sequence a of non-negative integers and all $x \in [0,1]$, we have $a = d_\beta(x)$ if and only if $\mathrm{val}_\beta(a) = x$ and for all $\ell \in \mathbb{N}$,*

$$\sum_{i=\ell+1}^{+\infty} \frac{a_i}{\beta^{i+1}} < \frac{1}{\beta^{\ell+1}}.$$

Proposition 1.4.5. *The β-expansion of a real number $x \in [0,1]$ is the greatest of all β-representations of x with respect to the lexicographic order.*

Proposition 1.4.6. *The function $d_\beta\colon [0,1] \to [\![0, \lfloor\beta\rfloor]\!]^{\mathbb{N}}, x \mapsto d_\beta(x)$ is increasing: for all $x, y \in [0,1]$, $x < y \iff d_\beta(x) <_{\mathrm{lex}} d_\beta(y)$.*

Proposition 1.4.7. *Let α and β be two real numbers greater than 1. Then $\alpha < \beta$ if and only if $d_\alpha(1) < d_\beta(1)$.*

The β-expansion of 1 plays a special role in the theory of β-expansions.

Proposition 1.4.8. *The β-expansion of 1 is never purely periodic.*

A β-representation is said to be *finite* if it ends with infinitely many zeros and *infinite* otherwise. If a β-representation is finite, we usually omit to write the tail of zeros. When $d_\beta(1)$ is finite, we modify it in order to have an infinite β-representation of 1 that is lexicographically maximal among all infinite β-representations of 1. As it will be seen later on, this new β-representation of 1 reveals its importance.

Definition 1.4.9. Let $d^*_\beta(1)$ denote the *quasi-greedy β-expansion of* 1 defined as follows:

$$d^*_\beta(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite} \\ (a_0 \cdots a_{\ell-2}(a_{\ell-1}-1))^\omega & \text{if } d_\beta(1) = a_0 \cdots a_{\ell-1} \text{ with } \ell \in \mathbb{N}_{\geq 1} \\ & \quad\quad \text{and } a_{\ell-1} \neq 0. \end{cases}$$

Example 1.4.10. Resuming Example 1.4.3, we get $d^*_3(1) = 2^\omega$, $d^*_\varphi(1) = (10)^\omega$ and $d^*_{\varphi^2}(1) = 21^\omega$.

Definition 1.4.11. A real number $\beta > 1$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic (equivalently, if $d^*_\beta(1)$ is ultimately periodic). Further, if $d_\beta(1)$ is finite, then we say that β is a *simple Parry number*.

Remark 1.4.12. An algebraic description of Parry numbers is not obvious. However, some links with Perron and Pisot number are known: any Pisot number is a Parry number [Ber77, Sch80], any Parry number is a Perron number (see for example [Fab95]) and neither of the statements can be reversed. Moreover, every quadratic Parry number is a Pisot number [Bas02]. More detailed information on Galois conjugates of a Parry number β was given by Solomyak [Sol94].

The following example illustrates the existence of Parry but non-Pisot numbers.

Example 1.4.13. The polynomial $x^4 - 3x^3 - 2x^2 - 3$ has the two real zeros $\beta \simeq 3.6164$ and $\gamma \simeq -1.0968$ and two complex zeros with modulus less than 1. Hence, the real number β is a non-Pisot number. However, it is easily checked that $d_\beta(1) = 32030^\omega$. So β is a Parry number.

Definition 1.4.14. The set D_β is the set of β-expansions of real numbers in the interval $[0,1)$ and the set S_β is the topological closure of D_β with respect to the prefix distance of infinite words:

$$D_\beta = \{d_\beta(x) : x \in [0,1)\} \quad \text{and} \quad S_\beta = \overline{D_\beta}.$$

In 1960, Parry [Par60] characterized those infinite words over $\mathbb{N}$ that belong to D_β thanks to the quasi-greedy β-expansion of 1. Such infinite words are sometimes called *β-admissible sequences*. The advantage of this characterization is that it is a purely combinatorial criterion on the sequences.

Theorem 1.4.15 (Parry's theorem, [Par60])**.** *Let a be an infinite sequence of non-negative integers. Then $a \in D_\beta$ if and only if $\sigma^n(a) <_{\mathrm{lex}} d^*_\beta(1)$ for all $n \in \mathbb{N}$.*

Corollary 1.4.16. *Let a be an infinite sequence of non-negative integers. Then $a \in S_\beta$ if and only if $\sigma^n(a) \leq_{\mathrm{lex}} d^*_\beta(1)$ for all $n \in \mathbb{N}$.*

Corollary 1.4.17. *Let a be an infinite sequence of non-negative integers such that $a_0 \geq 1$, $a_n \leq a_0$ for all $n \geq 1$ and $a \neq 10^\omega$. Then there exists a* *unique real number $\beta > 1$ such that $\sum_{i\in\mathbb{N}} \frac{a_i}{\beta^{i+1}} = 1$. Furthermore, $a = d_\beta(1)$ if and only if $\sigma^n(a) < a$ for all $n \in \mathbb{N}_{\geq 1}$.*

We now turn to the β-shift. To do so, let us recall the needed definitions.

Definition 1.4.18. Let A be an alphabet. A subset of $A^{\mathbb{N}}$ is a *subshift* of $A^{\mathbb{N}}$ if it is shift-invariant and closed with respect to the topology induced by the prefix distance. Let $S \subseteq A^{\mathbb{N}}$ be a subshift, $I(S) = A^+ \setminus \mathrm{Fac}(S)$ be the set of *factors avoided by* S and $X(S)$ be the set of words of $I(S)$ which have no proper factors in $I(S)$. A subshift $S \subseteq A^{\mathbb{N}}$ is *sofic* if $X(S)$ is regular, or equivalently if the language $\mathrm{Fac}(S)$ is regular. A subshift $S \subseteq A^{\mathbb{N}}$ is of *finite type* if $X(S)$ is finite.

In view of Corollary 1.4.16, the subset S_β of $[\![0, \lfloor\beta\rfloor]\!]^{\mathbb{N}}$ is a subshift, which we call the *β-shift*. The properties of D_β and S_β are recalled in the following.

Proposition 1.4.19. *Let $d^*_\beta(1) = t_0t_1t_2\cdots$. We have $D_\beta = Y^\omega$ where*

$$Y = \{t_0 \cdots t_{n-1}a : n \in \mathbb{N}, a \in [\![0, t_n - 1]\!]\}.$$

Theorem 1.4.20 ([IT74])**.** *The β-shift S_β is of finite type if and only if β is a simple Parry number.*

Theorem 1.4.21 (Bertrand-Mathis' theorem, [BM86])**.** *The β-shift S_β is sofic if and only if β is a Parry number.*

Let us describe the automaton, given in the proof of Bertrand-Mathis' theorem, accepting $\mathrm{Fac}(S_\beta)$ when β is a Parry number.

Definition 1.4.22. Suppose that $d^*_\beta(1)$ is ultimately periodic and denote

$$d^*_\beta(1) = t_0 \cdots t_{m-1}\big(t_m \cdots t_{m+n-1}\big)^\omega.$$

Let $\mathcal{A}_\beta$ be the deterministic finite automaton defined as follows and depicted in Figure 1.6. The set of states is $Q = \big\{q_i : i \in [\![0, m+n-1]\!]\big\}$. The initial state is q_0 and all states are final. The alphabet is $[\![0, \lfloor\beta\rfloor]\!]$ and the (partial) transition function $E \colon Q \times [\![0, \lfloor\beta\rfloor]\!] \to Q$ of the automaton $\mathcal{A}_\beta$ is defined as follows. For each $i \in [\![0, m+n-1]\!]$, we have

$$E(q_i, t_i) = \begin{cases} q_{i+1} & \text{if } i \neq m+n-1 \\ q_m & \text{else} \end{cases}$$

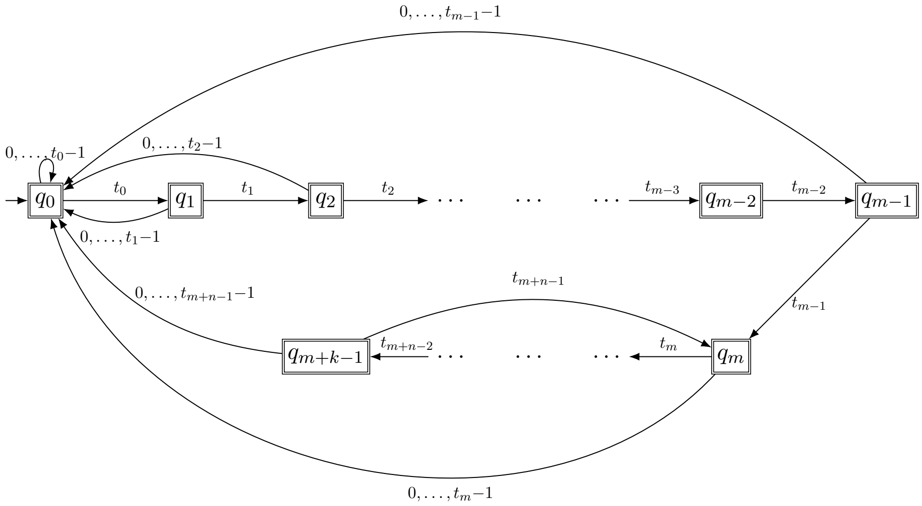

Figure 1.6: The automaton $\mathcal{A}_\beta$ for $d_\beta^*(1) = t_0 \cdots t_{m-1}\big(t_m \cdots t_{m+n-1}\big)^\omega$.

Figure 1.7: The automata $\mathcal{A}_\varphi$ (left) and $\mathcal{A}_{\varphi^2}$ (right).

and for all $s \in [\![0, t_i - 1]\!]$, we have $E(q_i, s) = q_0$.

Example 1.4.23. From Examples 1.4.3 and 1.4.10, we already know that the Golden ratio φ is a simple Parry number and its square φ^2 is a non-simple Parry number. The automata $\mathcal{A}_\varphi$ and $\mathcal{A}_{\varphi^2}$ are depicted in Figure 1.7.

Remark 1.4.24. For any Parry number β, the automaton $\mathcal{A}_\beta$ can be seen as a Büchi automaton accepting infinite words. In that case, it is easy to see that the corresponding accepted ω-language is exactly the β-shift S_β. Moreover, we can modify this Büchi automaton in order to accept D_β. Suppose that the Büchi automaton $\mathcal{A}_\beta$ had been constructed by considering the quasi-greedy β-expansion of 1 as non-purely periodic, that is, if $d_\beta^*(1)$ is purely periodic of period length ℓ it suffices to consider the first letter as the preperiod and the

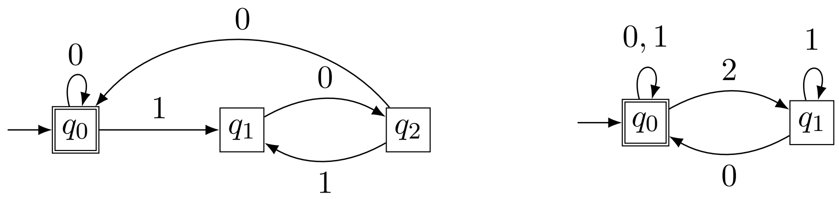

Figure 1.8: Büchi automata accepting D_φ (left) and D_{φ^2} (right).

next ℓ ones as the period. By taking only the initial state q_0 as unique final state, we obtain a Büchi automaton accepting D_β (see [BR10, Proposition 2.3.4]).

Example 1.4.25. The automata $\mathcal{A}_\varphi$ and $\mathcal{A}_{\varphi^2}$ from Figure 1.7 can be seen as Büchi automata accepting S_φ and S_{φ^2} respectively. Moreover, Büchi automata accepting D_φ and D_{φ^2} are depicted in Figure 1.8.

1.4.2 Spectrum and set of β-representations of 0

Considering a real base $\beta > 1$ and an alphabet of integers, one could ask if the set of infinite words having value 0 in base β can be the ω-language accepted by a finite Büchi automaton. This question reveals its importance, especially when the question of normalization (introduced in the next section) arises. This set of infinite words is intimately linked with a discrete set of real numbers called spectrum. Spectra were introduced by Erdős, Joo and Komornik in 1990 [EJK90] and have been gaining interest in recent years [AK13, Fen16, FP18, HMV18, Váv21].

In this context, we work with alphabets of integer digits. Hence, we extend the definition of β-representations of non-negative real numbers (see Definition 1.4.1) to the set of sequences over $\mathbb{Z}$. That is, we allow negative integer digits.

Definition 1.4.26. For a real number $\beta > 1$ and $d \in \mathbb{N}$, we let $Z(\beta, d)$ denote the *set of β-representations of zero over the alphabet* $[\![-d, d]\!]$:

$$Z(\beta, d) = \{a \in [\![-d, d]\!]^{\mathbb{N}} : \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{n+1}} = 0\}.$$

The *d-spectrum of β* is the set

$$X^d(\beta) = \{\sum_{n=0}^{\ell-1} a_n \beta^{\ell-1-n} : \ell \in \mathbb{N},\ a_0, a_1, \dots, a_{\ell-1} \in [\![-d, d]\!]\}.$$

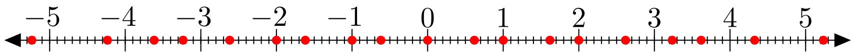

Figure 1.9: The 1-spectrum of φ.

Example 1.4.27. The 1-spectrum of φ in the neighbourhood of 0, namely $X^1(\varphi) \cap [-5.5, 5.5]$, is depicted in Figure 1.9.

Definition 1.4.28. Let X be a topological space and $Y \subseteq X$. A point $x \in X$ is an *accumulation point* of the set Y if every open neighborhood of x contains at least one point from Y distinct from x.

The following theorem linking these two sets was proved in [FP18].

Theorem 1.4.29. *Let $\beta > 1$ and $d \in \mathbb{N}$. Then $Z(\beta, d)$ is accepted by a finite Büchi automaton if and only if the spectrum $X^d(\beta)$ has no accumulation point in $\mathbb{R}$.*

The next result is due to Akiyama and Komornik [AK13] and Feng [Fen16].

Theorem 1.4.30. *Let $\beta > 1$ and $d \in \mathbb{N}$. The spectrum $X^d(\beta)$ has an accumulation point in $\mathbb{R}$ if and only if $\beta - 1 < d$ and β is not a Pisot number.*

The following result is a direct consequence of Theorems 1.4.29 and 1.4.30, as noticed in [FP18].

Theorem 1.4.31. *Let $\beta > 1$. The following assertions are equivalent.*

1. *The set $Z(\beta, d)$ is accepted by a finite Büchi automaton for all $d \geq 0$.*
2. *The set $Z(\beta, d)$ is accepted by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$.*
3. *β is a Pisot number.*

1.4.3 Normalization in real bases

A question on β-expansions that has raised a lot of interest all along the years is to characterize the real bases for which the normalization is computable by a finite 2-tape Büchi automaton.

Definition 1.4.32. Let A be an arbitrary alphabet of integer digits. The *normalization* $\nu_{\beta,A}$ in base β over the alphabet A is the partial function which maps any β-representation over A of a real number $x \in [0,1)$ onto $d_\beta(x)$.

This function $\nu_{\beta,A}$ is partial since, depending on the alphabet A, a word over A can have a value not in $[0,1)$.

Definition 1.4.33. Let A be an arbitrary alphabet of integer digits. A 2-tape Büchi automaton accepting the set

$$\{(u,v) \in (A \times [\![0, \lceil\beta\rceil - 1]\!])^{\mathbb{N}} : \mathrm{val}_\beta(u) \in [0,1),\ v = \nu_{\beta,A}(u)\}$$

is called a *normalizer*.

A normalizer can be constructed thanks to a Büchi automaton accepting the set of β-representations of 0 over the smallest symmetric alphabet containing A. We now describe the construction of a well-known Büchi automaton accepting $Z(\beta,d)$, called the zero automaton, and recall the related results.

Definition 1.4.34. For any positive integer d, we define the *zero automaton in base β over the alphabet $[\![-d,d]\!]$* by $\mathcal{Z}(\beta,d) = (Q_d, 0, Q_d, [\![-d,d]\!], E)$ where

$$Q_d = X^d(\beta) \cap [-\tfrac{d}{\beta-1}, \tfrac{d}{\beta-1}]$$

and for all $s,t \in X^d(\beta)$ and all $a \in \mathbb{Z}$, there is a transition

$$s \xrightarrow{a} t \quad \text{if and only if} \quad t = \beta s + a. \tag{1.3}$$

Proposition 1.4.35. *The zero automaton $\mathcal{Z}(\beta,d)$ accepts the set $Z(\beta,d)$.*

Example 1.4.36. The zero automaton in base φ over the alphabet $[\![-1,1]\!]$ is depicted in Figure 1.10. For example, the infinite words $1(\overline{1}0)^\omega$ and $\overline{1}01^\omega$ (where $\overline{1}$ designates the digit -1) are accepted by the Büchi automaton $\mathcal{Z}(\varphi,1)$. Therefore, the infinite words $1(\overline{1}0)^\omega$ and $\overline{1}01^\omega$ have value 0 in base φ.

The zero automaton $\mathcal{Z}(\beta,d)$ is the key element to build a normalizer. Hence, in order to understand when there exists a finite normalizer, we state the following result which is an improvement of the result from [FS10] about the finiteness of $\mathcal{Z}(\beta,d)$.

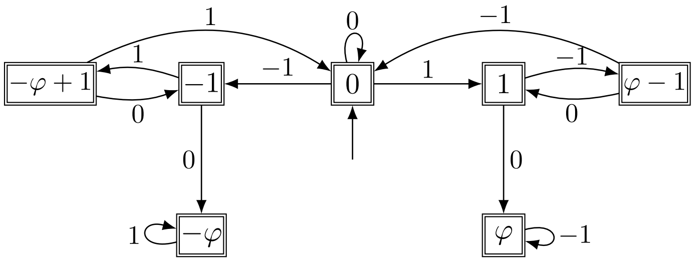

Figure 1.10: The zero automaton $\mathcal{Z}(\varphi, 1)$.

Theorem 1.4.37. *The following conditions are equivalent.*

(i) The zero automaton $\mathcal{Z}(\beta, d)$ is finite for every $d \geq 0$.

(ii) The zero automaton $\mathcal{Z}(\beta, d)$ is finite for one $d \geq \lceil\beta\rceil - 1$.

(iii) β is a Pisot number.

In order to get this revised result (compared to the one in [FS10]) it is sufficient to prove the following result.

Proposition 1.4.38. *The zero automaton $\mathcal{Z}(\beta, \lceil\beta\rceil - 1)$ is finite if and only if β is Pisot.*

Proof. The condition is sufficient by [FS10]. The condition is necessary since if the zero automaton $\mathcal{Z}(\beta, \lceil\beta\rceil - 1)$ is finite then the set $Z(\beta, \lceil\beta\rceil - 1)$ is accepted by a finite Büchi automaton and we get by Theorem 1.4.31 that β is a Pisot number. ∎

The second step is the construction of a converter.

Definition 1.4.39. Consider two finite alphabets of integers A and C and let $d = \max_{a\in A, c\in C} |a - c|$. We define the *converter* of β from A to C by

$$\mathcal{C}(\beta, A \times C) = (Q_d, 0, Q_d, A \times C, E')$$

where the transitions E' are defined as follows. Let $s, t \in Q_d$ and $a \in A$, $c \in C$, we define

$$s \xrightarrow[\mathcal{C}(\beta, A\times C)]{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} t \quad \text{if and only if} \quad s \xrightarrow[\mathcal{Z}(\beta, d)]{a-c} t.$$

Proposition 1.4.40. *The converter $\mathcal{C}(\beta, A \times C)$ accepts the set*

$$\{(u,v) \in (A \times C)^{\mathbb{N}} : \mathrm{val}_\beta(u) = \mathrm{val}_\beta(v)\}.$$

Example 1.4.41. Using the zero automaton in base φ over the alphabet $[\![-1,1]\!]$ depicted in Figure 1.10, we obtain the converter $\mathcal{C}(\beta, \{0,1\}^2)$ depicted in Figure 1.11.

By Theorem 1.4.37, if β is a Pisot number then the converter $\mathcal{C}(\beta, A \times C)$ is finite for all finite alphabets A and C. Moreover, since every Pisot number is Parry, by Remark 1.4.24, there exists a finite Büchi automaton accepting

$$A^{\mathbb{N}} \times D_\beta = \{(u,v) \in (A \times A_\beta)^{\mathbb{N}} : \exists x \in [0,1),\ v = d_\beta(x)\}.$$

Then, by computing the product of $\mathcal{C}(\beta, A \times A_\beta)$ and this finite Büchi automaton we obtain a finite normalizer.

In particular, we get the following result.

Theorem 1.4.42. *If β is a Pisot number then, for any finite alphabet A of integers, the normalization in base $\beta > 1$ over the alphabet A is computable by a finite 2-tape Büchi automaton.*

Example 1.4.43. We continue Examples 1.4.25, 1.4.36 and 1.4.41. By computing the product of the converter $\mathcal{C}(\varphi, \{0,1\}^2)$ and the Büchi automaton accepting $\{0,1\}^{\mathbb{N}} \times D_\varphi$ (obtained by modifying the Büchi automaton accepting D_φ), we obtain the normalizer in base φ depicted in Figure 1.12 (where only the accessible and co-accessible states are drawn). For example, the pair of words $\left[\begin{smallmatrix} 001^\omega \\ 10^\omega \end{smallmatrix}\right]$ is accepted by the normalizer depicted in Figure 1.12. Therefore, we get $\nu_{\varphi,\{0,1\}}(001^\omega) = 10^\omega$.

1.4.4 Dynamics of β-expansions

Real base expansions have also been studied through a dynamical point of view. This section is devoted to the study of their associated dynamical systems. We refer the reader to [DK21] for more details.

Definition 1.4.44. The *greedy β-transformation*, denoted T_β, is defined by

$$T_\beta \colon [0,1) \to [0,1),\ x \mapsto \beta x - \lfloor \beta x \rfloor .$$

The greedy β-expansion of a real number $x \in [0,1)$ can be obtained by setting $d_\beta(x) = a_0 a_1 a_2 \cdots$ with $a_n = \lfloor \beta T_\beta^n(x) \rfloor$ for all $n \in \mathbb{N}$.

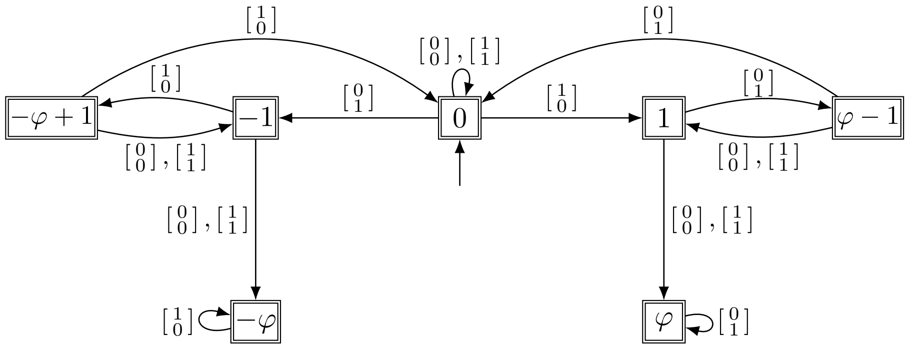

Figure 1.11: The converter $\mathcal{C}(\varphi, \{0,1\}^2)$.

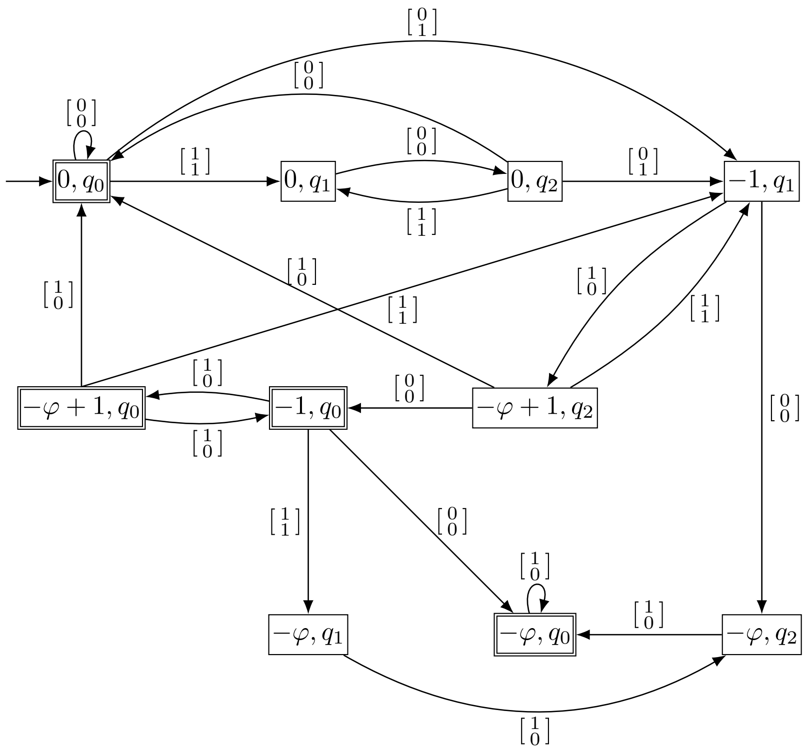

Figure 1.12: The normalizer in base φ over the alphabet $\{0, 1\}$.

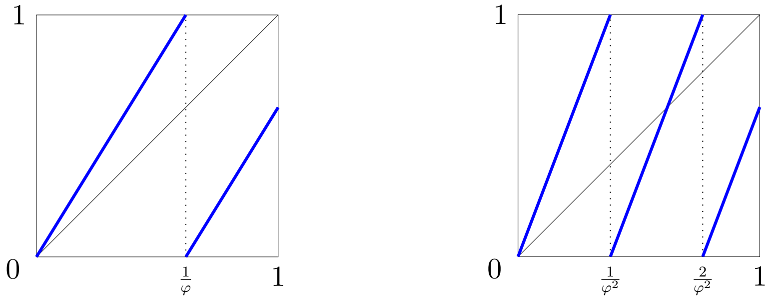

Figure 1.13: The transformations T_φ (left) and T_{φ^2} (right).

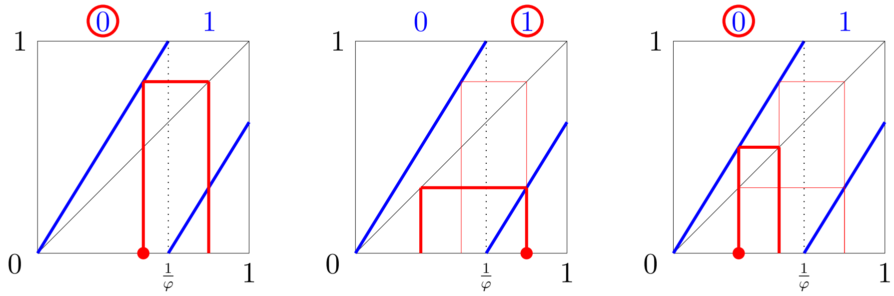

Figure 1.14: Three iterations of the map T_φ on the real number $\frac{1}{2}$.

As illustrated in the following example, β-transformations are usually represented by unit squares depicting the associated maps T_β. In addition, the diagonal is commonly represented in order to easily iterate the map. Each branch of the β-transformation corresponds to a digit in $[\![0, \lceil\beta\rceil - 1]\!]$, that is, for all $n \in \mathbb{N}$ and $x \in [0, 1)$, if $T_\beta^n(x)$ belongs to the preimage of the $(i+1)^{\text{st}}$ branch of the map with $i \in [\![0, \lceil\beta\rceil - 1]\!]$, then the $(n+1)^{\text{st}}$ digit of $d_\beta(x)$ is i.

Example 1.4.45. The transformations T_φ and T_{φ^2} are depicted in Figure 1.13. Moreover, one can see in Figure 1.14 three iterations of the map T_φ on the real number $\frac{1}{2}$. Since the third step gives the value $\frac{1}{2}$ again, we get $d_\varphi(\frac{1}{2}) = (010)^\omega$.

A fundamental dynamical result is the following. This summarizes results from [Par60, Rén57, Roh61].

Theorem 1.4.46. *There exists a unique T_β-invariant absolutely continuous probability measure μ_β on $\mathcal{B}([0, 1))$. Furthermore, the measure μ_β is* *equivalent to the Lebesgue measure on* $\mathcal{B}([0,1))$ *and the dynamical system* $([0,1),\mathcal{B}([0,1)),\mu_\beta,T_\beta)$ *is ergodic and has entropy* $\log(\beta)$.

Remark 1.4.47. It follows from Theorem 1.4.46 that T_β is *non-singular with respect to the Lebesgue measure.*

Rényi [Rén57] proved the existence of the measure $\mu_{\boldsymbol{\beta}}$ from Theorem 1.4.46 and Gel'fond [Gel59] and Parry [Par60] independently gave the following explicit formula for the density function of this measure.

Theorem 1.4.48. *The density function of the unique* T_β*-invariant absolutely continuous probability measure* μ_β *on* $\mathcal{B}([0,1))$ *is given by*

$$\frac{d\mu_\beta}{d\lambda}\colon [0,1)\to[0,1),\ x\mapsto \frac{1}{C}\sum_{n\in\mathbb{N}}\frac{1}{\beta^n}\chi_{[0,T_\beta^n(1))}(x)$$

where $C=\int_0^1\sum_{x<T_\beta^n(1)}\frac{1}{\beta^n}d\lambda$ *is a normalization constant.*

One can give a link between the combinatorial and dynamical properties of the greedy β-expansion. In fact, there exists an isomorphism between the dynamical system associated with the β-transformation and the β-shift S_β. In order to give this result, let us define a σ-algebra over infinite words, which will then be restricted to S_β (see Remark 1.3.24).

Definition 1.4.49. For an alphabet A, we let $\mathcal{C}_A$ denote the σ-algebra generated by the *cylinders* of the form

$$\mathcal{C}_A(a_0,\ldots,a_{\ell-1})=\{w\in A^{\mathbb{N}}: w_0=a_0,\ldots,w_{\ell-1}=a_{\ell-1}\}$$

with $\ell\in\mathbb{N}$ and $a_0,\ldots,a_{\ell-1}\in A$.

Theorem 1.4.50. *The map* $\psi_\beta\colon[0,1)\to S_\beta,\ x\mapsto d_\beta(x)$ *defines an isomorphism between the dynamical systems* $([0,1),\mathcal{B}([0,1)),\mu_\beta,T_\beta)$ *and* $(S_\beta,\mathcal{C}_{[\![0,\lceil\beta\rceil-1]\!]}\cap S_\beta,\mu_\beta\circ\psi_\beta^{-1},\sigma_{|S_\beta})$.

The β-transformation can be extended to a bigger interval than $[0,1)$.

Definition 1.4.51. Let

$$x_\beta=\frac{\lceil\beta\rceil-1}{\beta-1}$$

be the greatest real number that has a β-representation over the alphabet $[\![0,\lceil\beta\rceil-1]\!]$.

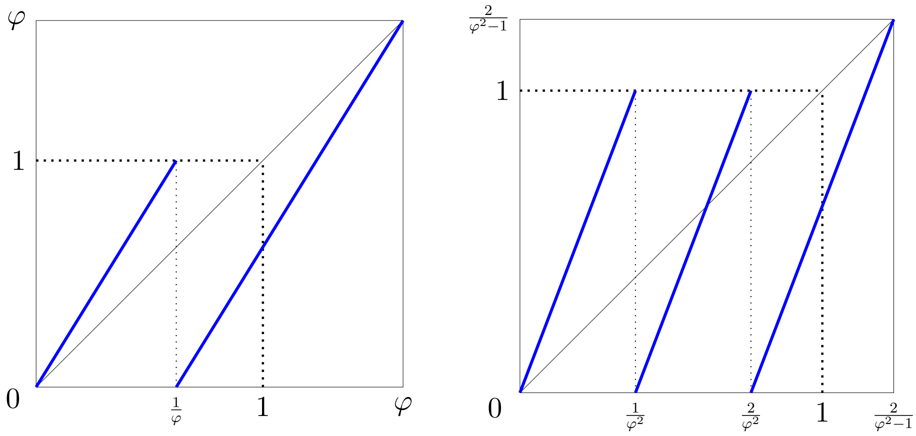

Figure 1.15: The extended transformations T_φ^{ext} (left) and $T_{\varphi^2}^{\mathrm{ext}}$ (right).

Clearly, we have $x_\beta \geq 1$ and $x_\beta = 1$ if and only if $\beta \in \mathbb{N}_{\geq 2}$. In [DK02b], the map T_β was extended to the interval $[0, x_\beta)$.

Definition 1.4.52. The *extended greedy β-transformation*, denoted T_β^{ext}, is defined by

$$T_\beta^{\mathrm{ext}} \colon [0, x_\beta) \to [0, x_\beta), \;\; x \mapsto \begin{cases} \beta x - \lfloor \beta x \rfloor & \text{if } x \in [0,1) \\ \beta x - (\lceil \beta \rceil - 1) & \text{if } x \in [1, x_\beta). \end{cases}$$

Example 1.4.53. We continue Example 1.4.45. The extended greedy transformations T_φ^{ext} and $T_{\varphi^2}^{\mathrm{ext}}$ are depicted in Figure 1.15.

Let us make some remarks.

Remark 1.4.54. For all $x \in \left[\frac{\lceil \beta \rceil - 1}{\beta}, \frac{\lceil \beta \rceil}{\beta}\right)$, the two cases of Definition 1.4.52 coincide since $\lfloor \beta x \rfloor = \lceil \beta \rceil - 1$. The extended β-transformation restricted to the interval $[0,1)$ gives back the classical greedy β-transformation from Definition 1.4.44. Moreover, for all $x \in [0, x_\beta)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $(T_\beta^{\mathrm{ext}})^n(x) \in [0,1)$.

Remark 1.4.55. It is important to note that if β is an integer, then the greedy β-expansion of 1 given in Section 1.4.1 is $\beta 0^\omega$ whereas the greedy β-expansion of 1 given thanks to the extended greedy β-transformation is $(\beta - 1)^\omega$ (corresponding to the quasi-greedy of 1 in base β in Section 1.4.1). Both definitions have their advantages in their area (combinatorics on words and dynamics) and a choice had been made in each theory.

Extending the measure μ_β on the Borel σ-algebra $\mathcal{B}([0,x_\beta))$ by

$$\mu_\beta^{\text{ext}}(B) = \mu_\beta(B \cap [0,1))$$

for all $B \in \mathcal{B}([0,x_\beta))$, we get the following extension of Theorem 1.4.46.

Theorem 1.4.56. *The extended measure $\mu_{\boldsymbol{\beta}}^{\text{ext}}$ is the unique T_β^{ext}-invariant probability measure absolutely continuous with respect to the Lebesgue measure on $\mathcal{B}([0,x_\beta))$. Furthermore, the measure μ_β^{ext} is equivalent to the Lebesgue measure on $\mathcal{B}([0,x_\beta))$ and the dynamical system $([0,x_\beta),\mathcal{B}([0,x_\beta)),\mu_\beta^{\text{ext}},T_\beta^{\text{ext}})$ is ergodic and has entropy $\log(\beta)$.*

In the greedy algorithm, one selects the largest digit among $0,1,\ldots,\lceil\beta\rceil-1$ at each step. Let us define the other extreme algorithm which chooses the least digit at each step [EJK90].

Definition 1.4.57. For $x \in (x_\beta - 1, x_\beta]$, define a β-representation of x thanks to the *lazy algorithm*: if the first N digits of the expansion of a real number $x \in (x_\beta - 1, x_\beta]$ are given by $a_0,\ldots,a_{N-1}$, then the next digit a_N is the least element in $[\![0,\lceil\beta\rceil-1]\!]$ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{+\infty} \frac{\lceil\beta\rceil-1}{\beta^{n+1}} \geq x,$$

or equivalently,

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \frac{x_\beta}{\beta^{N+1}} \geq x.$$

The so-obtained β-representation is called the *lazy β-expansion* of x and is denoted $\ell_\beta(x)$.

Dajani and Kraaikamp [DK02a] proved in 2002 that, as in the greedy case, the lazy β-expansion can be dynamically generated by a transformation.

Definition 1.4.58. The *lazy β-transformation*, denoted L_β, is defined by

$$L_\beta\colon (x_\beta-1,x_\beta] \to (x_\beta-1,x_\beta],\ x \mapsto \beta x - \lceil \beta x - x_\beta \rceil.$$

For all $x \in (x_\beta - 1, x_\beta]$, the lazy β-expansion of x can be obtained by setting $\ell_\beta(x) = a_0a_1a_2\cdots$ with $a_n = \lceil \beta L_\beta^n(x) - x_\beta \rceil$ for all $n \in \mathbb{N}$.

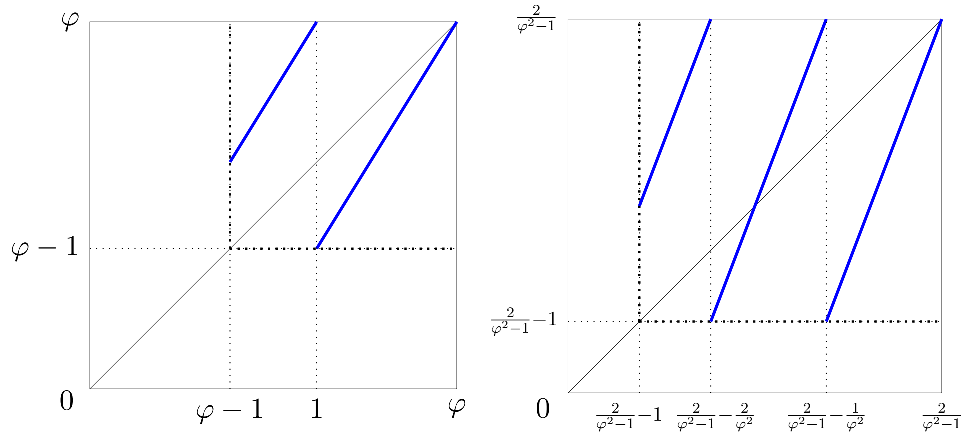

Figure 1.16: The transformations L_φ (left) and L_{φ^2} (right).

Example 1.4.59. The lazy transformations L_φ and L_{φ^2} are depicted in Figure 1.16.

Dajani and Kraaikamp [DK02b] proved that there is an isomorphism between the greedy and the lazy β-transformations.

Theorem 1.4.60. *The map $\phi_\beta\colon [0,1) \to (x_\beta - 1, x_\beta],\ x \mapsto x_\beta - x$ defines an isomorphism between the dynamical systems*

$$([0,1), \mathcal{B}([0,1)), \mu_\beta, T_\beta)$$

and

$$((x_\beta - 1, x_\beta], \mathcal{B}((x_\beta - 1, x_\beta]), \mu_\beta \circ \phi_\beta^{-1}, L_\beta).$$

As a direct consequence of this property, an analogue of Theorem 1.4.46 is obtained for the lazy transformation on $(x_\beta - 1, x_\beta]$.

Theorem 1.4.61. *The measure $\mu_{\boldsymbol{\beta}} \circ \phi_\beta^{-1}$ is the unique L_β-invariant probability measure absolutely continuous with respect to the Lebesgue measure on $\mathcal{B}((x_\beta - 1, x_\beta])$. Furthermore, the measure $\mu_\beta \circ \phi_\beta^{-1}$ is equivalent to the Lebesgue measure on $\mathcal{B}((x_\beta - 1, x_\beta])$ and the dynamical system $((x_\beta - 1, x_\beta], \mathcal{B}((x_\beta - 1, x_\beta]), \mu_\beta \circ \phi_\beta^{-1}, L_\beta)$ is ergodic and has entropy* $\log(\beta)$.

As in the greedy case, the lazy β-transformation L_β can be extended to the bigger interval $(0, x_\beta]$ as follows.

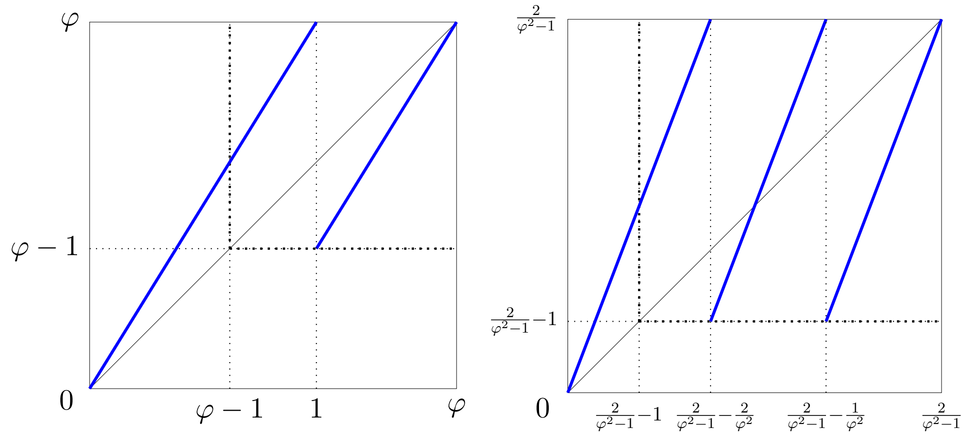

Figure 1.17: The extended transformations L_{φ}^{ext} (left) and $L_{\varphi^2}^{\text{ext}}$ (right).

Definition 1.4.62. The *extended lazy β-transformation*, denoted L_β^{ext}, is defined by

$$L_\beta^{\text{ext}}\colon (0, x_\beta] \to (0, x_\beta],\ \ x \mapsto \begin{cases} \beta x & \text{if } x \in (0, x_\beta - 1] \\ \beta x - \lceil \beta x - x_\beta \rceil & \text{if } x \in (x_\beta - 1, x_\beta]. \end{cases}$$

Remark 1.4.63. Observe that for all $x \in \left(\frac{x_\beta - 1}{\beta}, \frac{x_\beta}{\beta}\right]$, the two cases of the definition coincide since $\lceil \beta x - x_\beta \rceil = 0$. Moreover, since $L_\beta^{\text{ext}}\big((x_\beta - 1, x_\beta]\big) = (x_\beta - 1, x_\beta]$, the lazy transformation L_β^{ext} can be restricted to the length-one interval $(x_\beta - 1, x_\beta]$. This restriction gives back the lazy β-transformation L_β. Also note that for all $x \in (0, x_\beta]$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $(L_\beta^{\text{ext}})^n(x) \in (x_\beta - 1, x_\beta]$.

Example 1.4.64. The extended lazy transformations L_φ^{ext} and $L_{\varphi^2}^{\text{ext}}$ are depicted in Figure 1.17.

Theorem 1.4.65. *The map $\phi_\beta^{\text{ext}}\colon [0, x_\beta) \to (0, x_\beta],\ x \mapsto x_\beta - x$ defines an isomorphism between the dynamical systems*

$$([0, x_\beta), \mathcal{B}([0, x_\beta)), \mu_\beta^{\text{ext}}, T_\beta^{\text{ext}})$$

and

$$((0, x_\beta], \mathcal{B}((0, x_\beta]), \mu_\beta^{\text{ext}} \circ (\phi_\beta^{\text{ext}})^{-1}, L_\beta^{\text{ext}}).$$

Theorem 1.4.65 can be interpreted in Figures 1.15 and 1.17 as the rotation symmetry of 180 degrees.

As a direct consequence of Theorem 1.4.65, an analogue of Theorem 1.4.56 is obtained for the lazy transformation on $(0, x_\beta]$.

1.4.5 β-Representations over a general digit set

A generalization of β-representations is obtained by considering infinite words over arbitrary alphabets instead of the alphabet $[\![0, \lceil\beta\rceil - 1]\!]$. This generalization was originally defined by Pedicini in 2005 [Ped05].

Definition 1.4.66. Consider an arbitrary finite set $\Delta = \{d_0, d_1, \ldots, d_m\} \subset \mathbb{R}$ where $d_0 < d_1 < \cdots < d_m$. Then a (β, Δ)*-representation* of a real number x in the interval $[\frac{d_0}{\beta-1}, \frac{d_m}{\beta-1})$ is an infinite sequence $a_0a_1a_2\cdots$ over Δ such that $x = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{n+1}}$. Such a set Δ is called an *allowable digit set for* β if

$$\max_{k\in[\![0,m-1]\!]}(d_{k+1} - d_k) \leq \frac{d_m - d_0}{\beta - 1}. \tag{1.4}$$

In this case, every point in $[\frac{d_0}{\beta-1}, \frac{d_m}{\beta-1}]$ has a (β, Δ)-representation.

Considering an allowable digit set Δ for β, the greedy and lazy (β, Δ)-representation can be defined. Let us start with the greedy one.

Definition 1.4.67. Let Δ be an allowable digit set for β. The *greedy* (β, Δ)*-expansion* of a real number $x \in [\frac{d_0}{\beta-1}, \frac{d_m}{\beta-1})$ is defined recursively as follows: if the first N digits of the greedy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the greatest element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{+\infty} \frac{d_0}{\beta^{n+1}} \leq x.$$

From a dynamical point of view, let us define the transformation associated with these expansions [DK07].

Definition 1.4.68. Let Δ be an allowable digit set for β. The *greedy* (β, Δ)*-transformation*, denoted $T_{\beta,\Delta}$, is defined by

$$\begin{aligned} T_{\beta,\Delta}\colon [\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}) &\to [\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}), \\ x &\mapsto \begin{cases} \beta x - d_k & \text{if } x \in [\frac{d_0}{\beta-1} + \frac{d_k-d_0}{\beta}, \frac{d_0}{\beta-1} + \frac{d_{k+1}-d_0}{\beta}),\ k \in [\![0, m-1]\!], \\ \beta x - d_m & \text{if } x \in [\frac{d_0}{\beta-1} + \frac{d_m-d_0}{\beta}, \frac{d_m}{\beta-1}). \end{cases} \end{aligned}$$

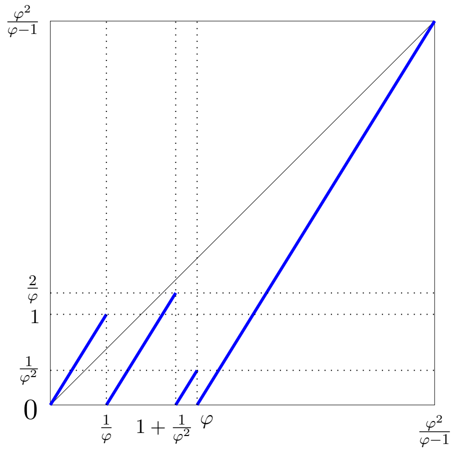

Figure 1.18: The transformation $T_{\varphi,\Delta}$ for $\Delta = \{0, 1, \frac{\varphi+1}{\varphi}, \varphi^2\}$.

The greedy (β, Δ)-expansion can also be obtained by iterating the greedy (β, Δ)-transformation as follows: for all $n \in \mathbb{N}$, a_n is the greatest digit d in Δ such that $\frac{d}{\beta} + \sum_{k=1}^{+\infty} \frac{d_0}{\beta^{k+1}} \leq T^n_{\beta,\Delta}(x)$.

Example 1.4.69. Consider the digit set $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$. It is easily checked that Δ is an allowable digit set for φ. The greedy (φ, Δ)-transformation

$$T_{\varphi,\Delta} \colon [0, \tfrac{\varphi^2}{\varphi-1}) \to [0, \tfrac{\varphi^2}{\varphi-1}),\ x \mapsto \begin{cases} \varphi x & \text{if } x \in [0, \frac{1}{\varphi}) \\ \varphi x - 1 & \text{if } x \in [\frac{1}{\varphi}, 1 + \frac{1}{\varphi^2}) \\ \varphi x - (\varphi + \frac{1}{\varphi}) & \text{if } x \in [1 + \frac{1}{\varphi^2}, \varphi) \\ \varphi x - \varphi^2 & \text{if } x \in [\varphi, \frac{\varphi^2}{\varphi-1}) \end{cases}$$

is depicted in Figure 1.18.

Similarly, if Δ is an allowable digit set for β, then the other extreme (β, Δ)-representation can be defined.

Definition 1.4.70. Let Δ be an allowable digit set for β. The *lazy* (β, Δ)-*expansion* of a real number $x \in (\frac{d_0}{\beta-1}, \frac{d_m}{\beta-1}]$ is defined recursively as follows: if the first N digits of the lazy (β, Δ)-expansion of x are given by $a_0, \ldots, a_{N-1}$, then the next digit a_N is the least element in Δ such that

$$\sum_{n=0}^{N} \frac{a_n}{\beta^{n+1}} + \sum_{n=N+1}^{+\infty} \frac{d_m}{\beta^{n+1}} \geq x.$$

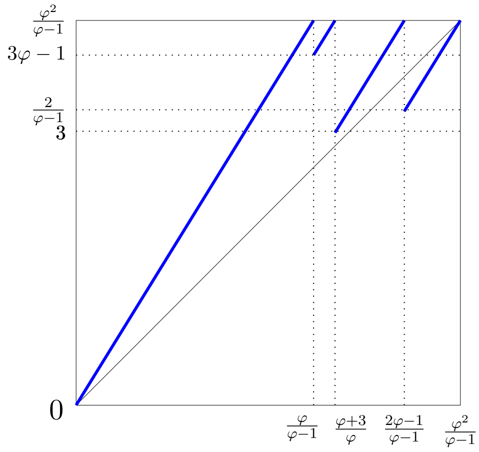

Figure 1.19: The transformation $L_{\varphi,\widetilde{\Delta}}$ for $\Delta = \{0, 1, \varphi + \frac{1}{\varphi}, \varphi^2\}$.

From a dynamical point of view, the lazy (β, Δ)-expansions can be generated by a transformation.

Definition 1.4.71. The *lazy* (β, Δ)*-transformation*

$$L_{\beta,\Delta} \colon (\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}] \to (\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}],$$
$$x \mapsto \begin{cases} \beta x & \text{if } x \in (\frac{d_0}{\beta-1}, \frac{d_m}{\beta-1} - \frac{d_m-d_0}{\beta}], \\ \beta x - d_k & \text{if } x \in (\frac{d_m}{\beta-1} - \frac{d_m-d_{k-1}}{\beta}, \frac{d_m}{\beta-1} - \frac{d_m-d_k}{\beta}],\ k \in [\![1, m]\!]. \end{cases}$$

The lazy (β, Δ)-transformation can be used to obtain the digits of the lazy (β, Δ)-expansions: for all $n \in \mathbb{N}$, a_n is the least digit d in Δ such that $\frac{d}{\beta} + \sum_{k=1}^{+\infty} \frac{d_m}{\beta^{k+1}} \geq L^n_{\beta,\Delta}(x)$.

The greedy and lazy (β, Δ)-transformations can be linked as in the real base expansions over the canonical alphabet $[\![0, \lceil\beta\rceil - 1]\!]$ (see Theorem 1.4.60).

Proposition 1.4.72. *If $\Delta = \{d_0, d_1, \ldots, d_m\} \subset \mathbb{R}$ where $d_0 < d_1 < \cdots < d_m$ is an allowable digit set for $\beta > 1$ then so is the set*

$$\widetilde{\Delta} = \{\widetilde{d_m}, \widetilde{d_{m-1}}, \ldots, \widetilde{d_0}\}$$

where for all $k \in [\![0, m]\!]$, $\widetilde{d_k} = d_0 + d_m - d_k$.

Theorem 1.4.73. *If Δ is an allowable digit set for $\beta > 1$ then the map*

$$\phi_{\beta,\Delta} \colon [\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}) \to (\tfrac{d_0}{\beta-1}, \tfrac{d_m}{\beta-1}],\ x \mapsto \tfrac{d_0+d_m}{\beta-1} - x$$

is a bicontinuous bijection satisfying $L_{\beta,\widetilde{\Delta}} \circ \phi_{\beta,\Delta} = \phi_{\beta,\Delta} \circ T_{\beta,\Delta}$.

Example 1.4.74. Consider the digit set $\widetilde{\Delta}$ where Δ is the digit set from Example 1.4.69. We get $\widetilde{\Delta} = \{0, 1 - \frac{1}{\varphi}, \varphi, \varphi^2\}$. The lazy $(\varphi, \widetilde{\Delta})$-transformation

$$L_{\varphi,\widetilde{\Delta}} \colon (0, \tfrac{\varphi^2}{\varphi-1}] \to (0, \tfrac{\varphi^2}{\varphi-1}],\ x \mapsto \begin{cases} \varphi x & \text{if } x \in (0, \frac{\varphi}{\varphi-1}] \\ \varphi x - (1 - \frac{1}{\varphi}) & \text{if } x \in (\frac{\varphi}{\varphi-1}, \frac{\varphi+3}{\varphi}] \\ \varphi x - \varphi & \text{if } x \in (\frac{\varphi+3}{\varphi}, \frac{2\varphi-1}{\varphi-1}] \\ \varphi x - \varphi^2 & \text{if } x \in (\frac{2\varphi-1}{\varphi-1}, \frac{\varphi^2}{\varphi-1}] \end{cases}$$

is depicted in Figure 1.19. It is conjugated to the greedy (φ, Δ)-transformation $T_{\varphi,\Delta}$ by $\phi_{\varphi,\Delta} \colon [0, \frac{\varphi^2}{\varphi-1}) \to (0, \frac{\varphi^2}{\varphi-1}],\ x \mapsto \frac{\varphi^2}{\varphi-1} - x$.

CHAPTER

2 COMBINATORIAL PROPERTIES OF CANTOR REAL BASE EXPANSIONS

In this chapter, we introduce and study series expansions of real numbers with an arbitrary Cantor real base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$, which we call $\boldsymbol{\beta}$-representations. In doing so, we generalize both representations of real numbers in real bases and through Cantor series.

First, we focus on the greedy algorithm and we show fundamental properties of $\boldsymbol{\beta}$-representations, each of which extends existing results on representations in a real base recalled in Section 1.4.1. In particular, we prove a generalization of Parry's theorem characterizing sequences of non-negative integers that are the greedy $\boldsymbol{\beta}$-representations of some real number in the interval $[0, 1)$.

Next, we define the lazy algorithm and we study the combinatorial properties of the lazy expansions in Cantor real bases. To do so, we prove that the lazy $\boldsymbol{\beta}$-expansions can be obtained by "flipping" the digits of the greedy ones. Hence, the combinatorial properties of the greedy $\boldsymbol{\beta}$-expansions we just obtained can be "flipped" to the lazy framework. In particular, a version of Parry's theorem in the lazy Cantor real base framework is proved.

The results presented in this chapter are from [CC21] and [Cis21]. Since this chapter generalizes the combinatorial properties of real base expansions to the Cantor real base framework, Sections 1.1, 1.2 and 1.4.1 are needed preliminaries for the good understanding of the contents of this chapter.

Contents of the Chapter

2.1 Definition of Cantor bases

Definition 2.1.1. A *Cantor real base*, or simply a *Cantor base*, is a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 such that $\prod_{n\in\mathbb{N}} \beta_n = +\infty$.

Example 2.1.2. For all $n \in \mathbb{N}$, let $\alpha_n = 1 + \frac{1}{2^{n+1}}$ and $\beta_n = 2 + \frac{1}{2^{n+1}}$. The sequence $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ is not a Cantor base since $\prod_{n\in\mathbb{N}} \alpha_n < +\infty$. In fact, for all $n \in \mathbb{N}$, we have $1 + \frac{1}{2^{n+1}} \le \exp\left(\frac{1}{2^{n+1}}\right)$ so, for all $N \in \mathbb{N}$, we get

$$\prod_{n=0}^{N} \alpha_n \le \prod_{n=0}^{N} \exp\big(\frac{1}{2^{n+1}}\big) = \exp\big(\sum_{n=0}^{N} \frac{1}{2^{n+1}}\big)$$

where the series $\sum_{n\in\mathbb{N}} \frac{1}{2^{n+1}}$ is a convergent geometric series. However, the sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ is indeed a Cantor base since $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. In fact, for all $N \in \mathbb{N}$, we have $\prod_{n=0}^{N} \beta_n \ge \prod_{n=0}^{N} 2$.

Proposition 2.1.3. *Any sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 that takes only finitely many values is a Cantor base.*

Proof. Consider a sequence $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ of real numbers greater than 1 that takes only finitely many values. There exists a real number $\beta > 1$ occuring infinitely many times in the sequence $\boldsymbol{\beta}$. Then, we get $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. ∎

In particular, by the previous proposition, any sequence $\boldsymbol{\beta} = (\beta, \beta, \ldots)$ with $\beta > 1$ is a Cantor base.

Definition 2.1.4. The *$\boldsymbol{\beta}$-value (partial) map* $\mathrm{val}_{\boldsymbol{\beta}} \colon (\mathbb{R}_{\geq 0})^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ is the map defined by

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \tag{2.1}$$

for any infinite word $a = a_0 a_1 a_2 \cdots$ over $\mathbb{R}_{\geq 0}$, provided that the series converges. A *$\boldsymbol{\beta}$-representation* of a non-negative real number x is an infinite word $a \in \mathbb{N}^{\mathbb{N}}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$. So we ask that the digits of a $\boldsymbol{\beta}$-representation are non-negative integers.

If $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, then for all $x \in [0, 1]$, a $\boldsymbol{\beta}$-representation of x is a β-representation of x as defined by Rényi [Rén57] (see Section 1.4.1). In this case, we do not distinguish the notation $\boldsymbol{\beta}$ and β: we write val_β and we talk about β-representations, as usual.

We will need to represent real numbers not only in a fixed Cantor base $\boldsymbol{\beta}$ but also in all Cantor bases obtained by shifting $\boldsymbol{\beta}$.

Definition 2.1.5. For all $n \in \mathbb{N}$, the n^{th} shift of the Cantor base $\boldsymbol{\beta}$ is denoted $\boldsymbol{\beta}^{(n)}$, that is, $\boldsymbol{\beta}^{(n)} = (\beta_n, \beta_{n+1}, \ldots)$. In particular, we have $\boldsymbol{\beta}^{(0)} = \boldsymbol{\beta}$.

2.2 Representations of 1

The $\boldsymbol{\beta}$-representations of 1 will be of interest in what follows, in particular the greedy and the quasi-greedy expansions of 1 (see Sections 2.3.1 and 2.3.3). We start our study by providing a characterization of those infinite words a over the alphabet $\mathbb{R}_{\geq 0}$ for which there exists a Cantor real base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$.

For any infinite word a over $\mathbb{N}$ satisfying some suitable conditions, the equation $\mathrm{val}_\beta(a) = 1$ admits a unique solution $\beta > 1$ (see Corollary 1.4.17). This classical result remains true for non-negative real digits and weaker conditions on the infinite word a.

Lemma 2.2.1. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $a_n \in O(n^d)$ for some $d \in \mathbb{N}$. There exists a real base β such that $\mathrm{val}_\beta(a) = 1$ if and only if $\sum_{n\in\mathbb{N}} a_n > 1$, in which case β is unique and $\beta \geq a_0$, and if moreover for all $n \in \mathbb{N}$, $a_n \leq a_0$, then $\beta \leq a_0 + 1$.*

Proof. If $\sum_{n\in\mathbb{N}} a_n \leq 1$ then for all real bases β, $\mathrm{val}_\beta(a) < 1$. Indeed, this is obvious if $a = 0^\omega$, and else $\mathrm{val}_\beta(a) < \sum_{n\in\mathbb{N}} a_n \leq 1$.

Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. Let $N \in \mathbb{N}$ be such that $\sum_{n=0}^{N} a_n > 1$. The function $f\colon [0,1) \to \mathbb{R},\ x \mapsto \sum_{n\in\mathbb{N}} a_n x^{n+1}$ is well-defined, continuous, increasing and such that $f(0) = 0$ and that for all $x \in [0,1)$, $f(x) \geq \sum_{n=0}^{N} a_n x^{n+1}$. The function $g\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{n=0}^{N} a_n x^{n+1}$ is continuous, increasing and such that $g(0) = 0$ and $g(1) > 1$. Therefore, there exists a unique $x_0 \in (0,1)$ such that $g(x_0) = 1$, and hence such that $f(x_0) \geq 1$. Now, there exists a unique $\gamma \in (0, x_0]$ such that $f(\gamma) = 1$. By setting $\beta = \frac{1}{\gamma}$, we get that $\beta \geq \frac{1}{x_0} > 1$ and $\mathrm{val}_\beta(a) = f\left(\frac{1}{\beta}\right) = 1$. Moreover, $\beta \geq a_0$ for otherwise $f\left(\frac{1}{\beta}\right) > f\left(\frac{1}{a_0}\right) \geq 1$.

If moreover for all $n \in \mathbb{N}$, $a_n \leq a_0$, then $\beta \leq a_0 + 1$ for otherwise we would have

$$\mathrm{val}_\beta(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\beta^{n+1}} < a_0 \sum_{n\in\mathbb{N}} \frac{1}{(a_0+1)^{n+1}} = 1.$$

■

No upper bound on the growth order of the digits a_n is needed in order to find a Cantor base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$.

Lemma 2.2.2. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $\sum_{n\in\mathbb{N}} a_n = +\infty$. Then there exists a Cantor base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$.*

Proof. First of all, observe that the hypothesis implies that a does not end in 0^ω and that $\prod_{n\in\mathbb{N}}(a_n + 1) = +\infty$.

We define two sequences of non-negative integers $(n_k)_{1\leq k\leq K}$ and $(m_k)_{1\leq k\leq K}$ where $K \in \mathbb{N} \cup \{+\infty\}$. The length K of these two sequences is the number of zero blocks in a, that is, the factors of the form 0^m which are neither preceded nor followed by 0 in a. Two cases stand out: either $K \in \mathbb{N}$ or $K = +\infty$. We describe the two cases at once. In order to do so, it should be understood that the parts of the definition where $k > K$ should just be ignored when $K \in \mathbb{N}$. Let n_1 denote the least $n \in \mathbb{N}$ such that $a_n = 0$ and let m_1 denote the least $m \in \mathbb{N}$ such that $a_{n_1+m} > 0$. Then for $k \geq 2$, let n_k denote the least integer $n > n_{k-1} + m_{k-1}$ such that $a_n = 0$ and let m_k denote the least $m \in \mathbb{N}$ such that $a_{n_k+m} > 0$. Thus, $(n_k)_{1 \le k \le K}$ is the sequence of positions of appearance of the successive zero blocks in a and $(m_k)_{1 \le k \le K}$ is the sequence of lengths of these blocks.

Next, for all $k \in [\![1, K]\!]$, we pick any α_k in the interval $(1, \sqrt[m_k]{a_{n_k+m_k} + 1})$. For all $n \in \mathbb{N}$, we define

$$\beta_n = \begin{cases} a_n + 1 & \text{if } n \in [\![0, n_1 - 1]\!] \text{ or } n \in \bigcup_{k=1}^{K} [\![n_k + m_k + 1, n_{k+1} - 1]\!] \\ \alpha_k & \text{if } n \in [\![n_k, n_k + m_k - 1]\!] \text{ for some } k \in [\![1, K]\!] \\ \frac{a_n+1}{\alpha_k^{m_k}} & \text{if } n = n_k + m_k \text{ for some } k \in [\![1, K]\!] \end{cases}$$

where we set $n_{K+1} = +\infty$ if $K \in \mathbb{N}$. In particular if $K = 0$, that is, if for all $n \in \mathbb{N}$, $a_n > 0$, then for all $n \in \mathbb{N}$, $\beta_n = a_n + 1$.

Let us show that in any case, the obtained sequence $\boldsymbol{\beta} = (\beta_n)_{n \in \mathbb{N}}$ is such that $\prod_{n \in \mathbb{N}} \beta_n = +\infty$ and $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. By construction,

$$\begin{aligned} \prod_{n \in \mathbb{N}} \beta_n &= \prod_{n=0}^{n_1-1} (a_n + 1) \cdot \prod_{k=1}^{K} \left(\alpha_k^{m_k} \cdot \frac{a_{n_k+m_k}+1}{\alpha_k^{m_k}} \cdot \prod_{n=n_k+m_k+1}^{n_{k+1}-1} (a_n + 1) \right) \\ &= \prod_{n \in \mathbb{N}} (a_n + 1). \end{aligned}$$

By induction we can show that

$$\sum_{n=0}^{n_k+m_k} \frac{a_n}{\prod_{i=0}^{n} \beta_i} = 1 - \frac{1}{\prod_{i=0}^{n_k+m_k} \beta_i} \quad \text{for all } k \in [\![1, K]\!].$$

If $K = +\infty$ then we obtain that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ by letting k tend to infinity. Otherwise, $K \in \mathbb{N}$. Set $n_0 = -1$ and $m_0 = 0$. By induction again, we can show that

$$\sum_{n=n_K+m_K+1}^{m} \frac{a_n}{\prod_{i=n_K+m_K+1}^{n} \beta_i} = 1 - \frac{1}{\prod_{i=n_K+m_K+1}^{m} \beta_i} \quad \text{for all } m \in \mathbb{N}.$$

By letting m tend to infinity, we get

$$\mathrm{val}_{\boldsymbol{\beta}^{(n_K+m_K+1)}}(\sigma^{n_K+m_K+1}(a)) = 1.$$

Finally, we obtain

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n=0}^{n_K+m_K} \frac{a_n}{\prod_{i=0}^{n} \beta_i} + \sum_{n=n_K+m_K+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i}$$

$$= 1 - \frac{1}{\prod_{i=0}^{n_K+m_K} \beta_i} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n_K+m_K+1)}}(\sigma^{n_K+m_K+1}(a))}{\prod_{i=0}^{n_K+m_K} \beta_i}$$
$$= 1.$$

■

Proposition 2.2.3. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$. There exists a Cantor base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ if and only if $\sum_{n\in\mathbb{N}} a_n > 1$.*

Proof. Similarly to the proof of Lemma 2.2.1, the condition $\sum_{n\in\mathbb{N}} a_n > 1$ is necessary. Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. If $\sum_{n\in\mathbb{N}} a_n = +\infty$ then we use Lemma 2.2.2. Otherwise, we have $1 < \sum_{n\in\mathbb{N}} a_n < +\infty$ and we apply Lemma 2.2.1. ■

2.3 Greedy $\boldsymbol{\beta}$-expansions

This section is concerned with the study of the greedy $\boldsymbol{\beta}$-expansions of real numbers smaller than or equal to 1. Properties of real base expansions from Section 1.4.1 will be generalized to the Cantor base framework.

2.3.1 The greedy algorithm

Definition 2.3.1. For $x \in [0,1]$, a distinguished $\boldsymbol{\beta}$-representation $\varepsilon_{\boldsymbol{\beta},0}(x)\varepsilon_{\boldsymbol{\beta},1}(x)\varepsilon_{\boldsymbol{\beta},2}(x)\cdots$ is computed thanks to the *greedy algorithm*:

- $\varepsilon_{\boldsymbol{\beta},0}(x) = \lfloor \beta_0 x \rfloor$ and $r_{\boldsymbol{\beta},0}(x) = \beta_0 x - \varepsilon_{\boldsymbol{\beta},0}(x)$
- $\varepsilon_{\boldsymbol{\beta},n}(x) = \lfloor \beta_n r_{\boldsymbol{\beta},n-1}(x) \rfloor$ and $r_n = \beta_n r_{\boldsymbol{\beta},n-1}(x) - \varepsilon_{\boldsymbol{\beta},n}(x)$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation of x is denoted by $d_{\boldsymbol{\beta}}(x)$ and is called the *greedy $\boldsymbol{\beta}$-expansion* of x. For all $n \in \mathbb{N}$, the value $r_{\boldsymbol{\beta},n}(x)$ belongs to the interval $[0,1)$ and we call $r_{\boldsymbol{\beta},n}(x)$ the $(n+1)^{st}$ *remainder of the greedy $\boldsymbol{\beta}$-expansion of x*.

We write $\varepsilon_n(x)$ and $r_n(x)$ instead of $\varepsilon_{\boldsymbol{\beta},n}(x)$ and $r_{\boldsymbol{\beta},n}(x)$ when the context is clear. The greedy $\boldsymbol{\beta}$-expansion of 1 will play a special role. For the sake of clarity, we let ε_n denote its digits instead of $\varepsilon_n(1)$.

As previously mentioned, if $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, then for all $x \in [0,1]$, the greedy $\boldsymbol{\beta}$-expansion of x is equal to the usual greedy β-expansion of x as defined by Rényi [Rén57] and we write indistinctly $\boldsymbol{\beta}$ or β.

Remark 2.3.2. The first digit $\varepsilon_0(x)$ belongs to $[\![0, \lfloor\beta_0\rfloor]\!]$ and for all $n \in \mathbb{N}_{\geq 1}$, the $(n+1)^{\text{st}}$ digit $\varepsilon_n(x)$ belongs to $[\![0, \lceil\beta_n\rceil - 1]\!]$. The letter $\lfloor\beta_0\rfloor$ differs from $\lceil\beta_0\rceil - 1$ only when $\beta_0 \in \mathbb{N}_{\geq 2}$. Moreover, in that case, the letter $\lfloor\beta_0\rfloor$ can only appear at position 0 of the $\boldsymbol{\beta}$-expansion of 1.

Example 2.3.3. If there exists $n \in \mathbb{N}$ such that β_n is an integer (without any restriction on the other β_m), then $d_{\boldsymbol{\beta}^{(n)}}(1) = \beta_n 0^\omega$.

Definition 2.3.4. We let $A_{\boldsymbol{\beta}}$ denote the (possibly infinite) alphabet $[\![0, \sup_{n\in\mathbb{N}}(\lceil\beta_n\rceil - 1)]\!]$.

The digits of the $\boldsymbol{\beta}$-expansions of real numbers in $[0,1]$ (resp. in $[0,1)$) belongs to $A_{\boldsymbol{\beta}} \cup \{\lfloor\beta_0\rfloor\}$ (resp., $A_{\boldsymbol{\beta}}$). Note that, if the supremum is infinite, the alphabet $A_{\boldsymbol{\beta}}$ is made of all non-negative integers.

The algorithm is called greedy since at each step it chooses the largest possible digit. Indeed, consider $x \in [0,1]$ and $N \in \mathbb{N}$, and suppose that the digits $\varepsilon_0(x), \ldots, \varepsilon_{N-1}(x)$ are already known. Then the digit $\varepsilon_N(x)$ is the largest element of $[\![0, \lceil\beta_N\rceil - 1]\!]$ ($[\![0, \lfloor\beta_0\rfloor]\!]$ if $N = 0$) such that $\sum_{n=0}^{N} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n}\beta_i} \leq x$. Thus

$$x = \sum_{n=0}^{N} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n}\beta_i} + \frac{r_N(x)}{\prod_{i=0}^{N}\beta_i}$$

where $r_N(x) \in [0,1)$. Note that since[1] a Cantor base satisfies $\prod_{n\in\mathbb{N}}\beta_n = +\infty$, the latter equality implies the convergence of the greedy algorithm and that $x = \mathrm{val}_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x))$.

Example 2.3.5. Consider the sequence $\boldsymbol{\alpha}$ (which is not a Cantor base) from Example 2.1.2. If we perform the greedy algorithm on $x = 1$ for the sequence $\boldsymbol{\alpha}$, we obtain the sequence of digits 10^ω, which is clearly not an $\boldsymbol{\alpha}$-representation of 1.

Example 2.3.6. Let $\alpha = \frac{1+\sqrt{13}}{2}$ and $\beta = \frac{5+\sqrt{13}}{6}$.

1. Consider $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ the Cantor base where the infinite word $\beta_0\beta_1\beta_2\cdots$ is the Thue-Morse word over the alphabet $\{\alpha, \beta\}$ (see Definition 1.2.15), that is, the Cantor base defined by

$$\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots). \tag{2.2}$$

[1]This is the reason why the condition $\prod_{n\in\mathbb{N}}\beta_n = +\infty$ appears in the definition of a Cantor base.

The greedy $\boldsymbol{\beta}$-expansion of $\frac{1}{2}$ has 10001 as a prefix and $d_{\boldsymbol{\beta}}(\frac{65-18\sqrt{13}}{6}) = 10020^\omega$. Moreover, we compute $d_{\boldsymbol{\beta}}(1) = 20010110^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 1010110^\omega$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 110^\omega$.

2. Consider $\boldsymbol{\beta} = (\sqrt{13}, \alpha, \beta, \alpha, \beta, \alpha, \beta, \ldots)$. It is easily checked that $d_{\boldsymbol{\beta}}(1) = 3(10)^\omega$ and that for all $m \in \mathbb{N}$, $d_{\boldsymbol{\beta}^{(2m+1)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(2m+2)}}(1) = 110^\omega$.

Definition 2.3.7. We call an *alternate base* a periodic Cantor base

$$\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1}, \beta_0, \ldots, \beta_{p-1}, \ldots),$$

that is, a Cantor base for which there exists $p \in \mathbb{N}_{\geq 1}$ such that for all $n \in \mathbb{N}$, $\beta_n = \beta_{n+p}$. In this case we simply note $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ and the integer p is called the *length* of the alternate base $\boldsymbol{\beta}$.

In what follows, most examples will be alternate bases and Chapters 3, 4 and 5 will be specifically devoted to their study.

Example 2.3.8. Let $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$ where φ still designates the Golden ratio $(1 + \sqrt{5})/2$. For all $m \in \mathbb{N}$, we have $d_{\boldsymbol{\beta}^{(3m)}}(1) = 30^\omega$, $d_{\boldsymbol{\beta}^{(3m+1)}}(1) = 110^\omega$ and $d_{\boldsymbol{\beta}^{(3m+2)}}(1) = 1(110)^\omega$.

Example 2.3.9. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. We have $d_{\boldsymbol{\beta}}(\frac{-5+2\sqrt{13}}{3}) = 110^\omega$ and $d_{\boldsymbol{\beta}}(\frac{2+\sqrt{13}}{9}) = (10)^\omega$. Moreover, the alternate base $\boldsymbol{\beta}^{(1)}$ equals the first shift of the Cantor base from the second item in Example 2.3.6. We get $d_{\boldsymbol{\beta}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$.

Note that both previous alternate bases will be running examples all along this text.

2.3.2 First properties of greedy expansions

Let us show that the classical properties of the greedy β-expansion theory are still valid for Cantor bases. Some are just an adaptation of the related proofs in [Lot02] but for the sake of completeness the details are written. From now on, unless otherwise stated, we consider a fixed Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$.

For all $x \in [0, 1)$ and $n \in \mathbb{N}$, we can express the digit $\varepsilon_n(x)$ and remainder $r_n(x)$ of the greedy $\boldsymbol{\beta}$-expansion of x thanks to the β_n-transformations from

Definition 1.4.44:

$$\varepsilon_n(x) = \left\lfloor \beta_n \big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)\big) \right\rfloor \quad \text{and} \quad r_n(x) = T_{\beta_n} \circ \cdots \circ T_{\beta_0}(x). \tag{2.3}$$

Proposition 2.3.10. *For all $x \in [0,1)$ and all $n \in \mathbb{N}$, we have*

$$\sigma^n \circ d_{\boldsymbol{\beta}}(x) = d_{\boldsymbol{\beta}^{(n)}} \circ T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x).$$

Proof. This follows from (2.3). In fact, for all $x \in [0,1)$ and all $n \in \mathbb{N}$, we have $\sigma^n \circ d_{\boldsymbol{\beta}}(x) = \varepsilon_{\boldsymbol{\beta},n}(x)\varepsilon_{\boldsymbol{\beta},n+1}(x)\cdots$ where, for all $m \in \mathbb{N}$,

$$\varepsilon_{\boldsymbol{\beta},n+m}(x) = \left\lfloor \beta_{n+m}\big(T_{\beta_{n+m-1}} \circ \cdots \circ T_{\beta_0}(x)\big) \right\rfloor .$$

Let y denote $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)$. We get

$$\varepsilon_{\boldsymbol{\beta},n+m}(x) = \left\lfloor \beta_{n+m}\big(T_{\beta_{n+m-1}} \circ \cdots \circ T_{\beta_n}(y)\big) \right\rfloor = \varepsilon_{\boldsymbol{\beta}^{(n)},m}(y).$$

We obtain $\sigma^n \circ d_{\boldsymbol{\beta}}(x) = d_{\boldsymbol{\beta}^{(n)}}(y)$. ∎

Definition 2.3.11. We let $D_{\boldsymbol{\beta}}$ denote the subset of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ of all greedy $\boldsymbol{\beta}$-expansions of real numbers in the interval $[0,1)$:

$$D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x) : x \in [0,1)\}.$$

Infinite words in $D_{\boldsymbol{\beta}}$ are said to be *greedy $\boldsymbol{\beta}$-admissible sequences*.

As in the real base framework, a goal of this study is to characterize $D_{\boldsymbol{\beta}}$.

Lemma 2.3.12. *For all infinite words a over $\mathbb{N}$ and all $x \in [0,1]$, $a = d_{\boldsymbol{\beta}}(x)$ if and only if $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and for all $k \in \mathbb{N}$,*

$$\sum_{n=k+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} < \frac{1}{\prod_{i=0}^{k} \beta_i}. \tag{2.4}$$

Proof. From the greedy algorithm, for all $x \in [0,1]$, $\mathrm{val}_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x)) = x$ and for all $k \in \mathbb{N}$,

$$\left(\sum_{n=k+1}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i}\right) \prod_{i=0}^{k} \beta_i = \left(x - \sum_{n=0}^{k} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i}\right) \prod_{i=0}^{k} \beta_i = r_k(x) < 1.$$

Conversely, suppose that a is an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and such that for all $k \in \mathbb{N}$, (2.4) holds. Let us show by induction that for all $m \in \mathbb{N}$, $a_m = \varepsilon_m(x)$. From (2.4) for $k = 0$, we get that $x - \frac{a_0}{\beta_0} < \frac{1}{\beta_0}$. Thus, $\beta_0 x - 1 < a_0$. Since $\frac{a_0}{\beta_0} \le x$, we get that $a_0 \le \beta_0 x$. Therefore, $a_0 = \lfloor \beta_0 x \rfloor = \varepsilon_0(x)$. Now, suppose that $m \in \mathbb{N}_{\ge 1}$ and that for $n \in [\![0, m-1]\!]$, $a_n = \varepsilon_n(x)$. Then

$$a_m + \left(\sum_{n=m+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \right) \prod_{i=0}^{m} \beta_i = \varepsilon_m(x) + r_m(x).$$

By using (2.4) for $k = m$, since $r_m(x) < 1$, we obtain that $a_m = \varepsilon_m(x)$. ∎

Proposition 2.3.13. *Let a be a $\boldsymbol{\beta}$-representation of some real number x in $[0,1]$. Then the following four assertions are equivalent.*

1. *The infinite word a is the greedy $\boldsymbol{\beta}$-expansion of x.*
2. *For all $n \in \mathbb{N}_{\ge 1}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.*
3. *The infinite word $\sigma(a)$ belongs to $D_{\boldsymbol{\beta}^{(1)}}$.*
4. *For all $n \in \mathbb{N}_{\ge 1}$, $\sigma^n(a)$ belongs to $D_{\boldsymbol{\beta}^{(n)}}$.*

Proof. Since $\mathrm{val}_{\boldsymbol{\beta}}(a) = x \in [0,1]$, it follows from Lemma 2.3.12 that $a = d_{\boldsymbol{\beta}}(x)$ if and only if for all $k \in \mathbb{N}$, (2.4) holds. In order to obtain the equivalences between the first three items, it suffices to note that the greedy condition (2.4) can be rewritten as $\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(\sigma^{k+1}(a)) < 1$. Clearly (4) implies (3). Finally we obtain that (3) implies (4) by iterating the implication (1) $\implies$ (3). ∎

Corollary 2.3.14. *An infinite word a over $\mathbb{N}$ belongs to $D_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.*

Proposition 2.3.15. *The greedy $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1]$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of x.*

Proof. Let $x \in [0,1]$ and $a \in \mathbb{N}^{\mathbb{N}}$ be a $\boldsymbol{\beta}$-representation of x. Proceed by contradiction and suppose that $a >_{\mathrm{lex}} d_{\boldsymbol{\beta}}(x)$. There exists $k \in \mathbb{N}$ such that $\varepsilon_0(x) \cdots \varepsilon_{k-1}(x) = a_0 \cdots a_{k-1}$ and $a_k > \varepsilon_k(x)$. Then

$$\sum_{n=k}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} = \sum_{n=k}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i} \ge \frac{\varepsilon_k(x)+1}{\prod_{i=0}^{k} \beta_i} + \sum_{n=k+1}^{+\infty} \frac{a_n}{\prod_{i=0}^{n} \beta_i}$$

and hence

$$\sum_{n=k+1}^{+\infty} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} \geq \frac{1}{\prod_{i=0}^{k} \beta_i}$$

which is impossible by Lemma 2.3.12. ■

Remark 2.3.16. In this section, we made a choice of definition for the greedy $\boldsymbol{\beta}$-expansion of 1. This choice was motivated firstly by conserving the same algorithm for the real numbers in $[0,1)$ and for the real number 1 and secondly by Proposition 2.3.15 we have just established. In fact, an other choice would have been to preserve the alphabet $A_{\boldsymbol{\beta}}$ by avoiding the digit $\lfloor \beta_0 \rfloor$ when β_0 is an integer. However, in that case, when $\beta_0 \in \mathbb{N}_{\geq 2}$, we would get that the greedy expansion of 1 is $(\beta_0 - 1) d_{\boldsymbol{\beta}^{(1)}}(1)$ which is not lexicographically maximal among all $\boldsymbol{\beta}$-representations of 1 since $\beta_0 0^\omega >_{\text{lex}} (\beta_0 - 1) d_{\boldsymbol{\beta}^{(1)}}(1)$. It is important to note that for the dynamical point of view in Chapter 5, the other choice will be made. This will be motivated differently. Note that this ambiguity with two possible definitions of the greedy β-expansion of 1 when $\beta \in \mathbb{N}_{\geq 2}$ already appears in the real base case as pointed out in Remark 1.4.55.

Proposition 2.3.17. *The function* $d_{\boldsymbol{\beta}} \colon [0,1] \to (A_{\boldsymbol{\beta}} \cup \{\lfloor \beta_0 \rfloor\})^{\mathbb{N}}$ *is increasing:*

$$\forall x, y \in [0,1], \quad x < y \iff d_{\boldsymbol{\beta}}(x) <_{\text{lex}} d_{\boldsymbol{\beta}}(y).$$

Proof. Suppose that $d_{\boldsymbol{\beta}}(x) <_{\text{lex}} d_{\boldsymbol{\beta}}(y)$. There exists $k \in \mathbb{N}$ such that

$$\varepsilon_0(x) \cdots \varepsilon_{k-1}(x) = \varepsilon_0(y) \cdots \varepsilon_{k-1}(y)$$

and $\varepsilon_k(x) < \varepsilon_k(y)$. By Lemma 2.3.12, we get

$$x = \sum_{n \in \mathbb{N}} \frac{\varepsilon_n(x)}{\prod_{i=0}^{n} \beta_i} < \sum_{n=0}^{k-1} \frac{\varepsilon_n(y)}{\prod_{i=0}^{n} \beta_i} + \frac{\varepsilon_k(y) - 1}{\prod_{i=0}^{k} \beta_i} + \frac{1}{\prod_{i=0}^{k} \beta_i} = \sum_{n=0}^{k} \frac{\varepsilon_n(y)}{\prod_{i=0}^{n} \beta_i} \leq y.$$

It follows immediately that $x < y$ implies $d_{\boldsymbol{\beta}}(x) <_{\text{lex}} d_{\boldsymbol{\beta}}(y)$. ■

Corollary 2.3.18. *If a is an infinite word over $\mathbb{N}$ such that $\operatorname{val}_{\boldsymbol{\beta}}(a) \leq 1$, then $a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. In particular, $d_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of all real numbers in $[0,1]$.*

Proof. Let a be an infinite word over $\mathbb{N}$ such that $\operatorname{val}_{\boldsymbol{\beta}}(a) \leq 1$. By Propositions 2.3.15 and 2.3.17, $a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(\operatorname{val}_{\boldsymbol{\beta}}(a)) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. ■

Recall the property of the greedy β-expansions stating that considering two bases α and β, we have $\alpha < \beta$ if and only if $d_\alpha(1) < d_\beta(1)$ (see Proposition 1.4.7). The following proposition provides a generalization of a weaker version of this property to Cantor bases.

Proposition 2.3.19. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ be two Cantor bases such that for all $n \in \mathbb{N}$, $\prod_{i=0}^{n} \alpha_i \le \prod_{i=0}^{n} \beta_i$. Then for all $x \in [0,1]$, we have $d_{\boldsymbol{\alpha}}(x) \le_{\text{lex}} d_{\boldsymbol{\beta}}(x)$.*

Proof. Let $x \in [0,1]$ and suppose to the contrary that $d_{\boldsymbol{\alpha}}(x) >_{\text{lex}} d_{\boldsymbol{\beta}}(x)$. Thus, there exists $k \in \mathbb{N}$ such that $\varepsilon_{\boldsymbol{\alpha},0}(x)\cdots\varepsilon_{\boldsymbol{\alpha},k-1}(x) = \varepsilon_{\boldsymbol{\beta},0}(x)\cdots\varepsilon_{\boldsymbol{\beta},k-1}(x)$ and $\varepsilon_{\boldsymbol{\alpha},k}(x) > \varepsilon_{\boldsymbol{\beta},k}(x)$. From Lemma 2.3.12 and from the hypothesis, we obtain that

$$
\begin{aligned}
x &\le \sum_{n=0}^{k-1} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\beta_i} + \frac{\varepsilon_{\boldsymbol{\alpha},k}(x)-1}{\prod_{i=0}^{k}\beta_i} + \sum_{n=k+1}^{+\infty} \frac{\varepsilon_{\boldsymbol{\beta},n}(x)}{\prod_{i=0}^{n}\beta_i} \\
&< \sum_{n=0}^{k} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\beta_i} \\
&\le \sum_{n=0}^{k} \frac{\varepsilon_{\boldsymbol{\alpha},n}(x)}{\prod_{i=0}^{n}\alpha_i} \\
&\le x,
\end{aligned}
$$

a contradiction. ■

Corollary 2.3.20. *Let $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ be two Cantor bases such that for all $n \in \mathbb{N}$, $\alpha_n \le \beta_n$. Then for all $x \in [0,1]$, we have $d_{\boldsymbol{\alpha}}(x) \le_{\text{lex}} d_{\boldsymbol{\beta}}(x)$.*

It is not true that $d_{\boldsymbol{\alpha}}(1) <_{\text{lex}} d_{\boldsymbol{\beta}}(1)$ implies that for all $n \in \mathbb{N}$, $\prod_{i=0}^{n} \alpha_i \le \prod_{i=0}^{n} \beta_i$ as the following example shows. The same example shows that the lexicographic order on the Cantor bases is not sufficient either. Here, the term lexicographic order refers to the following order: $\boldsymbol{\alpha} < \boldsymbol{\beta}$ whenever there exists $k \in \mathbb{N}$ such that $\alpha_n = \beta_n$ for $n \in [\![0, k-1]\!]$ and $\alpha_k < \beta_k$.

Example 2.3.21. Let $\boldsymbol{\alpha} = \overline{(2+\sqrt{3}, 2)}$ and $\boldsymbol{\beta} = \overline{(2+\sqrt{2}, 5)}$. Then $d_{\boldsymbol{\alpha}}(1) = 31^\omega$ and $d_{\boldsymbol{\beta}}(1)$ starts with the prefix 32, hence $d_{\boldsymbol{\alpha}}(1) <_{\text{lex}} d_{\boldsymbol{\beta}}(1)$.

2.3.3 Quasi-greedy expansions

Definition 2.3.22. A $\boldsymbol{\beta}$-representation is said to be *finite* if it ends with infinitely many zeros, and *infinite* otherwise. The *length* of a finite $\boldsymbol{\beta}$-representation is the length of the longest prefix ending in a non-zero digit.

In this text, we usually omit to write the tail of zeros of finite $\boldsymbol{\beta}$-representations. When the greedy $\boldsymbol{\beta}$-expansion of 1 is finite, we show how to modify it in order to obtain an infinite $\boldsymbol{\beta}$-representation of 1 that is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of 1.

Definition 2.3.23. The *quasi-greedy $\boldsymbol{\beta}$-expansion* of 1 denoted by $d^*_{\boldsymbol{\beta}}(1)$ is defined recursively as follows:

$$d^*_{\boldsymbol{\beta}}(1) = \begin{cases} d_{\boldsymbol{\beta}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) \text{ is infinite} \\ \varepsilon_0 \cdots \varepsilon_{n-2}(\varepsilon_{n-1}-1)d^*_{\boldsymbol{\beta}^{(n)}}(1) & \text{if } d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{n-1} \\ & \quad \text{with } n \in \mathbb{N}_{\ge 1},\ \varepsilon_{n-1} > 0. \end{cases} \tag{2.5}$$

By construction, the quasi-greedy $\boldsymbol{\beta}$-expansion of 1 is an infinite word over the alphabet $A_{\boldsymbol{\beta}}$.

When $\boldsymbol{\beta} = (\beta, \beta, \ldots)$, we recover the usual definition of the quasi-greedy β-expansion. In particular, it is easy to check that in this case, if $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{n-1}$ with $n \in \mathbb{N}_{\ge 1}$ and $\varepsilon_{n-1} > 0$, then the quasi-greedy expansion is purely periodic and $d^*_{\boldsymbol{\beta}}(1) = (\varepsilon_0 \ldots \varepsilon_{n-2}(\varepsilon_{n-1}-1))^\omega$.

Example 2.3.24. Let $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$ the alternate base already considered in Example 2.3.8. Then we directly have that $d^*_{\boldsymbol{\beta}^{(2)}}(1) = d_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$. In order to compute $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$, we need to go through the definition several times. We compute $d^*_{\boldsymbol{\beta}}(1) = 2d^*_{\boldsymbol{\beta}^{(1)}}(1) = 210d^*_{\boldsymbol{\beta}}(1) = (210)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 10d^*_{\boldsymbol{\beta}}(1) = 10(210)^\omega = (102)^\omega$. The computation of $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$ can be interpreted thanks to Figure 2.1.

Example 2.3.25. Let $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ be an alternate base such that for all $i \in [\![0, p-1]\!]$, $\beta_i \in \mathbb{N}_{\ge 2}$. Then for all $i \in [\![0, p-1]\!]$, $d_{\boldsymbol{\beta}^{(i)}}(1) = \beta_i 0^\omega$ and

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = ((\beta_i - 1) \cdots (\beta_{p-1} - 1)(\beta_0 - 1) \ldots (\beta_{i-1} - 1))^\omega.$$

The recursive calls to the definition (2.5) are illustrated in Figure 2.2.

$$d^*_{\boldsymbol{\beta}}(1) = 2\underbrace{\qquad\quad}$$

$$d^*_{\boldsymbol{\beta}^{(1)}}(1) = 10\underbrace{\qquad\quad}$$

Figure 2.1: Symbolic computation of $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$ for $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$.

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = (\beta_i - 1)\underbrace{\qquad\qquad}$$

$$d^*_{\boldsymbol{\beta}^{(i+1)}}(1) = (\beta_{i+1} - 1)\underbrace{\qquad\qquad}$$

$$\vdots$$

$$d^*_{\boldsymbol{\beta}^{(i-1)}}(1) = (\beta_{i-1} - 1)\underbrace{\qquad\qquad}$$

Figure 2.2: Symbolic computation of $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ for $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}}) \in (\mathbb{N}_{\geq 2})^{\mathbb{N}}$.

Remark 2.3.26. As explained in Remark 2.3.16, a choice was made for the definition of $d_{\boldsymbol{\beta}}(1)$ when $\beta_0 \in \mathbb{N}_{\geq 2}$. It is important to note that even if the other choice were made, meaning by avoiding the letter $\lfloor \beta_0 \rfloor$, the obtained quasi-greedy $\boldsymbol{\beta}$-expansion of 1 would coincide with that obtained with our choice of definition of the greedy $\boldsymbol{\beta}$-expansion of 1.

Contrary to the real base case, for arbitrary Cantor bases, when the greedy expansion of 1 is finite, the quasi-greedy expansion of 1 can be not purely periodic.

Example 2.3.27. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. In Example 2.3.9, we computed $d_{\boldsymbol{\beta}}(1) = 201$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 11$. Then $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$ and $d^*_{\boldsymbol{\beta}}(1) = 200 d^*_{\boldsymbol{\beta}^{(1)}}(1) = 200(10)^\omega$. Figure 2.3 symbolically depicts the computation of $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$.

$$d^*_{\boldsymbol{\beta}}(1) = 200\underbrace{\qquad}$$

$$d^*_{\boldsymbol{\beta}^{(1)}}(1) = 10\underbrace{\qquad}$$

Figure 2.3: Symbolic computation of $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$ for $\boldsymbol{\beta} = \overline{\left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)}$.

Moreover, even if the greedy $\boldsymbol{\beta}$-expansion is finite, the quasi-greedy $\boldsymbol{\beta}$-representation can be infinite not ultimately periodic. Suppose that $d_{\boldsymbol{\beta}}(1)$ is finite and that an infinite quasi-greedy is involved during the computation of $d^*_{\boldsymbol{\beta}}(1)$. Let $n \in \mathbb{N}_{\geq 1}$ be the positive integer such that $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is the involved infinite expansion. Then $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if so is $d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Example 2.3.28. Consider the Cantor base $\boldsymbol{\beta} = (3, \beta, \beta, \beta, \beta, \ldots)$ where $\beta = \sqrt{6}(2+\sqrt{6})$. We get $d_{\boldsymbol{\beta}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = d_\beta(1)$ is infinite not ultimately periodic since β is a non-Pisot quadratic number (see Remark 1.4.12). Therefore, the quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1) = 2d^*_{\boldsymbol{\beta}^{(1)}}(1)$ is not ultimately periodic.

Before using the quasi-greedy $\boldsymbol{\beta}$-expansion of 1 in order to study the greedy admissible sequences, let us prove that the so-defined quasi-greedy $\boldsymbol{\beta}$-expansion of 1 is a $\boldsymbol{\beta}$-representation of 1 and is precisely the lexicographically maximal infinite one.

Proposition 2.3.29. *The quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1)$ is a $\boldsymbol{\beta}$-representation of* 1*.*

Proof. If $d^*_{\boldsymbol{\beta}}(1) = d_{\boldsymbol{\beta}}(1)$ the result is immediate. Thus, we suppose that $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{n-1}$ with $n \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{n-1} > 0$ and

$$d^*_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{n-2}(\varepsilon_{n-1} - 1)d^*_{\boldsymbol{\beta}^{(n)}}(1).$$

We get

$$\mathrm{val}_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1)) = \mathrm{val}_{\boldsymbol{\beta}}(\varepsilon_0 \cdots \varepsilon_{n-2}(\varepsilon_{n-1} - 1)) + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1))}{\prod_{k=0}^{n-1} \beta_k}$$

$$= 1 - \frac{1}{\prod_{k=0}^{n-1} \beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1))}{\prod_{k=0}^{n-1} \beta_k}.$$

Hence, it is sufficient to now prove that $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1)) = 1$. Again, if $d^*_{\boldsymbol{\beta}^{(n)}}(1) = d_{\boldsymbol{\beta}^{(n)}}(1)$ then it is immediate, otherwise, there exists $m \in \mathbb{N}_{\geq 1}$ such that

$$\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1)) = 1 - \frac{1}{\prod_{k=0}^{m-1} \beta_{n+k}} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n+m)}}(d^*_{\boldsymbol{\beta}^{(n+m)}}(1))}{\prod_{k=0}^{m-1} \beta_{n+k}}.$$

We get

$$\mathrm{val}_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1)) = 1 - \frac{1}{\prod_{k=0}^{n+m-1} \beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n+m)}}(d^*_{\boldsymbol{\beta}^{(n+m)}}(1))}{\prod_{k=0}^{n+m-1} \beta_k}.$$

The result follows by iterating the reasoning since either we have an equality at one step or we conclude since $\prod_{n\in\mathbb{N}} \beta_n = +\infty$. ■

Proposition 2.3.30. *If a is an infinite word over $\mathbb{N}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) < 1$, then $a <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. Furthermore, $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of all real numbers in $[0, 1]$.*

Proof. If $d_{\boldsymbol{\beta}}(1)$ is infinite then the result follows from Corollary 2.3.18. Thus, we suppose that there exists $k \in \mathbb{N}_{\geq 1}$ such that $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{k-1}$ and $\varepsilon_{k-1} > 0$.

First, let $a \in \mathbb{N}^{\mathbb{N}}$ be such that $\mathrm{val}_{\boldsymbol{\beta}}(a) < 1$ and suppose to the contrary that $a \geq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. By Corollary 2.3.18, $a <_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. Then $a_0 \cdots a_{k-2} = \varepsilon_0 \cdots \varepsilon_{k-2}$, $a_{k-1} = \varepsilon_{k-1} - 1$ and $\sigma^k(a) \geq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(k)}}(1)$. Since

$$\begin{aligned}
\mathrm{val}_{\boldsymbol{\beta}}(a) &= \sum_{n=0}^{k-2} \frac{\varepsilon_n}{\prod_{i=0}^{n} \beta_i} + \frac{\varepsilon_{k-1} - 1}{\prod_{i=0}^{k-1} \beta_i} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k)}}\big(\sigma^k(a)\big)}{\prod_{i=0}^{k-1} \beta_i} \\
&= 1 - \frac{1}{\prod_{i=0}^{k-1} \beta_i} \Big(1 - \mathrm{val}_{\boldsymbol{\beta}^{(k)}}\big(\sigma^k(a)\big)\Big),
\end{aligned}$$

we get that $\mathrm{val}_{\boldsymbol{\beta}^{(k)}}\big(\sigma^k(a)\big) < 1$. By Corollary 2.3.18 again, $\sigma^k(a) <_{\mathrm{lex}} d_{\boldsymbol{\beta}^{(k)}}(1)$. Therefore $d_{\boldsymbol{\beta}^{(k)}}(1)$ must be finite and we obtain that $a = d^*_{\boldsymbol{\beta}}(1)$ by iterating the reasoning. But then $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$, a contradiction.

We now turn to the second part. Suppose that $a \in \mathbb{N}^{\mathbb{N}}$ does not end in 0^ω and is such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \leq 1$. Our aim is to show that $a \leq_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. We know from Corollary 2.3.18 that $a \leq_{\mathrm{lex}} d_{\boldsymbol{\beta}}(1)$. Now, suppose to the contrary that

$a >_{\rm lex} d^*_{\boldsymbol{\beta}}(1)$. Then $a_0 \cdots a_{k-2} = \varepsilon_0 \cdots \varepsilon_{k-2}$, $a_{k-1} = \varepsilon_{k-1} - 1$, and $\sigma^k(a) >_{\rm lex} d^*_{\boldsymbol{\beta}^{(k)}}(1)$. As in the first part of the proof, we obtain that $\mathrm{val}_{\boldsymbol{\beta}^{(k)}}(\sigma^k(a)) \le 1$ and that $d_{\boldsymbol{\beta}^{(k)}}(1)$ must be finite. By iterating the reasoning, we obtain that $a = d^*_{\boldsymbol{\beta}}(1)$, a contradiction. ■

2.3.4 Greedy admissible sequences

In this section, we generalize Theorem 1.4.15, namely Parry's theorem, to Cantor bases by characterizing greedy $\boldsymbol{\beta}$-admissible sequences.

Lemma 2.3.31. *Let a be an infinite word over $\mathbb{N}$ and for each $n \in \mathbb{N}$, let $b^{(n)}$ be a $\boldsymbol{\beta}^{(n)}$-representation of 1. Suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) \le_{\rm lex} b^{(n)}$. Then for all $k, l, m, n \in \mathbb{N}$ with $l \ge 1$, the following implication holds:*

$$\begin{gathered} a_k \cdots a_{k+l-1} <_{\rm lex} b^{(n)}_m \cdots b^{(n)}_{m+l-1} \\ \Longrightarrow \\ \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(a_k \cdots a_{k+l-1}) \le \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(b^{(n)}_m \cdots b^{(n)}_{m+l-1}). \end{gathered} \tag{2.6}$$

Consequently, for all $k, m, n \in \mathbb{N}$, the following implication holds:

$$\sigma^k(a) <_{\rm lex} \sigma^m(b^{(n)}) \implies \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(\sigma^k(a)) \le \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(\sigma^m(b^{(n)})). \tag{2.7}$$

Proof. Proceed by induction on l. The base case $l = 1$ is clear. Let $l \ge 2$ and suppose that for all $l' < l$ and all $k, m, n \in \mathbb{N}$, the implication (2.6) is true. Now let $k, m, n \in \mathbb{N}$ and suppose that $a_k \cdots a_{k+l-1} <_{\rm lex} b^{(n)}_m \cdots b^{(n)}_{m+l-1}$. Two cases are possible.

Case 1: $a_k = b^{(n)}_m$. Then $a_{k+1} \cdots a_{k+l-1} <_{\rm lex} b^{(n)}_{m+1} \cdots b^{(n)}_{m+l-1}$ and by induction hypothesis, we obtain that

$$\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+l-1}) \le \mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(n)}_{m+1} \cdots b^{(n)}_{m+l-1}).$$

Therefore

$$\begin{aligned} \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(a_k \cdots a_{k+l-1}) &= \frac{a_k}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1} \cdots a_{k+l-1})}{\beta_k} \\ &\le \frac{b^{(n)}_m}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b^{(n)}_{m+1} \cdots b^{(n)}_{m+l-1})}{\beta_k} \\ &= \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(b^{(n)}_m \cdots b^{(n)}_{m+l-1}). \end{aligned}$$

Case 2: $a_k < b^{(n)}_m$. Since $\sigma^{k+1}(a) \le_{\rm lex} b^{(k+1)}$ by hypothesis, we have

$$a_{k+1} \cdots a_{k+l-1} \le_{\rm lex} b^{(k+1)}_0 \cdots b^{(k+1)}_{l-2}.$$

By induction hypothesis,

$$\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1}\cdots a_{k+l-1}) \leq \mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b_0^{(k+1)}\cdots b_{l-2}^{(k+1)}) \leq 1.$$

Then

$$\begin{aligned}\mathrm{val}_{\boldsymbol{\beta}^{(k)}}(a_k\cdots a_{k+l-1}) &= \frac{a_k}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(a_{k+1}\cdots a_{k+l-1})}{\beta_k} \\ &\leq \frac{b_m^{(n)}-1}{\beta_k} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(k+1)}}(b_0^{(k+1)}\cdots b_{l-2}^{(k+1)})}{\beta_k} \\ &\leq \mathrm{val}_{\boldsymbol{\beta}^{(k)}}(b_m^{(n)}\cdots b_{m+l-1}^{(n)}).\end{aligned}$$

Thus, the implication (2.6) is proved. The implication (2.7) immediately follows. ■

Lemma 2.3.32. *Let a be an infinite word over $\mathbb{N}$ and for each $n \in \mathbb{N}$, let $b^{(n)}$ be a $\boldsymbol{\beta}^{(n)}$-representation of 1. Suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\mathrm{lex}} b^{(n)}$. Then for all $n \in \mathbb{N}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$ unless there exists $l \in \mathbb{N}_{\geq 1}$ such that*

- $b^{(n)} = b_0^{(n)}\cdots b_{l-1}^{(n)}$ *with* $b_{l-1}^{(n)} > 0$
- $a_n a_{n+1}\cdots a_{n+l-1} = b_0^{(n)}\cdots b_{l-2}^{(n)}(b_{l-1}^{(n)}-1)$
- $\mathrm{val}_{\boldsymbol{\beta}^{(n+l)}}(\sigma^{n+l}(a)) = 1$

in which case $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) = 1$.

Proof. Let $n \in \mathbb{N}$. By hypothesis, $\sigma^n(a) <_{\mathrm{lex}} b^{(n)}$. So there exists $l \in \mathbb{N}_{\geq 1}$ such that $a_n\cdots a_{n+l-2} = b_0^{(n)}\cdots b_{l-2}^{(n)}$ and $a_{n+l-1} < b_{l-1}^{(n)}$. By hypothesis, we also have $\sigma^{n+l}(a) <_{\mathrm{lex}} b^{(n+l)}$. We get from Lemma 2.3.31 that

$$\mathrm{val}_{\boldsymbol{\beta}^{(n+l)}}(\sigma^{n+l}(a)) \leq \mathrm{val}_{\boldsymbol{\beta}^{(n+l)}}(b^{(n+l)}) = 1.$$

Then

$$\begin{aligned}\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) &= \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(a_n\cdots a_{n+l-2}) + \frac{a_{n+l-1}}{\prod_{i=n}^{n+l-1}\beta_i} + \frac{\mathrm{val}_{\boldsymbol{\beta}^{(n+l)}}(\sigma^{n+l}(a))}{\prod_{i=n}^{n+l-1}\beta_i} \\ &\leq \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(b_0^{(n)}\cdots b_{l-2}^{(n)}) + \frac{b_{l-1}^{(n)}-1}{\prod_{i=n}^{n+l-1}\beta_i} + \frac{1}{\prod_{i=n}^{n+l-1}\beta_i} \\ &= \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(b_0^{(n)}\cdots b_{l-1}^{(n)})\end{aligned}$$

$$\leq 1.$$

Moreover, the equality holds throughout if and only if $b^{(n)} = b_0^{(n)} \cdots b_{l-1}^{(n)}$, $a_{n+l-1} = b_{l-1}^{(n)} - 1$ and $\mathrm{val}_{\boldsymbol{\beta}^{(n+l)}}(\sigma^{n+l}(a)) = 1$. The conclusion follows. ∎

The following theorem generalizes Parry's theorem for real bases.

Theorem 2.3.33. *An infinite word a over $\mathbb{N}$ belongs to $D_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. In view of Corollary 2.3.14, it suffices to show that the following two assertions are equivalent.

1. For all $n \in \mathbb{N}$, $\mathrm{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a)) < 1$.
2. For all $n \in \mathbb{N}$, $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

The fact that (1) implies (2) follows from Proposition 2.3.30. Since any quasi-greedy expansion of 1 is infinite, we obtain that (2) implies (1) by Proposition 2.3.29 and Lemma 2.3.32. ∎

Example 2.3.34. Let $\boldsymbol{\beta} = (\overline{3, \varphi, \varphi})$ be the alternate base already studied in Examples 2.3.8 and 2.3.24. The sequence $a = 210(110)^\omega$ is the greedy $\boldsymbol{\beta}$-expansion of some $x \in (0,1)$. In fact, since $d^*_{\boldsymbol{\beta}^{(0)}}(1) = (210)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (102)^\omega$ and $d^*_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega$, by Theorem 2.3.33, there exists $x \in [0,1)$ such that $a = d_{\boldsymbol{\beta}}(x)$. We can compute that $a = d_{\boldsymbol{\beta}}(\mathrm{val}_{\boldsymbol{\beta}}(a)) = d_{\boldsymbol{\beta}}\big(\frac{19+9\sqrt{5}}{3(7+3\sqrt{5})}\big)$.

We obtain a corollary characterizing the greedy $\boldsymbol{\beta}$-expansions of a real number x in the interval $[0,1]$ among all its $\boldsymbol{\beta}$-representations.

Corollary 2.3.35. *A $\boldsymbol{\beta}$-representation a of some real number $x \in [0,1]$ is its greedy $\boldsymbol{\beta}$-expansion if and only if for all $n \in \mathbb{N}_{\geq 1}$, $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. Let $a \in \mathbb{N}^{\mathbb{N}}$ be such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \in [0,1]$. From Theorem 2.3.33, $\sigma(a)$ belongs to $D_{\boldsymbol{\beta}^{(1)}}$ if and only if for all $n \in \mathbb{N}_{\geq 1}$, $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. The conclusion then follows from Proposition 2.3.13. ∎

Example 2.3.36. Consider $\boldsymbol{\beta} = \big(\overline{\frac{16+5\sqrt{10}}{9}, 9}\big)$. Then $d_{\boldsymbol{\beta}}(1) = d^*_{\boldsymbol{\beta}}(1) = 34(27)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 90^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 834(27)^\omega$. For all $m \in \mathbb{N}_{\geq 1}$, we have

$\sigma^{2m}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ and $\sigma^{2m-1}(34(27)^\omega) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(1)}}(1)$ as prescribed by Corollary 2.3.35.

In comparison with the β-expansion theory, considering a Cantor base $\boldsymbol{\beta}$ and an infinite word a over $\mathbb{N}$, Corollary 2.3.35 does not give a purely combinatorial condition to check whether a is the greedy $\boldsymbol{\beta}$-expansion of 1. We will see in Chapter 3 that even though an improvement of this result in the context of alternate bases can be proved, a purely combinatorial condition cannot exist.

2.3.5 The greedy $\boldsymbol{\beta}$-shift

Definition 2.3.37. Let $S_{\boldsymbol{\beta}}$ denote the topological closure of $D_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words, that is, $S_{\boldsymbol{\beta}} = \overline{D_{\boldsymbol{\beta}}}$.

Proposition 2.3.38. *Let $a, b \in S_{\boldsymbol{\beta}}$.*

1. *If $a <_{\text{lex}} b$ then $\operatorname{val}_{\boldsymbol{\beta}}(a) \le \operatorname{val}_{\boldsymbol{\beta}}(b)$.*
2. *If $\operatorname{val}_{\boldsymbol{\beta}}(a) < \operatorname{val}_{\boldsymbol{\beta}}(b)$ then $a <_{\text{lex}} b$.*

Proof. Consider two sequences $(a^{(n)})_{n\in\mathbb{N}}$ and $(b^{(n)})_{n\in\mathbb{N}}$ of $D_{\boldsymbol{\beta}}$ such that $\lim_{n\to+\infty} a^{(n)} = a$ and $\lim_{n\to+\infty} b^{(n)} = b$. Suppose that $a <_{\text{lex}} b$. Then there exists $k \in \mathbb{N}_{\ge 1}$ such that $a_0 \cdots a_{k-1} = b_0 \cdots b_{k-1}$ and $a_k < b_k$. By definition of the prefix distance, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $a_0^{(n)} \cdots a_k^{(n)} = a_0 \cdots a_k$ and $b_0^{(n)} \cdots b_k^{(n)} = b_0 \cdots b_k$. Therefore, for all $n \ge N$, we have $a^{(n)} <_{\text{lex}} b^{(n)}$, and then by Proposition 2.3.17, $\operatorname{val}_{\boldsymbol{\beta}}(a^{(n)}) < \operatorname{val}_{\boldsymbol{\beta}}(b^{(n)})$. Since the function $\operatorname{val}_{\boldsymbol{\beta}}$ is continuous, by letting k tend to infinity, we obtain $\operatorname{val}_{\boldsymbol{\beta}}(a) \le \operatorname{val}_{\boldsymbol{\beta}}(b)$. This proves the first item. The second item follows immediately. ∎

Thanks to the generalization of Parry's theorem in Theorem 2.3.33, we get the following combinatorial characterization of the set $S_{\boldsymbol{\beta}}$.

Proposition 2.3.39. *An infinite word a over $\mathbb{N}$ belongs to $S_{\boldsymbol{\beta}}$ if and only if for all $n \in \mathbb{N}$, $\sigma^n(a) \le_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.*

Proof. Suppose that $a \in S_{\boldsymbol{\beta}}$. Then there exists a sequence $(a^{(k)})_{k\in\mathbb{N}}$ of $D_{\boldsymbol{\beta}}$ converging to a. By Theorem 2.3.33, for all $k, n \in \mathbb{N}$, we have $\sigma^n(a^{(k)}) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. By letting k tend to infinity, we get that for all $n \in \mathbb{N}$, $\sigma^n(a) \le_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Conversely, suppose that for all $n \in \mathbb{N}$, $\sigma^n(a) \le_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. For each $k \in \mathbb{N}$, let $a^{(k)} = a_0 \cdots a_k 0^\omega$. Then $\lim_{k\to+\infty} a^{(k)} = a$ and for all $k, n \in \mathbb{N}$, $\sigma^n(a^{(k)}) \le_{\text{lex}} \sigma^n(a) \le_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Since $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is infinite, for all $k, n \in \mathbb{N}$, $\sigma^n(a^{(k)}) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. By Theorem 2.3.33, we deduce that for all $k \in \mathbb{N}$, $a^{(k)} \in D_{\boldsymbol{\beta}}$. Therefore $a \in S_{\boldsymbol{\beta}}$. ■

Corollary 2.3.40. *For all $a \in S_{\boldsymbol{\beta}}$, we have $\sigma(a) \in S_{\boldsymbol{\beta}^{(1)}}$.*

Example 2.3.41. Consider the alternate base $\boldsymbol{\beta} = \overline{\left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)}$. Since we have $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$, the infinite word 020^ω belongs to the set $S_{\boldsymbol{\beta}^{(1)}}$. The infinite word $\sigma(020^\omega) = 20^\omega$ belongs to the set $S_{\boldsymbol{\beta}}$. However, it does not belong to the set $S_{\boldsymbol{\beta}^{(1)}}$.

As illustrated in the previous example, the closed set $S_{\boldsymbol{\beta}}$ is not shift-invariant. Let us define another closed set which will be proved to be shift-invariant.

Definition 2.3.42. We set

$$\Delta_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} D_{\boldsymbol{\beta}^{(n)}} \quad \text{and} \quad \Sigma_{\boldsymbol{\beta}} = \overline{\Delta_{\boldsymbol{\beta}}}.$$

Proposition 2.3.43. *The sets $\Delta_{\boldsymbol{\beta}}$ and $\Sigma_{\boldsymbol{\beta}}$ are both shift-invariant.*

Proof. Let a be an infinite word over $\mathbb{N}$ and $n \in \mathbb{N}$. It follows from Corollary 2.3.14 that if $a \in D_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in D_{\boldsymbol{\beta}^{(n+1)}}$. Then, it is easily seen that if $a \in S_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in S_{\boldsymbol{\beta}^{(n+1)}}$. ■

In view of Proposition 2.3.43, the subset $\Sigma_{\boldsymbol{\beta}}$ of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ is a subshift, which we call the *greedy $\boldsymbol{\beta}$-shift*.

Proposition 2.3.44. *We have* $\mathrm{Fac}(S_{\boldsymbol{\beta}}) = \mathrm{Fac}(D_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Delta_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$.

Proof. By definition, we have $\mathrm{Fac}(S_{\boldsymbol{\beta}}) = \mathrm{Fac}(D_{\boldsymbol{\beta}})$, $\mathrm{Fac}(\Delta_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ and $\mathrm{Fac}(D_{\boldsymbol{\beta}}) \subseteq \mathrm{Fac}(\Delta_{\boldsymbol{\beta}})$. Let us show that $\mathrm{Fac}(D_{\boldsymbol{\beta}}) \supseteq \mathrm{Fac}(\Delta_{\boldsymbol{\beta}})$. Let $f \in \mathrm{Fac}(\Delta_{\boldsymbol{\beta}})$. There exist $n \in \mathbb{N}$ and $a \in D_{\boldsymbol{\beta}^{(n)}}$ such that $f \in \mathrm{Fac}(a)$. It follows from Corollary 2.3.14 that $0^n a$ belongs to $D_{\boldsymbol{\beta}}$. Therefore, $f \in \mathrm{Fac}(D_{\boldsymbol{\beta}})$. ■

We define sets of finite words $X_{\boldsymbol{\beta},n}$ for $n \in \mathbb{N}_{\ge 1}$ as follows.

Definition 2.3.45. If $d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 \cdots$ then we let

$$X_{\boldsymbol{\beta},n} = \{t_0 \cdots t_{n-2} s : s \in [\![0, t_{n-1} - 1]\!]\}.$$

Note that $X_{\boldsymbol{\beta},n}$ is empty if and only if $t_{n-1} = 0$.

Proposition 2.3.46. *We have*

$$D_{\boldsymbol{\beta}} = \bigcup_{n_0 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},n_0} \Bigg(\bigcup_{n_1 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta}^{(n_0)},n_1} \Bigg(\bigcup_{n_2 \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta}^{(n_0+n_1)},n_2} \Bigg(\quad \cdots \quad \Bigg)\Bigg)\Bigg).$$

Proof. For the sake of conciseness, we let $X_{\boldsymbol{\beta}}$ denote the right-hand set of the equality. For $n \in \mathbb{N}$, write $d^*_{\boldsymbol{\beta}^{(n)}}(1) = t_0^{(n)} t_1^{(n)} \cdots$.

Let $a \in D_{\boldsymbol{\beta}}$. By Theorem 2.3.33, for all $n \in \mathbb{N}$, $\sigma^n(a) < d^*_{\boldsymbol{\beta}^{(n)}}(1)$. In particular, $a < d^*_{\boldsymbol{\beta}}(1)$. Thus, there exist $n_0 \in \mathbb{N}_{\geq 1}$ such that $t^{(0)}_{n_0-1} > 0$ and $s_0 \in [\![0, t^{(0)}_{n_0-1} - 1]\!]$ such that $a = t_0 \cdots t_{n_0-2} s_0 \sigma^{n_0}(a)$. Next, we also have $\sigma^{n_0}(a) < d^*_{\boldsymbol{\beta}^{(n_0)}}(1)$. Then there exist $n_1 \in \mathbb{N}_{\geq 1}$ such that $t^{(n_0)}_{n_1-1} > 0$ and $s_1 \in [\![0, t^{(n_0)}_{n_1-1} - 1]\!]$ such that $\sigma^{n_0}(a) = t_0^{(n_0)} \cdots t^{(n_0)}_{n_1-2} s_1 \sigma^{n_0+n_1}(a)$. We get that $a \in X_{\boldsymbol{\beta}}$ by iterating the process.

Now, let $a \in X_{\boldsymbol{\beta}}$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of $\mathbb{N}_{\geq 1}$ such that $a = u_0 u_1 u_2 \cdots$ where for all $k \in \mathbb{N}$, $u_k \in X_{\boldsymbol{\beta}^{(n_0+\cdots n_{k-1})},n_k}$. By Theorem 2.3.33, in order to prove that $a \in D_{\boldsymbol{\beta}}$, it suffices to show that for all $n \in \mathbb{N}$, $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Let thus $n \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and finite words x and y such that $u_k = xy$, $y \neq \varepsilon$ and $\sigma^n(a) = y u_{k+1} u_{k+2} \cdots$. Then $n = n_0 + \cdots + n_{k-1} + |x|$ and $\sigma^n(a) <_{\mathrm{lex}} \sigma^{|x|}\big(d^*_{\boldsymbol{\beta}^{(n_0+\cdots n_{k-1})}}(1)\big)$. If $x = \varepsilon$ then we obtain $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n_0+\cdots n_{k-1})}}(1) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$. Otherwise it follows from Corollary 2.3.35 that $\sigma^{|x|}\big(d_{\boldsymbol{\beta}^{(n_0+\cdots n_{k-1})}}(1)\big) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n_0+\cdots n_{k-1}+|x|)}}(1) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$, hence we get $\sigma^n(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ as well. ■

Corollary 2.3.47. *We have* $D_{\boldsymbol{\beta}} = \bigcup_{n \in \mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},n} D_{\boldsymbol{\beta}^{(n)}}$.

Corollary 2.3.48. *Any prefix of* $d^*_{\boldsymbol{\beta}}(1)$ *belongs to* $\mathrm{Pref}(D_{\boldsymbol{\beta}})$.

Proof. Write $d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 t_2 \cdots$ and let $n \in \mathbb{N}_{\geq 1}$. Since $d^*_{\boldsymbol{\beta}}(1)$ is infinite, there exists $k > n$ such that $t_{k-1} > 0$. Choose the least such k and let $s \in [\![0, t_{k-1} - 1]\!]$. Then $t_0 \cdots t_{n-1} 0^{k-n-1} s$ belongs to $X_{\boldsymbol{\beta},k}$. The conclusion follows from Proposition 2.3.46. ■

Remark 2.3.49. The analogue of Definition 2.3.45 in the real base case splits the definition of the set depending on the finiteness of the greedy $\boldsymbol{\beta}$-expansion of 1 (see for example [Lot02, Chapter 7]). In this text, for the sake of simplicity in the proofs, we did not split in the same way since more than one expansion is involved while computing greedy admissible $\boldsymbol{\beta}$-expansions.

We get an equivalent definition of the quasi-greedy $\boldsymbol{\beta}$-expansion of 1.

Proposition 2.3.50. *We have*

$$d^*_{\boldsymbol{\beta}}(1) = \lim_{x \to 1^-} d_{\boldsymbol{\beta}}(x). \tag{2.8}$$

Proof. Let $t_0 t_1 \cdots$ denote $d^*_{\boldsymbol{\beta}}(1)$. By Corollary 2.3.48, for all $n \in \mathbb{N}$, the word $t_0 \cdots t_{n-1} 0^\omega$ is the greedy $\boldsymbol{\beta}$-expansion of a real number $x_n \in [0,1)$. For all $n \in \mathbb{N}$, we have $x_{n+1} = x_n + \frac{t_n}{\prod_{k=0}^{n} \beta_k}$. Hence, we have $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$. There exists $r \in [0,1]$ such that $\lim_{n\to+\infty} x_n = r$. We now prove that $r = 1$. Suppose $r < 1$. For all $x \in (r,1)$, we have $x > x_n$ for all $n \in \mathbb{N}$. By Proposition 2.3.17, we have $d_{\boldsymbol{\beta}}(x) >_{\text{lex}} t_0 \cdots t_{n-1} 0^\omega$ for all $n \in \mathbb{N}$. This is absurd since by Theorem 2.3.33, we have $d_{\boldsymbol{\beta}}(x) < d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 \cdots$. Hence, we have $\lim_{n\to+\infty} x_n = 1$. Now, consider $x \in [0,1)$ and let N denote the maximal index $n \in \mathbb{N}$ such that $x \ge x_n$. Let a denote a $\boldsymbol{\beta}^{(N)}$-representation of $(x - x_N) \prod_{k=0}^{N-1} \beta_k$. The infinite word $t_0 \cdots t_{N-1} a$, is a $\boldsymbol{\beta}$-representation of x. By Proposition 2.3.15, we have $d_{\boldsymbol{\beta}}(x) \ge_{\text{lex}} t_0 \cdots t_{N-1} a$. Moreover, by Theorem 2.3.33, we have $d_{\boldsymbol{\beta}}(x) <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$. We obtain that the length-N prefix of $d_{\boldsymbol{\beta}}(x)$ is $t_0 \cdots t_{N-1}$. Hence, the result follows. ∎

In Section 2.3.1, we made a choice of definition for the greedy $\boldsymbol{\beta}$-expansion of 1 and, in Section 2.3.3, we defined $d^*_{\boldsymbol{\beta}}(1)$ accordingly. One could define the quasi-greedy $\boldsymbol{\beta}$-expansion of 1 immediately as (2.8).

2.4 Lazy $\boldsymbol{\beta}$-expansions

This section is concerned with the combinatorial study of lazy $\boldsymbol{\beta}$-expansions of real numbers. Recall that the lazy real base expansions had been studied only in the dynamical point of view and not in the combinatorial one. Therefore, even if both are somehow related, results of this section can be considered as unprecedented.

2.4.1 Definition of $x_{\boldsymbol{\beta}}$

Let $\bigotimes_{n\in\mathbb{N}}[\![0,\lceil\beta_n\rceil-1]\!]$ denote the set of infinite words $a\in A_{\boldsymbol{\beta}}^{\mathbb{N}}$ such that, for all $n\in\mathbb{N}$, the letter a_n belongs to $[\![0,\lceil\beta_n\rceil-1]\!]$ (see Remark 1.2.8). We now define (if it exists) the greatest real number that has a $\boldsymbol{\beta}$-representation in $\bigotimes_{n\in\mathbb{N}}[\![0,\lceil\beta_n\rceil-1]\!]$.

Definition 2.4.1. Let

$$x_{\boldsymbol{\beta}}=\sum_{n\in\mathbb{N}}\frac{\lceil\beta_n\rceil-1}{\prod_{k=0}^{n}\beta_k}. \tag{2.9}$$

Either this series converges or $x_{\boldsymbol{\beta}}=+\infty$. If $x_{\boldsymbol{\beta}}<+\infty$, then $x_{\boldsymbol{\beta}}$ is the greatest real number that has a $\boldsymbol{\beta}$-representation in $\bigotimes_{n\in\mathbb{N}}[\![0,\lceil\beta_n\rceil-1]\!]$.

Example 2.4.2. Consider the sequence $\boldsymbol{\beta}=(\frac{n+2}{n+1})_{n\in\mathbb{N}}$. The sequence $\boldsymbol{\beta}$ is a Cantor base since $\prod_{n\in\mathbb{N}}\frac{n+2}{n+1}=+\infty$. We get

$$x_{\boldsymbol{\beta}}=\sum_{n\in\mathbb{N}}\frac{1}{\prod_{k=0}^{n}\frac{k+2}{k+1}}=\sum_{n\in\mathbb{N}}\frac{1}{n+2}=+\infty.$$

The following result gives a sufficient condition in order to have $x_{\boldsymbol{\beta}}<+\infty$.

Proposition 2.4.3. *Any Cantor base $\boldsymbol{\beta}$ that takes only finitely many values has a finite corresponding $x_{\boldsymbol{\beta}}$.*

Proof. Consider a Cantor base $\boldsymbol{\beta}$ that takes only finitely many values. There exist $m,M\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, we have $\beta_m\le\beta_n\le\beta_M$. We get

$$x_{\boldsymbol{\beta}}=\sum_{n\in\mathbb{N}}\frac{\lceil\beta_n\rceil-1}{\prod_{k=0}^{n}\beta_k}\le\sum_{n\in\mathbb{N}}\frac{\lceil\beta_M\rceil-1}{(\beta_m)^{n+1}}=\frac{\lceil\beta_M\rceil-1}{\beta_m-1}.$$

■

Corollary 2.4.4. *Any alternate base $\boldsymbol{\beta}$ has a finite corresponding $x_{\boldsymbol{\beta}}$.*

We now link the values $x_{\boldsymbol{\beta}^{(n)}}$ and $x_{\boldsymbol{\beta}^{(n+1)}}$ for all $n\in\mathbb{N}$.

Proposition 2.4.5. *Let $n\in\mathbb{N}$. Suppose that $x_{\boldsymbol{\beta}^{(n)}}<+\infty$. We have*

$$x_{\boldsymbol{\beta}^{(n)}}=\frac{x_{\boldsymbol{\beta}^{(n+1)}}+\lceil\beta_n\rceil-1}{\beta_n}. \tag{2.10}$$

Proof. Let $n \in \mathbb{N}$ and suppose that $x_{\boldsymbol{\beta}^{(n)}} < +\infty$. We have

$$x_{\boldsymbol{\beta}^{(n)}} = \sum_{m \in \mathbb{N}} \frac{\lceil \beta_{n+m} \rceil - 1}{\prod_{k=0}^{m} \beta_{n+k}}.$$

Therefore, we obtain

$$\begin{aligned} x_{\boldsymbol{\beta}^{(n)}} &= \frac{\lceil \beta_n \rceil - 1}{\beta_n} + \sum_{m=1}^{+\infty} \frac{\lceil \beta_{n+m} \rceil - 1}{\prod_{k=0}^{m} \beta_{n+k}} \\ &= \frac{\lceil \beta_n \rceil - 1}{\beta_n} + \frac{x_{\boldsymbol{\beta}^{(n+1)}}}{\beta_n}. \end{aligned}$$

The conclusion follows. ∎

Since the greedy algorithm converges on $[0,1]$ (see Section 2.3.1), it can be easily seen that $x_{\boldsymbol{\beta}} \geq 1$. In fact, since $d_{\boldsymbol{\beta}}(1)$ belongs to the set of words $\bigotimes_{n \in \mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$, we have

$$x_{\boldsymbol{\beta}} = \sum_{n \in \mathbb{N}} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k} \geq \sum_{n \in \mathbb{N}} \frac{\varepsilon_n(1)}{\prod_{k=0}^{n} \beta_k} = 1.$$

Lemma 2.4.6. *We have $x_{\boldsymbol{\beta}} = 1$ if and only if $\beta_n \in \mathbb{N}_{\geq 2}$ for all $n \in \mathbb{N}$.*

Proof. By (2.10), we have $x_{\boldsymbol{\beta}} = 1$ if and only if

$$1 = \frac{x_{\boldsymbol{\beta}^{(1)}} + \lceil \beta_0 \rceil - 1}{\beta_0}.$$

However, we have $x_{\boldsymbol{\beta}^{(1)}} \geq 1$ so $x_{\boldsymbol{\beta}^{(1)}} + \lceil \beta_0 \rceil - 1 \geq \lceil \beta_0 \rceil$. Hence, we get $x_{\boldsymbol{\beta}} = 1$ if and only if $x_{\boldsymbol{\beta}^{(1)}} = 1$ and $\lceil \beta_0 \rceil = \beta_0$. The conclusion follows by induction. ∎

Example 2.4.7. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ already widely studied in Section 2.3. We get $x_{\boldsymbol{\beta}} = \frac{5+7\sqrt{13}}{18} \simeq 1.67$ and $x_{\boldsymbol{\beta}^{(1)}} = \frac{2+\sqrt{13}}{3} \simeq 1.86$.

Example 2.4.8. [2] Let $\alpha, \beta > 1$ and let $\boldsymbol{\beta} = (\alpha, \beta, \beta, \alpha, \beta, \alpha, \alpha, \beta, \ldots)$ be the Thue-Morse Cantor base on $\{\alpha, \beta\}$ defined as (2.2). For all $n \geq 1$, let

$$x_n = \sum_{m=0}^{2^n - 1} \frac{\lceil \beta_m \rceil - 1}{\prod_{k=0}^{m} \beta_k}.$$

[2] I thank Jean-Pierre Schneiders for suggesting the way to approximate the value of $x_{\boldsymbol{\beta}}$ in this example.

We get $x_{\boldsymbol{\beta}} = \lim_{n\to+\infty} x_n$. Similarly, let $\overline{\boldsymbol{\beta}}$ denote the Cantor base $\overline{\boldsymbol{\beta}} = (\overline{\beta_n})_{n\in\mathbb{N}}$ where $\overline{\alpha} = \beta$ and $\overline{\beta} = \alpha$. We get $\overline{\boldsymbol{\beta}} = (\beta, \alpha, \alpha, \beta, \alpha, \beta, \beta, \alpha, \ldots)$. For all $n \geq 1$, write

$$y_n = \sum_{m=0}^{2^n-1} \frac{\lceil \overline{\beta_m} \rceil - 1}{\prod_{k=0}^{m} \overline{\beta_k}}.$$

By definition of the Thue-Morse sequence (see Definition 1.2.15), for all $n \in \mathbb{N}$ we have

$$(\beta_{2^n}, \beta_{2^n+1}, \ldots, \beta_{2^{n+1}-1}) = (\overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_{2^n-1}}).$$

Moreover, for all $n \geq 1$, the sequence $(\beta_0, \ldots, \beta_{2^n-1})$ has the same number of α and β. We get $\prod_{k=0}^{2^n-1} \beta_k = (\alpha\beta)^{2^{n-1}}$. Hence, we have

$$\begin{cases} x_1 = \frac{\lceil \alpha \rceil - 1}{\alpha} + \frac{\lceil \beta \rceil - 1}{\alpha\beta}, \\ y_1 = \frac{\lceil \beta \rceil - 1}{\beta} + \frac{\lceil \alpha \rceil - 1}{\beta\alpha}, \\ x_{n+1} = x_n + \frac{1}{(\alpha\beta)^{2^{n-1}}} y_n, \quad \forall n \geq 1 \\ y_{n+1} = y_n + \frac{1}{(\alpha\beta)^{2^{n-1}}} x_n, \quad \forall n \geq 1. \end{cases}$$

That is, for all $n \geq 1$, we have

$$v_{n+1} = A_n v_n$$

where,

$$v_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} 1 & \frac{1}{(\alpha\beta)^{2^{n-1}}} \\ \frac{1}{(\alpha\beta)^{2^{n-1}}} & 1 \end{pmatrix}.$$

For all $n \geq 1$, the eigenvalues of the matrix A_n are $1+\frac{1}{(\alpha\beta)^{2^{n-1}}}$ and $1-\frac{1}{(\alpha\beta)^{2^{n-1}}}$ of eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. Moreover, we have

$$v_1 = \tfrac{x_1+y_1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{x_1-y_1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We obtain

$$\begin{aligned} v_{n+1} &= A_n A_{n-1} \cdots A_1 v_1 \\ &= \tfrac{x_1+y_1}{2} A_n A_{n-1} \cdots A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{x_1-y_1}{2} A_n A_{n-1} \cdots A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \tfrac{x_1+y_1}{2} \prod_{k=1}^{n} \left(1 + \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{x_1-y_1}{2} \prod_{k=1}^{n} \left(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Then, the value of $x_{\boldsymbol{\beta}}$ can be computed by

$$x_{\boldsymbol{\beta}} = \lim_{n \to +\infty} x_n = \tfrac{x_1+y_1}{2} \prod_{k \in \mathbb{N}_{\geq 1}} \big(1 + \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big) + \tfrac{x_1-y_1}{2} \prod_{k \in \mathbb{N}_{\geq 1}} \big(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big).$$

We now study the first[3] infinite product in the above formula. We have

$$\begin{aligned} &\Big(\prod_{k \in \mathbb{N}_{\geq 1}} \big(1 + \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big)\Big(\prod_{k \in \mathbb{N}_{\geq 1}} \big(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big) \\ &= \prod_{k \in \mathbb{N}_{\geq 1}} \Big(\big(1 + \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\big(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big)\Big) \\ &= \prod_{k=2}^{+\infty} \big(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big). \end{aligned}$$

Hence, we get

$$\prod_{k \in \mathbb{N}_{\geq 1}} \big(1 + \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big) = \frac{1}{1 - \frac{1}{\alpha\beta}}.$$

Then, the value of $x_{\boldsymbol{\beta}}$ can be computed by

$$x_{\boldsymbol{\beta}} = \tfrac{x_1+y_1}{2} \Big(\frac{1}{1 - \frac{1}{\alpha\beta}}\Big) + \tfrac{x_1-y_1}{2} \prod_{k \in \mathbb{N}_{\geq 1}} \big(1 - \tfrac{1}{(\alpha\beta)^{2^{k-1}}}\big).$$

In particular, by considering the Cantor base from Example 2.3.6, a computer approximation of $\prod_{k \in \mathbb{N}_{\geq 1}} \big(1 - \frac{1}{(\alpha\beta)^{2^{k-1}}}\big)$ gives 0.627941. Hence, we get $x_{\boldsymbol{\beta}} \simeq 1.73295$.

2.4.2 The lazy algorithm

If $x_{\boldsymbol{\beta}} < +\infty$ the *lazy $\boldsymbol{\beta}$-expansions* are defined. Hence, from now on, when dealing with the lazy algorithm, we consider Cantor bases $\boldsymbol{\beta} = (\beta_n)_{n \in \mathbb{N}}$ such that $x_{\boldsymbol{\beta}} < +\infty$.

In the greedy algorithm, each digit is chosen as the largest possible at the considered position. On the contrary, in the lazy algorithm, each digit is chosen as the least possible at each step.

[3]Note that the infinite product $\prod_{k \in \mathbb{N}_{\geq 1}} \big(1 - z^{2^{k-1}}\big)$ (which cannot be simplified in general) is equal to the generating function of the sequence $\big((-1)^{t_n}\big)_{n \in \mathbb{N}}$ where $t_0 t_1 t_2 \cdots$ is the Thue-Morse sequence over the alphabet $\{0, 1\}$.

Definition 2.4.9. The *lazy algorithm* is defined as follows: for $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$, if the first N digits of the lazy $\boldsymbol{\beta}$-expansion of x are given by $\xi_{\boldsymbol{\beta},0}, \ldots, \xi_{\boldsymbol{\beta},N-1}$, then the next digit $\xi_{\boldsymbol{\beta},N}$ is the least element in $[\![0, \lceil \beta_N \rceil - 1]\!]$ such that

$$\sum_{n=0}^{N} \frac{\xi_{\boldsymbol{\beta},n}}{\prod_{k=0}^{n} \beta_k} + \sum_{n=N+1}^{+\infty} \frac{\lceil \beta_n \rceil - 1}{\prod_{k=0}^{n} \beta_k} \geq x,$$

or equivalently,

$$\sum_{n=0}^{N} \frac{\xi_{\boldsymbol{\beta},n}}{\prod_{k=0}^{n} \beta_k} + \frac{x_{\boldsymbol{\beta}^{(N+1)}}}{\prod_{k=0}^{N} \beta_k} \geq x.$$

The lazy algorithm over $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ can be equivalently defined as follows:

- $\xi_{\boldsymbol{\beta},0}(x) = \lceil \beta_0 x - x_{\boldsymbol{\beta}^{(1)}} \rceil$ and $s_{\boldsymbol{\beta},0}(x) = \beta_0 x - \xi_{\boldsymbol{\beta},0}(x)$
- $\xi_{\boldsymbol{\beta},n}(x) = \lceil \beta_n s_{\boldsymbol{\beta},n-1}(x) - x_{\boldsymbol{\beta}^{(n+1)}} \rceil$ and $s_{\boldsymbol{\beta},n}(x) = \beta_n s_{\boldsymbol{\beta},n-1}(x) - \xi_{\boldsymbol{\beta},n}(x)$ for $n \in \mathbb{N}_{\geq 1}$.

The obtained $\boldsymbol{\beta}$-representation of $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is denoted by $\ell_{\boldsymbol{\beta}}(x)$ and is called the *lazy $\boldsymbol{\beta}$-expansion* of x.

As before, if the context is clear, the index $\boldsymbol{\beta}$ in the writings $\xi_{\boldsymbol{\beta},n}(x)$ and $s_{\boldsymbol{\beta},n}(x)$ are omitted.

Example 2.4.10. We continue Examples 2.3.9 and 2.4.7. The first 5 digits of $\ell_{\boldsymbol{\beta}}(\frac{35-5\sqrt{13}}{18})$ are 10212.

Any greedy $\boldsymbol{\beta}$-expansions of real numbers in $[0,1)$ and lazy $\boldsymbol{\beta}$-expansions of real numbers in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ belong to $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ and more precisely to the set of words $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$.

2.4.3 Flip greedy and get lazy

In this section greedy and lazy Cantor base expansions are compared.

Definition 2.4.11. Let $\theta_{\boldsymbol{\beta}}$ be the map defined by

$$\begin{aligned} \theta_{\boldsymbol{\beta}} \colon \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!] &\to \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!], \\ a_0 a_1 \cdots &\mapsto (\lceil \beta_0 \rceil - 1 - a_0)(\lceil \beta_1 \rceil - 1 - a_1) \cdots . \end{aligned}$$

The map $\theta_{\boldsymbol{\beta}}$ is continuous with respect to the topology induced by the prefix distance, bijective and the inverse map $\theta_{\boldsymbol{\beta}}^{-1}$ is the map $\theta_{\boldsymbol{\beta}}$ itself. For any infinite word $a \in \bigotimes_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, we get

$$\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - \mathrm{val}_{\boldsymbol{\beta}}(a). \tag{2.11}$$

Moreover, the map $\theta_{\boldsymbol{\beta}}$ is decreasing with respect to the lexicographic order, that is, for all infinite words a and b in $\bigotimes_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$, we get

$$a <_{\mathrm{lex}} b \iff \theta_{\boldsymbol{\beta}}(a) >_{\mathrm{lex}} \theta_{\boldsymbol{\beta}}(b). \tag{2.12}$$

The map $\theta_{\boldsymbol{\beta}}$ is the key of the reasoning in order to link the greedy and the lazy $\boldsymbol{\beta}$-expansions. In fact, as shown in the following result, it will allow us to "flip" the greedy expansions in order to get the lazy ones.

Proposition 2.4.12. *For all $x \in [0,1)$ and all $n \in \mathbb{N}$, we have $\xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}}-x) = \lceil\beta_n\rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)$ and $s_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) = x_{\boldsymbol{\beta}^{(n+1)}} - r_{\boldsymbol{\beta},n}(x)$. In particular, we get*

$$\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x)).$$

Proof. Consider $x \in [0,1)$. We proceed by induction on n. By (2.10), we have

$$\begin{aligned}
\xi_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}} - x) =& \lceil\beta_0(x_{\boldsymbol{\beta}} - x) - x_{\boldsymbol{\beta}^{(1)}}\rceil \\
=& \lceil\lceil\beta_0\rceil - 1 - \beta_0 x\rceil \\
=& \lceil\beta_0\rceil - 1 + \lceil-\beta_0 x\rceil \\
=& \lceil\beta_0\rceil - 1 - \lfloor\beta_0 x\rfloor \\
=& \lceil\beta_0\rceil - 1 - \varepsilon_{\boldsymbol{\beta},0}(x).
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
s_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}} - x) =& \beta_0(x_{\boldsymbol{\beta}} - x) - (\lceil\beta_0\rceil - 1 - \varepsilon_{\boldsymbol{\beta},0}(x)) \\
=& \beta_0 x_{\boldsymbol{\beta}} - (\lceil\beta_0\rceil - 1) - (\beta_0 x - \varepsilon_{\boldsymbol{\beta},0}(x)) \\
=& x_{\boldsymbol{\beta}^{(1)}} - r_{\boldsymbol{\beta},0}(x)
\end{aligned}$$

where (2.10) is used again in the last equality. By induction, for all $n \in \mathbb{N}_{\geq 1}$, we have

$$\begin{aligned}
\xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) =& \lceil\beta_n s_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) - x_{\boldsymbol{\beta}^{(n+1)}}\rceil \\
=& \lceil\beta_n(x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x)) - x_{\boldsymbol{\beta}^{(n+1)}}\rceil \\
=& \lceil\lceil\beta_n\rceil - 1 - \beta_n r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x)\rceil
\end{aligned}$$

$$
\begin{aligned}
&= \lceil \beta_n \rceil - 1 - \lfloor \beta_n r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) \rfloor \\
&= \lceil \beta_n \rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
s_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) =& \beta_n s_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x) - \xi_{\boldsymbol{\beta},n}(x_{\boldsymbol{\beta}} - x) \\
=& \beta_n (x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x)) - (\lceil \beta_n \rceil - 1 - \varepsilon_{\boldsymbol{\beta},n}(x)) \\
=& x_{\boldsymbol{\beta}^{(n+1)}} - r_{\boldsymbol{\beta},n}(x).
\end{aligned}
$$

In particular, we can conclude that $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x))$. ■

Example 2.4.13. Let $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ be the alternate base considered in Example 2.3.9. By Proposition 2.4.12, the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - \frac{-5+2\sqrt{13}}{3} = \frac{25-5\sqrt{13}}{18}$ equals $10(21)^\omega$ since $d_{\boldsymbol{\beta}}(\frac{-5+2\sqrt{13}}{3}) = 11$. This coincides with Example 2.4.10.

Example 2.4.14. We continue Examples 2.3.6 and 2.4.8 where $\boldsymbol{\beta}$ is the Thue-Morse Cantor base. By Proposition 2.4.12, the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - \frac{1}{2} \simeq 1.23295$ has 11120 as a prefix since 10001 is a prefix of $d_{\boldsymbol{\beta}}(\frac{1}{2})$.

2.4.4 First properties of lazy expansions

Thanks to Proposition 2.4.12 results from Section 2.3 on greedy $\boldsymbol{\beta}$-expansions will be translated in terms of lazy $\boldsymbol{\beta}$-expansions. The differences between the greedy and lazy $\boldsymbol{\beta}$-expansions will be highlighted in the text.

Lemma 2.4.15. *For all $n \in \mathbb{N}$, we have*

$$\sigma^n \circ \theta_{\boldsymbol{\beta}} = \theta_{\boldsymbol{\beta}^{(n)}} \circ \sigma^n$$

on $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$.

Proof. Consider $n \in \mathbb{N}$ and $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$. We have

$$
\begin{aligned}
\sigma^n \circ \theta_{\boldsymbol{\beta}}(a) &= \sigma^n\big((\lceil \beta_0 \rceil - 1 - a_0)(\lceil \beta_1 \rceil - 1 - a_1) \cdots\big) \\
&= (\lceil \beta_n \rceil - 1 - a_n)(\lceil \beta_{n+1} \rceil - 1 - a_{n+1}) \cdots \\
&= \theta_{\boldsymbol{\beta}^{(n)}}(a_n a_{n+1} \cdots) \\
&= \theta_{\boldsymbol{\beta}^{(n)}} \circ \sigma^n(a).
\end{aligned}
$$

■

Proposition 2.4.16. *For all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and all $n \in \mathbb{N}$, we have*

$$\sigma^n(\ell_{\boldsymbol{\beta}}(x)) = \ell_{\boldsymbol{\beta}^{(n)}}(s_{\boldsymbol{\beta},n-1}(x)).$$

Proof. This is a consequence of Proposition 2.4.12, Lemma 2.4.15 and Proposition 2.3.10 since for all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ we have

$$\begin{aligned}
\sigma^n(\ell_{\boldsymbol{\beta}}(x)) &= \sigma^n \circ \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)) \\
&= \theta_{\boldsymbol{\beta}^{(n)}} \circ \sigma^n(d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)) \\
&= \theta_{\boldsymbol{\beta}^{(n)}}(d_{\boldsymbol{\beta}^{(n)}}(r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x))) \\
&= \ell_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - r_{\boldsymbol{\beta},n-1}(x_{\boldsymbol{\beta}} - x)) \\
&= \ell_{\boldsymbol{\beta}^{(n)}}(s_{\boldsymbol{\beta},n-1}(x)).
\end{aligned}$$

■

Proposition 2.4.17. *Let a be an infinite word over $\mathbb{N}$ and $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. We have $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$, $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$ and for all $N \in \mathbb{N}$,*

$$\sum_{n=N+1}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} > \frac{x_{\boldsymbol{\beta}^{(N+1)}} - 1}{\prod_{k=0}^{N} \beta_k}.$$

Proof. Consider $a \in \mathbb{N}^{\mathbb{N}}$ and $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. By Proposition 2.4.12, we have $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ and $\theta_{\boldsymbol{\beta}}(a) = d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$. By Lemma 2.3.12, we get $a = \ell_{\boldsymbol{\beta}}(x)$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$, $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - x$ and for all $N \in \mathbb{N}$,

$$\sum_{n=N+1}^{+\infty} \frac{\lceil\beta_n\rceil - 1 - a_n}{\prod_{k=0}^{n} \beta_k} < \frac{1}{\prod_{k=0}^{N} \beta_k}.$$

We conclude the proof by (2.11) and by definition of $x_{\boldsymbol{\beta}^{(N+1)}}$. ■

Proposition 2.4.18. *The lazy $\boldsymbol{\beta}$-expansion of a real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is lexicographically minimal among all $\boldsymbol{\beta}$-representations of x in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$.*

Proof. Consider a real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and an infinite word $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ be a $\boldsymbol{\beta}$-representation of x. Suppose that $a <_{\mathrm{lex}} \ell_{\boldsymbol{\beta}}(x)$. By (2.12), we get $\theta_{\boldsymbol{\beta}}(a) >_{\mathrm{lex}} \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x))$. By (2.11), $\theta_{\boldsymbol{\beta}}(a)$ is a $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - x$. Moreover, by Proposition 2.4.12 and since the inverse map $\theta_{\boldsymbol{\beta}}^{-1}$

is the map $\theta_{\boldsymbol{\beta}}$ itself, we have $\theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x)) = d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$. This is absurd since, by Proposition 2.3.15, $d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$ is lexicographically maximal among all $\boldsymbol{\beta}$-representations of $x_{\boldsymbol{\beta}} - x$. ■

Note that, contrary to Proposition 2.3.15, it cannot be stated that "the lazy $\boldsymbol{\beta}$-expansion of a real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ is lexicographically minimal among all $\boldsymbol{\beta}$-representations of x". In fact, the alphabet of the $\boldsymbol{\beta}$-representations of x must be fixed as shown in the following example.

Example 2.4.19. Let $\boldsymbol{\beta}$ be the alternate base from Example 2.3.9 and consider $x = 8 - 2\sqrt{13}$. We have $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and the lazy $\boldsymbol{\beta}$-expansion of x has 01 as a prefix. However, the infinite word 003330^ω is a $\boldsymbol{\beta}$-representation of x and $003330^\omega <_{\text{lex}} \ell_{\boldsymbol{\beta}}(x)$. This does not contradict Proposition 2.4.18 since the infinite word 003330^ω does not belong to $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$.

Proposition 2.4.20. *The function $\ell_{\boldsymbol{\beta}} \colon (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}] \to A_{\boldsymbol{\beta}}{}^{\mathbb{N}}$ is increasing:*

$$\forall x, y \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}], \quad x < y \iff \ell_{\boldsymbol{\beta}}(x) <_{\text{lex}} \ell_{\boldsymbol{\beta}}(y).$$

Proof. Consider $x, y \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. By Propositions 2.3.17 and 2.4.12 and by (2.12), we have

$$\begin{aligned} x < y &\iff x_{\boldsymbol{\beta}} - x > x_{\boldsymbol{\beta}} - y \\ &\iff d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) >_{\text{lex}} d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - y) \\ &\iff \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(x)) >_{\text{lex}} \theta_{\boldsymbol{\beta}}(\ell_{\boldsymbol{\beta}}(y)) \\ &\iff \ell_{\boldsymbol{\beta}}(x) <_{\text{lex}} \ell_{\boldsymbol{\beta}}(y). \end{aligned}$$

■

Remark 2.4.21. Considering two Cantor bases $\boldsymbol{\alpha} = (\alpha_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $\prod_{i=0}^{n} \alpha_i \leq \prod_{i=0}^{n} \beta_i$, by Proposition 2.3.19, we have $d_{\boldsymbol{\alpha}}(x) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(x)$ for all $x \in [0, 1)$. However, an analogous result cannot be obtained for the lazy expansions. In fact, since the interval of definition of the lazy expansions depends on the considered Cantor base, it is not possible to state a result of the form "for all $x \in I$, we have $\ell_{\boldsymbol{\alpha}}(x) \leq_{\text{lex}} \ell_{\boldsymbol{\beta}}(x)$ (or $\ell_{\boldsymbol{\alpha}}(x) \geq_{\text{lex}} \ell_{\boldsymbol{\beta}}(x)$)" where I is a fixed interval. Moreover, it is neither correct to say "for all $x \in [0, 1)$, we have $\ell_{\boldsymbol{\alpha}}(x_{\boldsymbol{\alpha}} - x) \leq_{\text{lex}} \ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)$ (or $\ell_{\boldsymbol{\alpha}}(x_{\boldsymbol{\alpha}} - x) \geq_{\text{lex}} \ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\alpha}} - x)$)". Indeed, this can already be seen while considering real bases, that is, $\boldsymbol{\beta} = (\beta, \beta, \ldots)$ with $\beta > 1$, as illustrated in Figure 2.4 (where the notation β, x_β and $\ell_\beta(\cdot)$ are used instead of $\boldsymbol{\beta}$, $x_{\boldsymbol{\beta}}$ and $\ell_{\boldsymbol{\beta}}(\cdot)$).

β	2	$\frac{11}{5}$	$\frac{5}{2}$
x_β	$\frac{\lceil 2\rceil-1}{2-1}=1$	$\frac{\lceil\frac{11}{5}\rceil-1}{\frac{11}{5}-1}=\frac{5}{3}$	$\frac{\lceil\frac{5}{2}\rceil-1}{\frac{5}{2}-1}=\frac{4}{3}$
$\ell_\beta(x_\beta-\frac{1}{2})$	01^ω	$1221\cdots$	$1211\cdots$

Figure 2.4: Some lazy $\boldsymbol{\beta}$-expansions when $\boldsymbol{\beta}=(\beta,\beta,\ldots)$ with $\beta>1$.

Remark 2.4.22. Note that, some results as Propositions 2.4.16 and 2.3.17 could also have been proved easily without any prerequisite from Section 2.3. In this section, a choice has been made, that is, to use as much as possible Proposition 2.4.12 and results from Section 2.3.

2.4.5 A word on the lazy expansion of $x_{\boldsymbol{\beta}}-1$

In Section 2.3, as explained in Remark 2.3.16, we made a choice of definition for the greedy $\boldsymbol{\beta}$-expansion of 1. One could have expected to define the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}}-1$ analogously, which is not done in Section 2.4.2. In this section, we will define it and compare it with the greedy $\boldsymbol{\beta}$-expansion of 1.

By definition of the lazy algorithm from Section 2.4.2, the digits are picked as the smallest possible at each step. Hence, by extending the lazy algorithm to the real number $x_{\boldsymbol{\beta}}-1$, we get that digit $\xi_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}}-1)$ should be the least element in $[\![0,\lceil\beta_0\rceil-1]\!]$ such that

$$\frac{\xi_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}}-1)}{\beta_0}+\frac{x_{\boldsymbol{\beta}^{(1)}}}{\beta_0}\geq x_{\boldsymbol{\beta}}-1.$$

By (2.10), we get that $\xi_{\boldsymbol{\beta},0}(x_{\boldsymbol{\beta}}-1)=0$ satisfies the wanted inequality. Hence, the first digit of $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ is 0 and its suffix starting at position 1 is the lazy $\boldsymbol{\beta}^{(1)}$-expansion of $\beta_0(x_{\boldsymbol{\beta}}-1)$ which, by (2.10), is equal to $x_{\boldsymbol{\beta}^{(1)}}+\lceil\beta_0\rceil-1-\beta_0$ in $[x_{\boldsymbol{\beta}^{(1)}}-1,x_{\boldsymbol{\beta}^{(1)}})$. That is,

$$\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)=0\,\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}}+\lceil\beta_0\rceil-1-\beta_0).$$

By construction, we obtain that the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}}-1$ is lexicographically minimal among all $\boldsymbol{\beta}$-representations of $x_{\boldsymbol{\beta}}-1$ in $\bigotimes_{n\in\mathbb{N}}[\![0,\lceil\beta_n\rceil-1]\!]$. This extends Proposition 2.4.18.

However, it is important to note that the equivalent definition of the lazy algorithm over $(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]$ using the ceiling function, given in Section 2.4.2, is not valid for the real number $x_{\boldsymbol{\beta}}-1$ when $\beta_0\in\mathbb{N}_{\geq 2}$. In fact, we have $\lceil\beta_0(x_{\boldsymbol{\beta}}-1)-x_{\boldsymbol{\beta}^{(1)}}\rceil=-1$ where -1 cannot appear as a letter of a $\boldsymbol{\beta}$-representation. Hence, $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ would have not been the image of $d_{\boldsymbol{\beta}}(1)$

by the map $\theta_{\boldsymbol{\beta}}$ when β_0 is an integer since if $\beta_0 \in \mathbb{N}_{\geq 2}$, we have $d_{\boldsymbol{\beta}}(1) = \beta_0 0^\omega$ whereas the first letter of $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is 0. However, if $\beta_0 \notin \mathbb{N}_{\geq 2}$, the lazy algorithm using the ceiling function can be used in order to compute the digits of $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$. As a consequence, it can be proved similarly to Proposition 2.4.12 that in that case we have $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(1))$.

As a conclusion, if we decided to define $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$, we would not have been able to extend the algorithm with the ceilings from Section 2.4.2, and moreover, we could not have given the property $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(1))$ since we would have had to separate the statement into two cases: if β_0 is an integer or not. Therefore, to avoid this ambiguity, I have decided to no longer work with $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$.

2.4.6 Quasi-lazy expansions

In this section, we define the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ in order to obtain similar results from Section 2.3.4 for lazy expansions and more precisely an analogue of Parry's theorem characterizing the lazy expansions of real numbers in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$.

Proposition 2.4.23. *The limit* $\lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} \ell_{\boldsymbol{\beta}}(x)$ *exists and is equal to* $\theta_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1))$.

Proof. By Proposition 2.4.12 and by continuity of $\theta_{\boldsymbol{\beta}}$, we get

$$\begin{aligned}
\lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} \ell_{\boldsymbol{\beta}}(x) &= \lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} \theta_{\boldsymbol{\beta}}(d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)) \\
&= \theta_{\boldsymbol{\beta}}\big(\lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x)\big) \\
&= \theta_{\boldsymbol{\beta}}\big(\lim_{y \to 1^-} d_{\boldsymbol{\beta}}(y)\big) \\
&= \theta_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1))
\end{aligned}$$

where the last equality is due to Proposition 2.3.50. ∎

Definition 2.4.24. The *quasi-lazy* $\boldsymbol{\beta}$*-expansion* of $x_{\boldsymbol{\beta}} - 1$ is the infinite word defined as follows:

$$\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \lim_{x \to (x_{\boldsymbol{\beta}}-1)^+} \ell_{\boldsymbol{\beta}}(x). \tag{2.13}$$

By Proposition 2.4.23, this limit exists and, similarly to Proposition 2.4.12, the "flip" of the quasi-greedy $\boldsymbol{\beta}$-expansions of 1 is the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$.

Proposition 2.4.25. *We have* $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \theta_{\boldsymbol{\beta}}(d^*_{\boldsymbol{\beta}}(1))$.

Proof. This is immediate by definition of the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}}-1$ and by Proposition 2.4.23. ■

Example 2.4.26. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. By Example 2.3.27, we have $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$. Hence, by Proposition 2.4.25, we get $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1) = 012(02)^\omega$ and $\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}}-1) = (02)^\omega$.

Proposition 2.4.27. *The quasi-lazy expansion* $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1)$ *is a* $\boldsymbol{\beta}$*-representation of* $x_{\boldsymbol{\beta}} - 1$.

Proof. This is direct by Propositions 2.4.25 and 2.3.29 and by (2.11). ■

Note that, in comparison with the quasi-greedy $\boldsymbol{\beta}$-expansion of 1 which is always infinite, the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ can be either finite or infinite.

Example 2.4.28. Consider an alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ such that for all $i \in [\![0, p-1]\!]$, $\beta_i \in \mathbb{N}_{\geq 2}$. From Example 2.3.25, we get $d^*_{\boldsymbol{\beta}^{(i)}}(1) = ((\beta_i - 1)\cdots(\beta_{p-1} - 1)(\beta_0 - 1)\cdots(\beta_{i-1} - 1))^\omega$ and since, by Lemma 2.4.6, $x_{\boldsymbol{\beta}^{(i)}} = 1$ for all $i \in [\![0, p-1]\!]$, we have $\ell^*_{\boldsymbol{\beta}^{(i)}}(0) = 0^\omega$.

The following result gives a necessary condition on the Cantor base $\boldsymbol{\beta}$ for having a finite quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$.

Proposition 2.4.29. *If the quasi-lazy* $\boldsymbol{\beta}$*-expansion of* $x_{\boldsymbol{\beta}} - 1$ *is finite of length* $n \in \mathbb{N}$*, then* $x_{\boldsymbol{\beta}^{(n)}} = 1$.

Proof. Suppose that $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0 \cdots \ell_{n-1} 0^\omega$ with $n \in \mathbb{N}$ and $\ell_{n-1} \neq 0$ (if it exists, that is, if $n \neq 0$). By Proposition 2.4.25, we get that

$$d^*_{\boldsymbol{\beta}}(1) = (\lceil \beta_0 \rceil - 1 - \ell_0)\cdots(\lceil \beta_{n-1} \rceil - 1 - \ell_{n-1})(\lceil \beta_n \rceil - 1)(\lceil \beta_{n+1} \rceil - 1)\cdots .$$

However, by Proposition 2.3.39, we know that

$$\sigma^n(d^*_{\boldsymbol{\beta}}(1)) = (\lceil \beta_n \rceil - 1)(\lceil \beta_{n+1} \rceil - 1)\cdots \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1).$$

Hence, we obtain that $\sigma^n(d^*_{\boldsymbol{\beta}}(1)) = d^*_{\boldsymbol{\beta}^{(n)}}(1)$. We conclude that

$$x_{\boldsymbol{\beta}^{(n)}} = \text{val}_{\boldsymbol{\beta}^{(n)}}(\sigma^n(d^*_{\boldsymbol{\beta}}(1)))$$

$$= \mathrm{val}_{\boldsymbol{\beta}^{(n)}}(d^*_{\boldsymbol{\beta}^{(n)}}(1))$$
$$= 1$$

where the last equality is due to Proposition 2.3.29. ■

Corollary 2.4.30. *If the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is finite of length $n \in \mathbb{N}$, then $\beta_k \in \mathbb{N}_{\geq 2}$ for all $k \geq n$.*

Proof. This immediately follows from Proposition 2.4.29 and Lemma 2.4.6. ■

As the following example shows, the necessary conditions given by the previous proposition and corollary are not sufficient.

Example 2.4.31. Consider the Cantor base $\boldsymbol{\beta} = (\frac{4}{3}, 2, 2, 2, 2, 2 \cdots)$. We have $x_{\boldsymbol{\beta}} = \frac{3}{2}$ and $x_{\boldsymbol{\beta}^{(n)}} = 1$ for all $n \geq 1$. However, we have $d^*_{\boldsymbol{\beta}}(1) = (10)^\omega$ and $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = (01)^\omega$.

Definition 2.4.32. An infinite word $w \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ is said *ultimately maximal* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $w_n = \lceil\beta_n\rceil - 1$.

Lemma 2.4.33. *The infinite word $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ cannot be ultimately maximal.*

Proof. This is a direct consequence of Proposition 2.4.25 since $d^*_{\boldsymbol{\beta}}(1)$ is infinite. ■

We now prove that $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is lexicographically smaller than all other $\boldsymbol{\beta}$-representations of real numbers in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ belonging to $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$.

Proposition 2.4.34. *If a is an infinite word in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$, then $a >_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$.*

Proof. Let a be an infinite word in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$. Then $\theta_{\boldsymbol{\beta}}(a)$ is an infinite word over $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ and by (2.11), we have $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = x_{\boldsymbol{\beta}} - \mathrm{val}_{\boldsymbol{\beta}}(a) \in [0, 1)$. By Proposition 2.3.30, we get that $\theta_{\boldsymbol{\beta}}(a) <_{\mathrm{lex}} d^*_{\boldsymbol{\beta}}(1)$. Moreover, by Proposition 2.4.25, we have $d^*_{\boldsymbol{\beta}}(1) = \theta_{\boldsymbol{\beta}}(\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1))$. Hence, by (2.12), we conclude that $a >_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$. ■

Note that, similarly to Proposition 2.4.18, Proposition 2.4.34 is weaker than its analogous greedy one, that is, Proposition 2.3.30, since we fix the alphabet of the $\boldsymbol{\beta}$-representations. A stronger result cannot be stated as illustrated in the next example.

Example 2.4.35. Continuing Examples 2.4.19 and 2.4.26, the infinite word 003330^ω is a $\boldsymbol{\beta}$-representation of $8-2\sqrt{13}$. However $003330^\omega <_{\text{lex}} 012(02)^\omega = \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$.

By Proposition 2.3.30, the word $d^*_{\boldsymbol{\beta}}(1)$ is lexicographically maximal among all infinite $\boldsymbol{\beta}$-representations of all real numbers in $[0,1]$. The following result gives the translation of this property in terms of the lazy representations.

Proposition 2.4.36. *The quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is the lexicographically least $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - 1$ in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ that is not ultimately maximal.*

Proof. By Proposition 2.4.27 and Lemma 2.4.33, the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ is a $\boldsymbol{\beta}$-representation of $x_{\boldsymbol{\beta}} - 1$ in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ which is not ultimately maximal. Moreover, let a be an infinite word in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x_{\boldsymbol{\beta}} - 1$ and suppose that $a <_{\text{lex}} \ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$. As above, we get $\theta_{\boldsymbol{\beta}}(a) >_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ with $\mathrm{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) = 1$. By Proposition 2.3.30, the word $\theta_{\boldsymbol{\beta}}(a)$ must be a finite $\boldsymbol{\beta}$-representation of 1. By setting N to the length of the longest prefix of $\theta_{\boldsymbol{\beta}}(a)$ not ending with 0, we get $a_n = \lceil\beta_n\rceil - 1$ for all $n \geq N$, that is, a is ultimately maximal in $\bigotimes_{n\in\mathbb{N}} [\![0, \lceil\beta_n\rceil - 1]\!]$. ∎

Remark 2.4.37. In order to "directly" compute the quasi-lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ without using a limit on words (as for the quasi-greedy $\boldsymbol{\beta}$-expansion, see (2.5)), one could have define the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - 1$ (which was not the choice made in this text as explained in Section 2.4.5) and define $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ respectively as follows: $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ if $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ is not ultimately periodic and $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \xi_0 \cdots \xi_{n-2}(\xi_{n-1} + 1)\ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1)$ if $\xi_{n-1} < \lceil\beta_{n-1}\rceil - 1$ and for all $m \geq n$, we have $\xi_m = \lceil\beta_m\rceil - 1$.

2.4.7 Lazy admissible sequences

Definition 2.4.38. We let $D'_{\boldsymbol{\beta}}$ denote the subset of $A^{\mathbb{N}}_{\boldsymbol{\beta}}$ of all lazy $\boldsymbol{\beta}$-expansions of real numbers in the interval $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and let $S'_{\boldsymbol{\beta}}$ denote the topological closure of $D'_{\boldsymbol{\beta}}$ with respect to the prefix distance of infinite words:

$$D'_{\boldsymbol{\beta}} = \{\ell_{\boldsymbol{\beta}}(x) : x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]\} \quad \text{and} \quad S'_{\boldsymbol{\beta}} = \overline{D'_{\boldsymbol{\beta}}}.$$

The following result links the sets $D'_{\boldsymbol{\beta}}$ and $S'_{\boldsymbol{\beta}}$ with their analogous greedy ones $D_{\boldsymbol{\beta}} = \{d_{\boldsymbol{\beta}}(x) : x \in [0,1)\}$ and $S_{\boldsymbol{\beta}} = \overline{D_{\boldsymbol{\beta}}}$.

Proposition 2.4.39. *The maps $\theta_{\boldsymbol{\beta}}|_{D_{\boldsymbol{\beta}}} : D_{\boldsymbol{\beta}} \to D'_{\boldsymbol{\beta}}$ and $\theta_{\boldsymbol{\beta}}|_{S_{\boldsymbol{\beta}}} : S_{\boldsymbol{\beta}} \to S'_{\boldsymbol{\beta}}$ are both bijective.*

Proof. By Proposition 2.4.12, the map $\theta_{\boldsymbol{\beta}}|_{D_{\boldsymbol{\beta}}}$ is well defined and surjective. Hence, by continuity of the map $\theta_{\boldsymbol{\beta}}$, the map $\theta_{\boldsymbol{\beta}}|_{S_{\boldsymbol{\beta}}}$ is also well defined and surjective. Moreover, since the map $\theta_{\boldsymbol{\beta}}$ is injective, so are the maps $\theta_{\boldsymbol{\beta}}|_{D_{\boldsymbol{\beta}}}$ and $\theta_{\boldsymbol{\beta}}|_{S_{\boldsymbol{\beta}}}$. ■

Proposition 2.4.40. *Let $a, b \in S'_{\boldsymbol{\beta}}$.*

1. *If $a <_{\text{lex}} b$ then $\text{val}_{\boldsymbol{\beta}}(a) \le \text{val}_{\boldsymbol{\beta}}(b)$.*
2. *If $\text{val}_{\boldsymbol{\beta}}(a) < \text{val}_{\boldsymbol{\beta}}(b)$ then $a <_{\text{lex}} b$.*

Proof. Suppose that $a, b \in S'_{\boldsymbol{\beta}}$ are such that $a <_{\text{lex}} b$. By Proposition 2.4.39 and (2.12), we have $\theta_{\boldsymbol{\beta}}(a), \theta_{\boldsymbol{\beta}}(b) \in S_{\boldsymbol{\beta}}$ and $\theta_{\boldsymbol{\beta}}(a) >_{\text{lex}} \theta_{\boldsymbol{\beta}}(b)$. By Proposition 2.3.38, we $\text{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(a)) \ge \text{val}_{\boldsymbol{\beta}}(\theta_{\boldsymbol{\beta}}(b))$. We conclude the proof of the first item by (2.11). The second item immediately follows. ■

We are now able to state a Parry-like theorem for Cantor real bases in the lazy framework.

Theorem 2.4.41. *Let a be an infinite word over $\mathbb{N}$.*

1. *The word a belongs to $D'_{\boldsymbol{\beta}}$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$ and for all $n \in \mathbb{N}$,*
$$\sigma^n(a) >_{\text{lex}} \ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1).$$
2. *The word a belongs to $S'_{\boldsymbol{\beta}}$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$ and for all $n \in \mathbb{N}$,*
$$\sigma^n(a) \ge_{\text{lex}} \ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1).$$

Proof. Let a be an infinite word. We have $a \in D'_{\boldsymbol{\beta}}$ if and only if $a \in \bigotimes_{n\in\mathbb{N}} [\![0, \lceil \beta_n \rceil - 1]\!]$ and $\theta_{\boldsymbol{\beta}}(a) \in D_{\boldsymbol{\beta}}$. Moreover, by Theorem 2.3.33, we have $\theta_{\boldsymbol{\beta}}(a) \in D_{\boldsymbol{\beta}}$ if and only if $\sigma^n(\theta_{\boldsymbol{\beta}}(a)) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(n)}}(1)$ for all $n \in \mathbb{N}$. However, for all $n \in \mathbb{N}$, by Lemma 2.4.15, we have $\sigma^n(\theta_{\boldsymbol{\beta}}(a)) = \theta_{\boldsymbol{\beta}^{(n)}}(\sigma^n(a))$ and by

Proposition 2.4.25, we have $d^*_{\boldsymbol{\beta}^{(n)}}(1) = \theta_{\boldsymbol{\beta}^{(n)}}(\ell^*_{\boldsymbol{\beta}^{(n)}}(x_{\boldsymbol{\beta}^{(n)}} - 1))$. Hence, the first item follows from (2.12). The second item can be proved in a similar fashion by using Proposition 2.3.39. ∎

Example 2.4.42. Consider $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$. In view of Example 2.4.26, the sequence $a = (2120)^\omega$ belongs to $D'_{\boldsymbol{\beta}}$.

Note that in Theorem 2.4.41, the hypothesis that a belongs to $\bigotimes_{n\in\mathbb{N}}[\![0, \lceil\beta_n\rceil - 1]\!]$ is required. For otherwise, any sequence a such that $a_n > \lceil\beta_n\rceil - 1$ for all $n \in \mathbb{N}$ would belong to $D'_{\boldsymbol{\beta}}$.

As a consequence of Theorem 2.4.41, we can characterize the set $D'_{\boldsymbol{\beta}}$ by translating Proposition 2.3.46 and Corollaries 2.3.47 and 2.3.48 to the lazy framework. To do so, we define sets of finite words $X'_{\boldsymbol{\beta},n}$ for $n \in \mathbb{N}_{\geq 1}$ as follows.

Definition 2.4.43. If $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0\ell_1\cdots$ then, for all $n \in \mathbb{N}_{\geq 1}$, we let

$$X'_{\boldsymbol{\beta},n} = \{\ell_0\cdots\ell_{n-2}s : s \in [\![\ell_{n-1}+1, \lceil\beta_{n-1}\rceil - 1]\!]\}.$$

Note that $X'_{\boldsymbol{\beta},n}$ is empty if and only if $\ell_{n-1} = \lceil\beta_{n-1}\rceil - 1$.

Proposition 2.4.44. *We have*

$$D'_{\boldsymbol{\beta}} = \bigcup_{n_0\in\mathbb{N}_{\geq 1}} X'_{\boldsymbol{\beta},n_0}\left(\bigcup_{n_1\in\mathbb{N}_{\geq 1}} X'_{\boldsymbol{\beta}^{(n_0)},n_1}\left(\bigcup_{n_2\in\mathbb{N}_{\geq 1}} X'_{\boldsymbol{\beta}^{(n_0+n_1)},n_2}\left(\quad\cdots\quad\right)\right)\right).$$

Therefore, we have $D'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}_{\geq 1}} X'_{\boldsymbol{\beta},n} D'_{\boldsymbol{\beta}^{(n)}}$ *and any prefix of* $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1)$ *belongs to* $\mathrm{Pref}(D'_{\boldsymbol{\beta}})$.

Proof. This follows from Propositions 2.4.25, 2.4.39 and 2.3.46 since

$$w_0w_1\cdots w_{n-1} \in X'_{\boldsymbol{\beta},n}$$

if and only if

$$(\lceil\beta_0\rceil - 1 - w_0)(\lceil\beta_1\rceil - 1 - w_1)\cdots(\lceil\beta_{n-1}\rceil - 1 - w_{n-1}) \in X_{\boldsymbol{\beta},n}.$$

∎

2.4.8 The lazy $\boldsymbol{\beta}$-shift

We end this chapter by defining and studying the lazy $\boldsymbol{\beta}$-shift.

Definition 2.4.45. We define

$$\Delta'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} D'_{\boldsymbol{\beta}^{(n)}} \quad \text{and} \quad \Sigma'_{\boldsymbol{\beta}} = \overline{\Delta'_{\boldsymbol{\beta}}}.$$

By Proposition 2.4.39, we get

$$\Delta'_{\boldsymbol{\beta}} = \bigcup_{n\in\mathbb{N}} \theta_{\boldsymbol{\beta}^{(n)}}(D_{\boldsymbol{\beta}^{(n)}}). \tag{2.14}$$

Proposition 2.4.46. *The sets $\Delta'_{\boldsymbol{\beta}}$ and $\Sigma'_{\boldsymbol{\beta}}$ are both shift-invariant.*

Proof. Let a be an infinite word over $\mathbb{N}$. By (2.14), if a belongs to $\Delta'_{\boldsymbol{\beta}}$, then there exists $n \in \mathbb{N}$ and an infinite word $b \in D_{\boldsymbol{\beta}^{(n)}}$ such that $a = \theta_{\boldsymbol{\beta}^{(n)}}(b)$. We obtain that $\sigma(a) = \sigma(\theta_{\boldsymbol{\beta}^{(n)}}(b)) = \theta_{\boldsymbol{\beta}^{(n+1)}}(\sigma(b))$ by Lemma 2.4.15. By Theorem 2.3.33, $\sigma(b) \in D_{\boldsymbol{\beta}^{(n+1)}}$ so $\sigma(a) \in D'_{\boldsymbol{\beta}^{(n+1)}}$. Then, it is easily seen that if $a \in S'_{\boldsymbol{\beta}^{(n)}}$ then $\sigma(a) \in S'_{\boldsymbol{\beta}^{(n+1)}}$. ■

Since the set $\Sigma'_{\boldsymbol{\beta}}$ is shift-invariant and closed with respect to the topology induced by the prefix distance on infinite words, we conclude that the subset $\Sigma'_{\boldsymbol{\beta}}$ of $A_{\boldsymbol{\beta}}^{\mathbb{N}}$ is a subshift, which we call the *lazy $\boldsymbol{\beta}$-shift*.

Let us now study the factors of the lazy $\boldsymbol{\beta}$-shift.

Proposition 2.4.47. *We have* $\mathrm{Fac}(D'_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Delta'_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$.

Proof. By definition, we have $\mathrm{Fac}(D'_{\boldsymbol{\beta}}) \subseteq \mathrm{Fac}(\Delta'_{\boldsymbol{\beta}}) = \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$. It remains to show that $\mathrm{Fac}(D'_{\boldsymbol{\beta}}) \supseteq \mathrm{Fac}(\Delta'_{\boldsymbol{\beta}})$. Let $f \in \mathrm{Fac}(\Delta'_{\boldsymbol{\beta}})$. By (2.14), there exist $n \in \mathbb{N}$ and $b \in D_{\boldsymbol{\beta}^{(n)}}$ such that $f \in \mathrm{Fac}(\theta_{\boldsymbol{\beta}^{(n)}}(b))$. In particular, $f \in \mathrm{Fac}(\theta_{\boldsymbol{\beta}}(0^n b))$ where, by Theorem 2.3.33, $0^n b \in D_{\boldsymbol{\beta}}$. We obtain that $f \in \mathrm{Fac}(\theta_{\boldsymbol{\beta}}(D_{\boldsymbol{\beta}})) = \mathrm{Fac}(D'_{\boldsymbol{\beta}})$ by Proposition 2.4.39. ■

Corollary 2.4.48. *We have*

$$\mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}}) = \bigcup_{n\in\mathbb{N}} \theta_{\boldsymbol{\beta}^{(n)}}\big(\mathrm{Pref}(D_{\boldsymbol{\beta}^{(n)}})\big).$$

Proof. By Propositions 2.4.46 and 2.4.47, we have $\mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}}) = \mathrm{Pref}(\Delta'_{\boldsymbol{\beta}}) = \bigcup_{n\in\mathbb{N}} \mathrm{Pref}(D'_{\boldsymbol{\beta}^{(n)}})$. The conclusion follows from Proposition 2.4.39. ■

CHAPTER

3

MORE COMBINATORIAL PROPERTIES OF ALTERNATE BASE EXPANSIONS

The aim of this chapter is to pay special attention to periodic Cantor real bases, referred to as alternate bases, and discuss some results that are specific to these particular Cantor real bases.

First, we improve some results from Chapter 2 about greedy and quasi-greedy $\boldsymbol{\beta}$-expansions of 1. In particular, generalizing Parry's result (see Corollary 1.4.17), we obtain a characterization of the greedy $\boldsymbol{\beta}$-expansion of 1 among all $\boldsymbol{\beta}$-representations of 1.

Second, we define Parry alternate bases and characterize them in terms of the periodicity of the greedy, quasi-greedy and quasi-lazy expansions.

Third, we study the alternate base greedy and lazy $\boldsymbol{\beta}$-shifts. In particular, we generalize Bertrand-Mathis' theorem by proving that the greedy (resp., lazy) $\boldsymbol{\beta}$-shift is sofic if and only if $\boldsymbol{\beta}$ is a Parry alternate base. However, a counterexample shows that, contrarily to the real base case, the greedy $\boldsymbol{\beta}$-shifts of finite type cannot be characterized thanks to the finiteness of the greedy $\boldsymbol{\beta}^{(i)}$-expansions.

The results presented in this chapter are from [CC21] and [Cis21].

Contents of the Chapter

3.1 Value function and representations of 1

We start with a few elementary observations. Consider an alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$. First, by Proposition 2.1.3, the condition $\prod_{n\in\mathbb{N}} \beta_n = +\infty$ from Definition 2.1.1 is trivially satisfied in the context of alternate bases. Then, for an alternate base $\boldsymbol{\beta}$ of length p, the $\boldsymbol{\beta}$-value (2.1) of an infinite word a over $\mathbb{R}_{\geq 0}$ can be rewritten as

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{n\in\mathbb{N}} \frac{a_n}{\left(\prod_{i=0}^{p-1} \beta_i\right)^{\lfloor \frac{n}{p} \rfloor} \prod_{i=0}^{n \bmod p} \beta_i}$$

or as

$$\mathrm{val}_{\boldsymbol{\beta}}(a) = \sum_{m\in\mathbb{N}} \frac{1}{\left(\prod_{i=0}^{p-1} \beta_i\right)^{m}} \sum_{j=0}^{p-1} \frac{a_{pm+j}}{\prod_{i=0}^{j} \beta_i}. \tag{3.1}$$

Further, the alphabet $A_{\boldsymbol{\beta}}$ is finite since $A_{\boldsymbol{\beta}} = [\![0, \max_{i\in[\![0,p-1]\!]} (\lceil \beta_i \rceil - 1)]\!]$. Finally, note that a Cantor base of the form $(\beta, \beta, \ldots)$ is an alternate base of length 1, in which case, as already mentioned, all definitions introduced so far coincide with those of Rényi for real bases β.

In Proposition 2.2.3, we gave a characterization of those infinite words $a \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ for which there exists a Cantor base $\boldsymbol{\beta}$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$. Here, we are interested in the stronger condition of the existence of an alternate base $\boldsymbol{\beta}$ satisfying $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$.

Proposition 3.1.1. *Let a be an infinite word over $\mathbb{R}_{\geq 0}$ such that $a_n \in O(n^d)$ for some $d \in \mathbb{N}$ and let $p \in \mathbb{N}_{\geq 1}$. There exists an alternate base $\boldsymbol{\beta}$ of length*

p such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = 1$ *if and only if* $\sum_{n\in\mathbb{N}} a_n > 1$. *If moreover* $p \geq 2$, *then there exist uncountably many such alternate bases.*

Proof. From Proposition 2.2.3, we already know that the condition $\sum_{n\in\mathbb{N}} a_n > 1$ is necessary. Now, suppose that $\sum_{n\in\mathbb{N}} a_n > 1$. If $p = 1$ then the result follows from Lemma 2.2.1. Suppose that $p \geq 2$. Consider any $(p-1)$-tuple $(\beta_1, \ldots, \beta_{p-1}) \in (\mathbb{R}_{>1})^{p-1}$. For all $\beta_0 > 1$, we can write $\mathrm{val}_{\boldsymbol{\beta}}(a) = \mathrm{val}_{\beta_0}(c)$ with $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})$ and

$$c_m = \frac{1}{\left(\prod_{i=1}^{p-1} \beta_i\right)^m} \sum_{j=0}^{p-1} \frac{a_{pm+j}}{\prod_{i=1}^{j} \beta_i} \quad \text{for all } m \in \mathbb{N}.$$

Note that $c \in (\mathbb{R}_{\geq 0})^{\mathbb{N}}$ and that $c_m \in O(m^d)$. By hypothesis, there exists $N \in \mathbb{N}$ such that $\sum_{n=0}^{N} a_n > 1$. Then

$$\sum_{m=0}^{\left\lfloor \frac{N}{p} \right\rfloor} c_m > \frac{\sum_{m=0}^{\left\lfloor \frac{N}{p} \right\rfloor} \sum_{j=0}^{p-1} a_{pm+j}}{\left(\prod_{i=1}^{p-1} \beta_i\right)^{\left\lfloor \frac{N}{p} \right\rfloor + 1}} \geq \frac{\sum_{n=0}^{N} a_n}{\left(\prod_{i=1}^{p-1} \beta_i\right)^{\left\lfloor \frac{N}{p} \right\rfloor + 1}}.$$

Therefore, any $(p-1)$-tuple $(\beta_1, \ldots, \beta_{p-1}) \in (\mathbb{R}_{>1})^{p-1}$ satisfying

$$\left(\prod_{i=1}^{p-1} \beta_i\right)^{\left\lfloor \frac{N}{p} \right\rfloor + 1} \leq \sum_{n=0}^{N} a_n$$

is such that $\sum_{m=0}^{\left\lfloor \frac{N}{p} \right\rfloor} c_m > 1$, and hence there exist uncountably many of them. For such a $(p-1)$-tuple, the infinite word c satisfies the hypothesis of Lemma 2.2.1, so there exists $\beta_0 > 1$ such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = \mathrm{val}_{\beta_0}(c) = 1$. ■

3.2 Greedy and quasi-greedy alternate base expansions of 1

From now on, we let $\boldsymbol{\beta}$ be a fixed alternate base and we let p be its length. The greedy and the quasi-greedy $\boldsymbol{\beta}$-expansions of 1 enjoy specific properties whenever $\boldsymbol{\beta}$ is an alternate base.

3.2.1 Some properties on periodicity

Proposition 3.2.1. *The greedy $\boldsymbol{\beta}$-expansion of* 1 *is not purely periodic.*

Proof. Suppose to the contrary that there exists $q \in \mathbb{N}_{\geq 1}$ such that for all $n \in \mathbb{N}$, $\varepsilon_n = \varepsilon_{n+q}$. By considering $k = \operatorname{lcm}(p, q)$, we get that $\boldsymbol{\beta}^{(k)} = \boldsymbol{\beta}$ and for all $n \in \mathbb{N}$, $\varepsilon_n = \varepsilon_{n+k}$. Therefore

$$1 = \operatorname{val}_{\boldsymbol{\beta}}\big(\varepsilon_0 \cdots \varepsilon_{k-1}\big) + \frac{1}{\prod_{i=0}^{k-1} \beta_i} = \operatorname{val}_{\boldsymbol{\beta}}\big(\varepsilon_0 \cdots \varepsilon_{k-2}(\varepsilon_{k-1} + 1)\big).$$

Thus $\varepsilon_0 \cdots \varepsilon_{k-2}(\varepsilon_{k-1}+1)$ is a $\boldsymbol{\beta}$-representation of 1 lexicographically greater than $d_{\boldsymbol{\beta}}(1)$, which is impossible by Proposition 2.3.15. ∎

The following example shows that the greedy $\boldsymbol{\beta}$-expansion of 1 can be ultimately periodic with a period which is coprime with the length p of $\boldsymbol{\beta}$.

Example 3.2.2. Let $\boldsymbol{\beta} = (\overline{\sqrt{6}, 3, \frac{2+\sqrt{6}}{3}})$. It is easily checked that $d_{\boldsymbol{\beta}^{(0)}}(1) = 2(10)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 3$ and $d_{\boldsymbol{\beta}^{(2)}}(1) = 11002$.

Proposition 3.2.3. *The quasi-greedy expansion $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if, within the first p recursive calls to Definition 2.3.23, either an infinite ultimately periodic greedy expansion is reached or only finite greedy expansions are involved.*

Proof. If there exists $n \in \mathbb{N}$ such that the infinite greedy expansion $d^*_{\boldsymbol{\beta}^{(n)}}(1)$ is involved in the computation of $d^*_{\boldsymbol{\beta}}(1)$, then clearly $d^*_{\boldsymbol{\beta}}(1)$ is ultimately periodic if and only if so is $d^*_{\boldsymbol{\beta}^{(n)}}(1)$.

Now, suppose that only finite greedy expansions are involved within p recursive calls to the definition of $d^*_{\boldsymbol{\beta}}(1)$. Then $d_{\boldsymbol{\beta}}(1)$ is finite. Thus, $d_{\boldsymbol{\beta}}(1) = \varepsilon_{\boldsymbol{\beta},0} \cdots \varepsilon_{\boldsymbol{\beta},k_0-1}$ with $k_0 \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\boldsymbol{\beta},k_0-1} > 0$. Then

$$d^*_{\boldsymbol{\beta}}(1) = \varepsilon_{\boldsymbol{\beta},0} \cdots \varepsilon_{\boldsymbol{\beta},k_0-2}(\varepsilon_{\boldsymbol{\beta},k_0-1} - 1) d^*_{\boldsymbol{\beta}^{(i_1)}}(1)$$

where $i_1 = k_0 \bmod p$. By hypothesis, $d_{\boldsymbol{\beta}^{(i_1)}}(1)$ is finite as well. Thus we have $d_{\boldsymbol{\beta}^{(i_1)}}(1) = \varepsilon_{\boldsymbol{\beta}^{(i_1)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_1)},k_1-1}$ with $k_1 \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{\boldsymbol{\beta}^{(i_1)},k_1-1} > 0$. Repeating the same argument, we obtain

$$d^*_{\boldsymbol{\beta}^{(i_1)}}(1) = \varepsilon_{\boldsymbol{\beta}^{(i_1)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_1)},k_1-2}(\varepsilon_{\boldsymbol{\beta}^{(i_1)},k_1-1} - 1) d^*_{\boldsymbol{\beta}^{(i_2)}}(1)$$

where $i_2 = k_0 + k_1 \bmod p$. By continuing in the same fashion and by setting $i_0 = 0$, we obtain two sequences $(k_j)_{j \in [\![0,p-1]\!]}$ and $(i_j)_{j \in [\![0,p]\!]}$. Because for all $j \in [\![0,p]\!]$, we have $i_j \in [\![0,p-1]\!]$, there exist $j, j' \in [\![0,p]\!]$ such that $j < j'$ and $i_j = i_{j'}$. Then $d^*_{\boldsymbol{\beta}}(1) = xy^\omega$ where

$$x = \prod_{n=0}^{j-1} \varepsilon_{\boldsymbol{\beta}^{(i_n)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_n)},k_n-2}(\varepsilon_{\boldsymbol{\beta}^{(i_n)},k_n-1} - 1)$$

and

$$y = \prod_{n=j}^{j'-1} \varepsilon_{\boldsymbol{\beta}^{(i_n)},0} \cdots \varepsilon_{\boldsymbol{\beta}^{(i_n)},k_n-2}(\varepsilon_{\boldsymbol{\beta}^{(i_n)},k_n-1} - 1).$$

■

3.2.2 Characterization of the greedy alternate base expansions of 1

The condition given in Corollary 2.3.35 gives that, in order to know if a $\boldsymbol{\beta}$-representation $a = (a_n)_{n\in\mathbb{N}}$ of some real number in $[0,1]$ is the greedy one, we have to check if $\sigma^{np}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ for $n \in \mathbb{N}_{\geq 1}$. Thus this does not allow us to check whether a given $\boldsymbol{\beta}$-representation of 1 is the greedy $\boldsymbol{\beta}$-expansion of 1 without effectively computing the quasi-greedy $\boldsymbol{\beta}$-expansion of 1, and hence the greedy $\boldsymbol{\beta}$-expansion of 1 itself. The following proposition provides us with such a condition in the case of alternate bases, provided that we are given the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 for $i \in [\![1, p-1]\!]$.

Proposition 3.2.4. *A $\boldsymbol{\beta}$-representation $a = (a_n)_{n\in\mathbb{N}}$ of 1 is the greedy $\boldsymbol{\beta}$-expansion of 1 if and only if for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1, p-1]\!]$, $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i)}}(1)$.*

Proof. The condition is necessary by Corollary 2.3.35 and since $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. Let us show that the condition is sufficient.

Let a be a $\boldsymbol{\beta}$-representation of 1 such that for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1, p-1]\!]$, $\sigma^{pm+i}(a) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i)}}(1)$. By Proposition 2.3.15, $a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. By Theorem 2.3.33, if $a <_{\text{lex}} d^*_{\boldsymbol{\beta}}(1)$ then $\text{val}_{\boldsymbol{\beta}}(a) < 1$, which contradicts that a is a $\boldsymbol{\beta}$-representation of 1. Thus, $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} a \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. If $d_{\boldsymbol{\beta}}(1)$ is infinite, then $a = d_{\boldsymbol{\beta}}(1)$ as desired. Now, suppose that $d_{\boldsymbol{\beta}}(1) = \varepsilon_0 \cdots \varepsilon_{k-1}$ with $k \in \mathbb{N}_{\geq 1}$ and $\varepsilon_{k-1} > 0$. Then $a_0 \cdots a_{k-2} = \varepsilon_0 \cdots \varepsilon_{k-2}$ and $a_{k-1} \in \{\varepsilon_{k-1} - 1, \varepsilon_{k-1}\}$. Since $\text{val}_{\boldsymbol{\beta}}(a) = 1$, if $a_{k-1} = \varepsilon_{k-1}$ then $a = d_{\boldsymbol{\beta}}(1)$. Therefore, in order to conclude, it suffices to show that $a_{k-1} \neq \varepsilon_{k-1} - 1$.

Suppose to the contrary that $a_{k-1} = \varepsilon_{k-1} - 1$. Then $d^*_{\boldsymbol{\beta}^{(k)}}(1) \leq_{\text{lex}} \sigma^k(a)$. By hypothesis, $k \bmod p = 0$. Therefore $d^*_{\boldsymbol{\beta}}(1) \leq_{\text{lex}} \sigma^k(a) \leq_{\text{lex}} d_{\boldsymbol{\beta}}(1)$. By repeating the same argument, we obtain that $a_k \cdots a_{2k-2} = \varepsilon_0 \cdots \varepsilon_{k-2}$ and $a_{2k-1} \in \{\varepsilon_{k-1} - 1, \varepsilon_{k-1}\}$. Since $\sigma^k(a) <_{\text{lex}} a$ by hypothesis, we must have $a_{2k-1} = \varepsilon_{k-1} - 1$. By iterating the argument, we obtain that $a = (\varepsilon_0 \cdots \varepsilon_{k-2}(\varepsilon_{k-1} - 1))^\omega$, contradicting that $\sigma^k(a) <_{\text{lex}} a$. ■

When $p = 1$, Proposition 3.2.4 provides us with the purely combinatorial condition proved by Parry (see Corollary 1.4.17) in order to determine whether a given $\boldsymbol{\beta}$-representation of 1 is the greedy $\boldsymbol{\beta}$-expansion of 1. However, when $p \geq 2$, we need to compute the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 for every $i \in [\![1, p-1]\!]$ first. This might lead us to a circular computation, in which case the condition may seem not useful in practice. Indeed, suppose that $p = 2$ and that we are provided with a $\boldsymbol{\beta}$-representation a of 1 and a $\boldsymbol{\beta}^{(1)}$-representation b of 1. Then in order to check if $a = d_{\boldsymbol{\beta}}(1)$, we need to compute $d^*_{\boldsymbol{\beta}^{(1)}}(1)$, and hence $d_{\boldsymbol{\beta}^{(1)}}(1)$ first. But then, in order to check if $b = d_{\boldsymbol{\beta}^{(1)}}(1)$, we need to compute $d^*_{\boldsymbol{\beta}}(1)$, and hence $d_{\boldsymbol{\beta}}(1)$, which brings us back to the initial problem. Nevertheless, this condition can be useful to check if a specific $\boldsymbol{\beta}$-representation of 1 is the greedy $\boldsymbol{\beta}$-expansion of 1. For example, consider a $\boldsymbol{\beta}$-representation a of 1 such that for all $m \in \mathbb{N}_{\geq 1}$, $\sigma^{pm}(a) <_{\text{lex}} a$ and for all $m \in \mathbb{N}$ and $i \in [\![1, p-1]\!]$, $a_{pm+i} < \lfloor \beta_i \rfloor - 1$. Then the infinite word a satisfies the hypothesis of Proposition 3.2.4 and a is the greedy $\boldsymbol{\beta}$-expansion of 1.

We have seen that considering an infinite word a over $\mathbb{N}$ and a positive integer p, there may exist more than one alternate base $\boldsymbol{\beta}$ of length p such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$. Moreover, among all of these alternate bases, it may be that some are such that a is greedy and others are such that a is not. Thus, a purely combinatorial condition for checking whether a $\boldsymbol{\beta}$-representation is greedy cannot exist.

Example 3.2.5. Consider $a = 2(10)^\omega$. Then $\text{val}_{\boldsymbol{\alpha}}(a) = \text{val}_{\boldsymbol{\beta}}(a) = 1$ for both $\boldsymbol{\alpha} = (\overline{1+\varphi, 2})$ and $\boldsymbol{\beta} = (\overline{\frac{31}{10}, \frac{420}{341}})$. It can be checked that $d_{\boldsymbol{\alpha}}(1) = a$ and $d_{\boldsymbol{\beta}}(1) \neq a$.

Furthermore, an infinite word a over $\mathbb{N}$ can be greedy for more than one alternate base.

Example 3.2.6. The infinite word 110^ω is the greedy expansion of 1 with respect to the three alternate bases $(\overline{\varphi, \varphi})$, $(\overline{\frac{5+\sqrt{13}}{6}, \frac{1+\sqrt{13}}{2}})$ and $(\overline{1.7, \frac{1}{0.7}})$.

At the opposite, it may happen that an infinite word a is a $\boldsymbol{\beta}$-representation of 1 for different alternate bases $\boldsymbol{\beta}$ but that none of these are such that a is greedy. As an illustration, by Proposition 3.2.1, for all purely periodic infinite words a over $\mathbb{N}$, all alternate bases $\boldsymbol{\beta}$ such that $\text{val}_{\boldsymbol{\beta}}(a) = 1$ are such that a is not the greedy $\boldsymbol{\beta}$-expansion of 1.

Example 3.2.7. The infinite word $(10)^\omega$ is a representation of 1 with respect to the three alternate bases considered in Example 3.2.6. However, the infinite words $(10)^\omega$ is purely periodic therefore, by Proposition 3.2.1, it is not the greedy expansion of 1 in any alternate base.

3.3 Parry alternate bases

Generalizing the concept of Parry numbers from Definition 1.4.11, we define *Parry alternate bases.*

Definition 3.3.1. An alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}})$ is a *Parry alternate base* if $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic for all $i \in [\![0, p-1]\!]$.

Parry alternate bases will play an important role in the next section, while characterizing sofic greedy and lazy $\boldsymbol{\beta}$-shifts, and in Chapter 4.

One might think at first that if $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}})$ is a Parry alternate base, then $\beta = \prod_{i=0}^{p-1} \beta_i$ must be a Parry number, that is, $d^*_\beta(1)$ must be ultimately periodic. This is not the case, as the following example shows.

Example 3.3.2. Let $\boldsymbol{\beta} = (\overline{\sqrt{6}, 3, \frac{2+\sqrt{6}}{3}})$. This alternate base $\boldsymbol{\beta}$ is a Parry alternate base by Example 3.2.2. But the product $\beta = \prod_{i=0}^{p-1} \beta_i = \sqrt{6}(2+\sqrt{6})$ is not a Parry number as explained in Example 2.3.28.

In the real base case, it is equivalent to say that $d_\beta(1)$ is ultimately periodic if and only if so is $d^*_\beta(1)$. Similarly, by Proposition 3.2.3, we get the following equivalent definition of Parry alternate bases.

Proposition 3.3.3. *An alternate base $\boldsymbol{\beta}$ is a Parry alternate base if and only if $d_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic for all $i \in [\![0, p-1]\!]$.*

Proof. Suppose that for all $i \in [\![0, p-1]\!]$, the greedy $\boldsymbol{\beta}^{(i)}$-expansion of 1 is ultimately periodic. Then, for all $i \in [\![0, p-1]\!]$, within the first p recursive calls to Definition 2.3.23, either an infinite ultimately periodic greedy expansion is reached or only finite greedy expansions are involved. By Proposition 3.2.3, we conclude that, for all $i \in [\![0, p-1]\!]$, the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansion of 1 is ultimately periodic. Conversely, if there exists $i \in [\![0, p-1]\!]$ such that $d_{\boldsymbol{\beta}^{(i)}}(1)$ is not ultimately periodic, then $d^*_{\boldsymbol{\beta}^{(i)}}(1) = d_{\boldsymbol{\beta}^{(i)}}(1)$ and we get that $\boldsymbol{\beta}$ is not a Parry alternate base. ■

Recall that any alternate base $\boldsymbol{\beta}$ has a finite corresponding $x_{\boldsymbol{\beta}}$ (defined in (2.9)), hence it makes sense to consider the lazy $\boldsymbol{\beta}$-expansions. The following result shows that Parry alternate bases can equivalently be defined thanks to the periodicity of the quasi-lazy expansions.

Proposition 3.3.4. *An alternate base $\boldsymbol{\beta}$ is a Parry alternate base if and only if $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1)$ is ultimately periodic for all $i \in [\![0,p-1]\!]$.*

Proof. Suppose that for all $i \in [\![0,p-1]\!]$, $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1)$ is ultimately periodic and write[1]

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^{\omega}.$$

Without loss of generality, suppose that n_i is a multiple of p (it suffices to take the least common multiple of p and the length of the period). For all $i \in [\![0,p-1]\!]$, by Proposition 2.4.25, we get[2]

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^{\omega}$$

with $t_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - \ell_n^{(i)}$ for all $n \in [\![0, m_i+n_i-1]\!]$. Hence, all quasi-greedy expansions of 1 are ultimately periodic. The converse can be proved in a similar fashion. ∎

3.4 Alternate base shifts

We now give more characterizations of the greedy and lazy $\boldsymbol{\beta}$-shifts when $\boldsymbol{\beta}$ is an alternate base. In particular, we extend Bertrand-Mathis' theorem by characterizing alternate bases $\boldsymbol{\beta}$ having sofic greedy $\boldsymbol{\beta}$-shifts and we extend this characterization to the lazy framework.

3.4.1 Greedy shifts and characterization of sofic ones

We define sets of finite words $Y_{\boldsymbol{\beta},h}$ for $h \in [\![0,p-1]\!]$ as follows.

Definition 3.4.1. If $d^*_{\boldsymbol{\beta}}(1) = t_0 t_1 \cdots$ then we let

$$Y_{\boldsymbol{\beta},h} = \{t_0 \cdots t_{n-2}s : n \in \mathbb{N}_{\geq 1},\ n \bmod p = h,\ s \in [\![0, t_{n-1}-1]\!]\}.$$

[1] Recall that $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1)$ can be finite, hence, n_i can be equal to 1 and $\ell_{m_i}^{(i)} = 0$.

[2] Note that the preperiod and period m_i and n_i may be not minimal.

Note that $Y_{\boldsymbol{\beta},h}$ is empty if and only if for all $n \in \mathbb{N}_{\geq 1}$ such that $n \bmod p = h$, $t_{n-1} = 0$. So, unlike the sets $X_{\boldsymbol{\beta},n}$ defined in Section 2.3.5, the sets $Y_{\boldsymbol{\beta},h}$ can be infinite. More precisely, $Y_{\boldsymbol{\beta},h}$ is infinite if and only if there exist infinitely many $n \in \mathbb{N}_{\geq 1}$ such that $n \bmod p = h$ and $t_{n-1} > 0$.

Proposition 3.4.2. *We have*

$$D_{\boldsymbol{\beta}} = \bigcup_{h_0=0}^{p-1} Y_{\boldsymbol{\beta},h_0}\left(\bigcup_{h_1=0}^{p-1} Y_{\boldsymbol{\beta}^{(h_0)},h_1}\left(\bigcup_{h_2=0}^{p-1} Y_{\boldsymbol{\beta}^{(h_0+h_1)},h_2}\left(\quad\cdots\quad\right)\right)\right).$$

Proof. It is easily seen that for all $h \in [\![0, p-1]\!]$,

$$\bigcup_{h=0}^{p-1} Y_{\boldsymbol{\beta},h} = \bigcup_{n\in\mathbb{N}_{\geq 1}} X_{\boldsymbol{\beta},n}.$$

The conclusion follows from Proposition 2.3.46. ∎

Corollary 3.4.3. *We have* $D_{\boldsymbol{\beta}} = \bigcup_{h=0}^{p-1} Y_{\boldsymbol{\beta},h} D_{\boldsymbol{\beta}^{(h)}}$.

In the case of a Parry alternate base $\boldsymbol{\beta}$, following the same lines as in Definition 1.4.22, we define an automaton over the finite alphabet $A_{\boldsymbol{\beta}}$.

Definition 3.4.4. Suppose that, for all $i \in [\![0, p-1]\!]$, $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic and write

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)}\left(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\right)^{\omega}$$

Let $\mathcal{A}_{\boldsymbol{\beta}}$ be the automaton defined as follows. The set of states is

$$Q = \left\{q_{i,j,k} : i, j \in [\![0, p-1]\!],\ k \in [\![0, m_i + n_i - 1]\!]\right\}.$$

The set I of initial states and the set F of final states are defined as

$$I = \left\{q_{i,i,0} : i \in [\![0, p-1]\!]\right\} \quad \text{and} \quad F = Q.$$

The (partial) transition function $E\colon Q \times A_{\boldsymbol{\beta}} \to Q$ of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is defined as follows. For each $i, j \in [\![0, p-1]\!]$ and each $k \in [\![0, m_i + n_i - 1]\!]$, we have

$$E(q_{i,j,k}, t_k^{(i)}) = \begin{cases} q_{i,(j+1) \bmod p,k+1} & \text{if } k \neq m_i + n_i - 1 \\ q_{i,(j+1) \bmod p,m_i} & \text{else} \end{cases}$$

and for all $s \in [\![0, t_k^{(i)} - 1]\!]$, we have

$$E(q_{i,j,k}, s) = q_{(j+1) \bmod p, (j+1) \bmod p, 0}.$$

Example 3.4.5. Let $\boldsymbol{\beta} = (\overline{\varphi^2, 2\varphi^2})$. Then $d_{\boldsymbol{\beta}^{(0)}}(1) = 2(30)^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 5(03)^\omega$. The corresponding automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is depicted in Figure 3.1. By removing the non-accessible states, we obtain the automaton of Figure 3.2.

The following result extends Theorem 1.4.21.

Theorem 3.4.6. *The greedy $\boldsymbol{\beta}$-shift $\Sigma_{\boldsymbol{\beta}}$ is sofic if and only if $\boldsymbol{\beta}$ is a Parry alternate base.*

Proof. Suppose that for all $i \in [\![0, p-1]\!]$, $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic. We show that the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ accepts the language $\operatorname{Fac}(\Sigma_{\boldsymbol{\beta}})$. From Propositions 2.3.43 and 2.3.44, we obtain that

$$\operatorname{Fac}(\Sigma_{\boldsymbol{\beta}}) = \operatorname{Pref}(\Delta_{\boldsymbol{\beta}}) = \bigcup_{i=0}^{p-1} \operatorname{Pref}(D_{\boldsymbol{\beta}^{(i)}}). \tag{3.2}$$

Therefore, it suffices to show that for each $i \in [\![0, p-1]\!]$, the language accepted from the initial state $q_{i,i,0}$ is precisely $\operatorname{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Let thus $i \in [\![0, p-1]\!]$.

First, consider a word w accepted from $q_{i,i,0}$. By Corollary 2.3.48, if w is a prefix of $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ then $w \in \operatorname{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Otherwise, by construction of $\mathcal{A}_{\boldsymbol{\beta}}$, w starts with some $u \in Y_{\boldsymbol{\beta}^{(i)}, h_0}$ where $h_0 = |u| \bmod p$. Moreover, the state reached after reading u from $q_{i,i,0}$ is $q_{j,j,0}$ where $j = (i + h_0) \bmod p$. We obtain that $w \in \operatorname{Pref}(D_{\boldsymbol{\beta}^{(i)}})$ by iterating the reasoning and by using Proposition 3.4.2.

Conversely, let $w \in \operatorname{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. By Proposition 3.4.2, we know that there exists $k \in \mathbb{N}$ and $h_0, \ldots, h_k \in [\![0, p-1]\!]$ such that $w = u_0 \cdots u_{k-1} x$ with $u_n \in Y_{\boldsymbol{\beta}^{(i+h_0+\cdots h_{n-1})}, h_n}$ for all $n \in [\![0, k-1]\!]$ and x is a (possibly empty) prefix of $d^*_{\boldsymbol{\beta}^{(i_k)}}(1)$ where $i_k = (i + h_0 + \cdots + h_{k-1}) \bmod p$. By construction of $\mathcal{A}_{\boldsymbol{\beta}}$, by reading u_0 from the state $q_{i,0}^{(i)}$, we reach the state $q_{i_1,i_1,0}$ where $i_1 = (i + h_0) \bmod p$. Then, by reading u_1 from the latter state, we reach the state $q_{i_2,i_2,0}$ where $i_2 = (i + h_0 + h_1) \bmod p$. By iterating the argument, after reading $u_0 \cdots u_{k-1}$, we end up in the state $q_{i_k,i_k,0}$. Since x is a prefix of $d^*_{\boldsymbol{\beta}^{(i_k)}}(1)$, it is possible to read x from the state $q_{i_k,i_k,0}$ in $\mathcal{A}_{\boldsymbol{\beta}}$. Since all states of $\mathcal{A}_{\boldsymbol{\beta}}$ are final, we obtain that w is accepted from $q_{i,i,0}$.

We turn to the necessary condition. Let

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} t_1^{(i)} \cdots \quad \text{for every } i \in [\![0, p-1]\!].$$

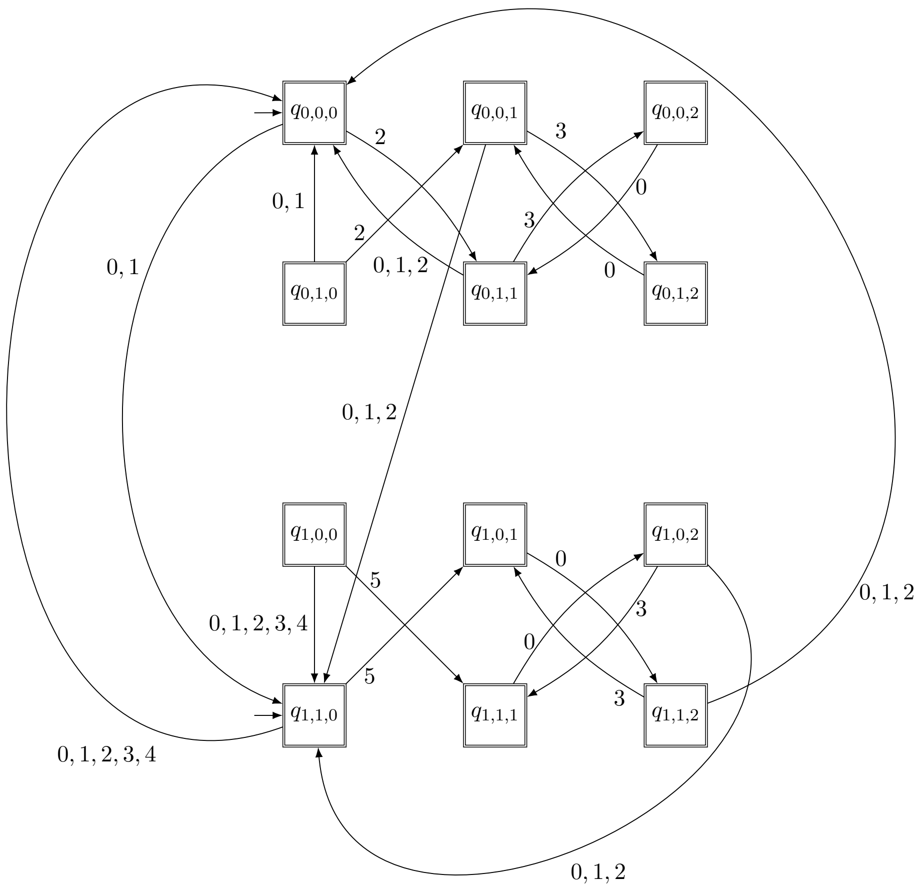

Figure 3.1: The automaton $\mathcal{A}_{\overline{(\varphi^2, 2\varphi^2)}}$.

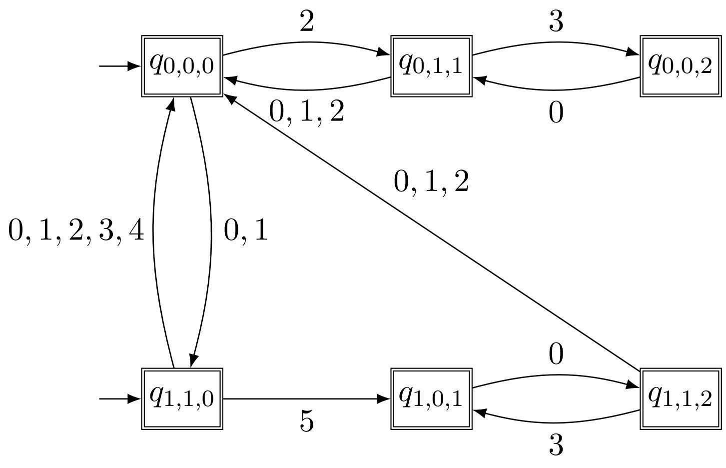

Figure 3.2: An accessible automaton accepting $\mathrm{Fac}(\Sigma_{\overline{(\varphi^2, 2\varphi^2)}})$.

Suppose that $j \in [\![0,p-1]\!]$ is such that $d^*_{\boldsymbol{\beta}^{(j)}}(1)$ is not ultimately periodic. Our aim is to find an infinite sequence $(w^{(m)})_{m\in\mathbb{N}}$ of finite words over $A_{\boldsymbol{\beta}}$ such that for all distinct $m,n \in \mathbb{N}$, the words $w^{(m)}$ and $w^{(n)}$ are not right congruent with respect to $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$. Recall that words x and y are not right congruent with respect to a language L if $x^{-1}L \neq y^{-1}L$, that is, if there exists some word z such that either $xz \in L$ and $yz \notin L$, or $xz \notin L$ and $yz \in L$. If we succeed then we will know that the number of right congruence classes is infinite and we will be able to conclude that $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ is not accepted by a finite automaton.

We define a partition $(G_1,\ldots,G_q)$ of $[\![0,p-1]\!]$ as follows. Let $r = \mathrm{Card}\{d^*_{\boldsymbol{\beta}^{(i)}}(1) : i \in [\![0,p-1]\!]\}$ and let $i_1,\ldots,i_r \in [\![0,p-1]\!]$ be such that $d^*_{\boldsymbol{\beta}^{(i_1)}}(1),\ldots,d^*_{\boldsymbol{\beta}^{(i_r)}}(1)$ are pairwise distinct. Without loss of generality, we can suppose that $d^*_{\boldsymbol{\beta}^{(i_1)}}(1) >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(i_r)}}(1)$. Let $q \in [\![1,r]\!]$ be the unique index such that $d^*_{\boldsymbol{\beta}^{(i_q)}}(1) = d^*_{\boldsymbol{\beta}^{(j)}}(1)$. We set

$$G_s = \{i \in [\![0,p-1]\!] : d^*_{\boldsymbol{\beta}^{(i)}}(1) = d^*_{\boldsymbol{\beta}^{(i_s)}}(1)\} \quad \text{for } s \in [\![1,q-1]\!]$$

and

$$G_q = \{i \in [\![0,p-1]\!] : d^*_{\boldsymbol{\beta}^{(i)}}(1) \leq d^*_{\boldsymbol{\beta}^{(j)}}(1)\}.$$

For each $s \in [\![1,q-1]\!]$, we write $G_s = \{i_{s,1},\ldots,i_{s,\alpha_s}\}$ where $i_{s,1} < \ldots < i_{s,\alpha_s}$ and we use the convention that $i_{s,\alpha_s+1} = i_{s+1,1}$ for $s \leq q-2$ and $i_{q-1,\alpha_{q-1}+1} = j$. Moreover, we let $g \in \mathbb{N}_{\geq 1}$ be such that for all $i,i' \in [\![0,p-1]\!]$ such that $d^*_{\boldsymbol{\beta}^{(i)}}(1) \neq d^*_{\boldsymbol{\beta}^{(i')}}(1)$, the length-$g$ prefixes of $d^*_{\boldsymbol{\beta}^{(i)}}(1)$ and $d^*_{\boldsymbol{\beta}^{(i')}}(1)$ are distinct. Then, for $s \in [\![1,q-1]\!]$, we define C_s to be the least $c \in \mathbb{N}_{\geq 1}$ such that $t^{(i_s)}_{g-1+c} > 0$. Finally, let $N \in \mathbb{N}_{\geq 1}$ be such that $pN \geq \max\{g,C_1,\ldots,C_{q-1}\}$.

For all $m \in \mathbb{N}$, consider

$$w^{(m)} = \left(\prod_{s=1}^{q-1}\prod_{k=1}^{\alpha_s} t^{(i_s)}_0 \cdots t^{(i_s)}_{g-1} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}\right) t^{(j)}_0 \cdots t^{(j)}_{m-1}.$$

For all $m \in \mathbb{N}$, $s \in [\![1,q-1]\!]$ and $k \in [\![1,\alpha_s]\!]$, the factor

$$t^{(i_s)}_0 \cdots t^{(i_s)}_{g-1} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}$$

has length $p(2N+1)+i_{s,k+1}-i_{s,k}$, and hence occurs at a position congruent to $i_{s,k}-i_{1,1}$ modulo p in $w^{(m)}$. Similarly, for all $m \in \mathbb{N}$, the factor $t^{(j)}_0 \cdots t^{(j)}_{m-1}$ occurs at a position congruent to $j-i_{1,1}$ modulo p in $w^{(m)}$. These observations will be crucial in what follows. The situation is illustrated in Figure 3.3.

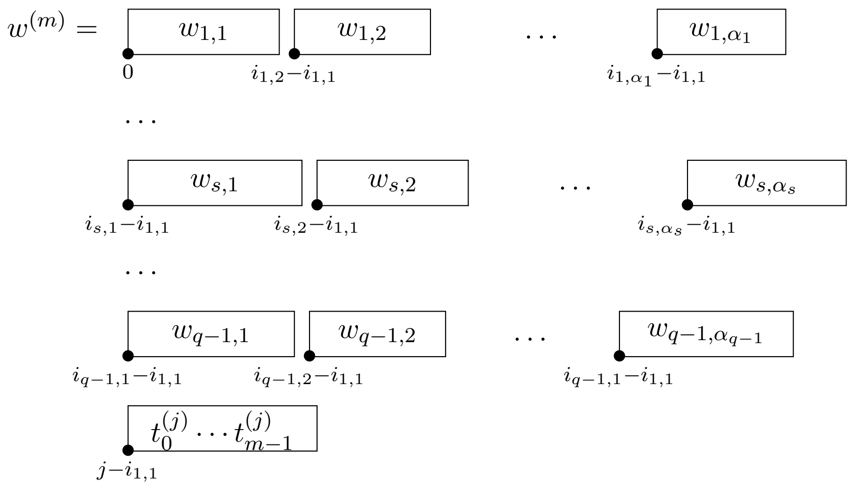

Figure 3.3: Positions modulo p of the occurrences of the factors $w_{k,s}$ and $t_0^{(j)} \cdots t_{m-1}^{(j)}$ in $w^{(m)}$, where $w_{k,s} = t_0^{(i_s)} \cdots t_{g-1}^{(i_s)} 0^{p(2N+1)-g+i_{s,k+1}-i_{s,k}}$.

Now, let $m, n \in \mathbb{N}$ be distinct. Since $d^*_{\boldsymbol{\beta}^{(j)}}(1)$ is not ultimately periodic, $\sigma^m\big(d^*_{\boldsymbol{\beta}^{(j)}}(1)\big) \neq \sigma^n\big(d^*_{\boldsymbol{\beta}^{(j)}}(1)\big)$. Thus, there exists $l \in \mathbb{N}_{\geq 1}$ such that $t_m^{(j)} \cdots t_{m+l-2}^{(j)} = t_n^{(j)} \cdots t_{n+l-2}^{(j)}$ and $t_{m+l-1}^{(j)} \neq t_{n+l-1}^{(j)}$. Without loss of generality, we suppose that $t_{m+l-1}^{(j)} > t_{n+l-1}^{(j)}$. Let $z = t_m^{(j)} \cdots t_{m+l-1}^{(j)}$. Our aim is to show that $w^{(m)}z \in \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ and $w^{(n)}z \notin \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$.

In order to obtain that $w^{(m)}z \in \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$, we show that $w^{(m)}z \in \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i_{1,1})}})$. First, for all $s \in [\![1, q-1]\!]$ and $k \in [\![1, \alpha_s]\!]$, $t_0^{(i_s)} \cdots t_{g-1}^{(i_s)} 0^{C_s} \in Y_{\boldsymbol{\beta}^{(i_{s,k})}, (g+C_s) \bmod p}$. Second, for all $i \in [\![0, p-1]\!]$, $0 \in Y_{\boldsymbol{\beta}^{(i)},1}$. Third, by Corollary 2.3.48, for all $h \in [\![0, p-1]\!]$, $t_0^{(j)} \cdots t_{m-1}^{(j)} z \in \mathrm{Pref}(Y_{\boldsymbol{\beta}^{(j)},h})$. The conclusion follows from Proposition 3.4.2.

In view of (3.2), in order to prove that $w^{(n)}z \notin \mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$, it suffices to show that for all $i \in [\![0, p-1]\!]$, $w^{(n)}z \notin \mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Proceed by contradiction and let $i \in [\![0, p-1]\!]$ and $w \in D_{\boldsymbol{\beta}^{(i)}}$ such that $w^{(n)}z$ is a prefix of w. By Theorem 2.3.33, for all $s \in [\![1, q]\!]$, the factor $t_0^{(i_s)} \cdots t_{g-1}^{(i_s)} 0^{C_s}$ occurs at a position e in w such that $(i+e) \bmod p$ belongs to $G_1 \cup \cdots \cup G_s$. For $s = 1$, we obtain that for all $k \in [\![1, \alpha_1]\!]$, $(i + i_{1,k} - i_{1,1}) \bmod p \in G_1$, and hence that

$$G_1 = \{(i + i_{1,1} - i_{1,1}) \bmod p, \ldots, (i + i_{1,\alpha_1} - i_{1,1}) \bmod p\}.$$

For $s = 2$, we get that for all $k \in [\![1, \alpha_2]\!]$, $(i + i_{2,k} - i_{1,1}) \bmod p \in G_1 \cup G_2$. If $(i + i_{2,k} - i_{1,1}) \bmod p \in G_1$ for some $k \in [\![1, \alpha_2]\!]$, then there exists $k' \in [\![1, \alpha_1]\!]$

such that $(i + i_{2,k} - i_{1,1}) \bmod p = (i + i_{1,k'} - i_{1,1}) \bmod p$, hence such that $i_{2,k} = i_{1,k'}$, which is impossible since G_1 and G_2 are disjoint. It follows that

$$G_2 = \{(i + i_{2,1} - i_{1,1}) \bmod p, \ldots, (i + i_{2,\alpha_2} - i_{1,1}) \bmod p\}.$$

By iterating the reasoning, we obtain that

$$G_s = \{(i+i_{s,1}-i_{1,1}) \bmod p, \ldots, (i+i_{s,\alpha_s}-i_{1,1}) \bmod p\} \quad \text{for all } s \in [\![1, q-1]\!].$$

We finally get that $(i+j-i_{1,1}) \bmod p$ belongs to G_q. Then $d^*_{\boldsymbol{\beta}^{((i+j-i_{1,1}) \bmod p)}}(1)$ $\leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(j)}}(1)$. Let e be the position where the factor $t_0^{(j)} \cdots t_{n-1}^{(j)}$ occurs in $w^{(n)}$, and hence also in w since $w^{(n)}z$ is a prefix of w. We have seen that $e \bmod p = j - i_{1,1} \bmod p$. Since $w \in D_{\boldsymbol{\beta}^{(i)}}$, it follows from Theorem 2.3.33 that

$$\sigma^e(w) <_{\text{lex}} d^*_{\boldsymbol{\beta}^{(i+e)}}(1) = d^*_{\boldsymbol{\beta}^{((i+j-i_{1,1}) \bmod p)}}(1) \leq_{\text{lex}} d^*_{\boldsymbol{\beta}^{(j)}}(1).$$

We have thus reached a contradiction since the factor $t_0^{(j)} \cdots t_{n-1}^{(j)} z$ is lexicographically greater than the length-$(n+l)$ prefix of $d^*_{\boldsymbol{\beta}^{(j)}}(1)$. ■

Note that, in the classical case $p = 1$, the previous proof is much shorter since $\Sigma_\beta = S_\beta$, $\text{Fac}(\Sigma_\beta) = \text{Pref}(D_\beta)$, and hence we can directly deduce that the words $t_0^{(j)} \cdots t_{m-1}^{(j)}$ and $t_0^{(j)} \cdots t_{n-1}^{(j)}$ (where in fact, $j = 0$) are not right congruent with respect to $\text{Fac}(\Sigma_\beta)$.

For $p = 1$, it is well known that the β-shift is of finite type if and only if $d_\beta(1)$ is finite (see Theorem 1.4.20). However, this result does not generalize to $p \geq 2$ as is illustrated by the following example.

Example 3.4.7. Consider the alternate base $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$ of Example 2.3.27. We have $d_{\boldsymbol{\beta}}(1) = 201$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 11$. We get $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$. By Theorem 2.3.33, we see that all words in $2(00)^*2$ are factors avoided by $\Sigma_{\boldsymbol{\beta}}$, so the greedy $\boldsymbol{\beta}$-shift $\Sigma_{\boldsymbol{\beta}}$ is not of finite type.

3.4.2 Lazy shifts and characterization of sofic ones

As in the greedy case, Proposition 2.4.44 can be straightened. To do so, we define sets of finite words $Y'_{\boldsymbol{\beta},h}$ for $h \in [\![0, p-1]\!]$ as follows.

Definition 3.4.8. If $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = \ell_0 \ell_1 \cdots$ then, for all $h \in [\![0, p-1]\!]$, we let

$$Y'_{\boldsymbol{\beta},h} = \{\ell_0 \cdots \ell_{n-2} s : n \in \mathbb{N}_{\geq 1},\ n \bmod p = h,\ s \in [\![\ell_{n-1} + 1, \lceil \beta_{n-1} \rceil - 1]\!]\}.$$

Note that $Y'_{\boldsymbol{\beta},h}$ is empty if and only if for all $n \in \mathbb{N}_{\geq 1}$ such that $n \bmod p = h$, $\ell_{n-1} = \lceil \beta_{n-1} \rceil - 1$. Moreover, unlike the sets $X'_{\boldsymbol{\beta},n}$ defined in Section 2.4.7, the sets $Y'_{\boldsymbol{\beta},h}$ can be infinite.

Proposition 3.4.9. *We have*

$$D'_{\boldsymbol{\beta}} = \bigcup_{h_0=0}^{p-1} Y'_{\boldsymbol{\beta},h_0} \left(\bigcup_{h_1=0}^{p-1} Y'_{\boldsymbol{\beta}^{(h_0)},h_1} \left(\bigcup_{h_2=0}^{p-1} Y'_{\boldsymbol{\beta}^{(h_0+h_1)},h_2} \left(\quad \cdots \quad \right)\right)\right).$$

Therefore, we have $D'_{\boldsymbol{\beta}} = \bigcup_{h=0}^{p-1} Y'_{\boldsymbol{\beta},h} D'_{\boldsymbol{\beta}^{(h)}}$.

In the lazy alternate base framework, an analogue of Bertrand-Mathis' theorem can be stated for the lazy $\boldsymbol{\beta}$-shift.

Theorem 3.4.10. *The lazy $\boldsymbol{\beta}$-shift $\Sigma'_{\boldsymbol{\beta}}$ is sofic if and only if $\boldsymbol{\beta}$ is a Parry alternate base.*

In order to prove this result, let us construct an automaton $\mathcal{A}'_{\boldsymbol{\beta}}$ in the case where all quasi-lazy expansions are ultimately periodic and state some results in order to link this automaton with the one used in the greedy case Theorem 3.4.6, namely the automaton $\mathcal{A}_{\boldsymbol{\beta}}$. Roughly, if all the quasi-lazy expansions are ultimately periodic, then so are the quasi-greedy expansions and the "image" of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ under the maps $\theta_{\boldsymbol{\beta}^{(i)}}$ with $i \in [\![0, p-1]\!]$ is an automaton accepting $\mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$. This notion of "image" of the automaton under the maps $\theta_{\boldsymbol{\beta}^{(i)}}$ will be clearer in what follows, more precisely in Lemmas 3.4.13 and 3.4.15.

Henceforth, suppose that for all $i \in [\![0, p-1]\!]$, $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1)$ is ultimately periodic and write

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^{\omega}.$$

As done in the proof of Proposition 3.3.4, without loss of generality, from now on, suppose that n_i is a multiple of p. For all $i \in [\![0, p-1]\!]$, by Proposition 2.4.25, we get

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^{\omega}$$

with $t_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - \ell_n^{(i)}$ for all $n \in [\![0, m_i + n_i - 1]\!]$.

Definition 3.4.11. Let $\mathcal{A}_{\boldsymbol{\beta}} = (Q, I, F, A_{\boldsymbol{\beta}}, E)$ be the automaton over the alphabet $A_{\boldsymbol{\beta}}$ from Section 3.4.1 which accepts $\mathrm{Fac}(\Sigma_{\boldsymbol{\beta}})$ (see Theorem 3.4.6). Define the automaton $\mathcal{A}'_{\boldsymbol{\beta}} = (Q, I, F, A_{\boldsymbol{\beta}}, E')$ where for each $i, j \in [\![0, p-1]\!]$ and each $k \in [\![0, m_i + n_i - 1]\!]$, we have

$$E'(q_{i,j,k}, \ell_k^{(i)}) = \begin{cases} q_{i,(j+1) \bmod p, k+1} & \text{if } k \neq m_i + n_i - 1 \\ q_{i,(j+1) \bmod p, m_i} & \text{else} \end{cases} \tag{3.3}$$

and for all $s \in [\![\ell_k^{(i)} + 1, \lceil \beta_j \rceil - 1]\!]$, we have

$$E'(q_{i,j,k}, s) = q_{(j+1) \bmod p, (j+1) \bmod p, 0}. \tag{3.4}$$

Since we supposed that the parameters n_i, with $i \in [\![0, p-1]\!]$, were multiples of p, we get the following result.

Lemma 3.4.12. *In the automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$, for all $i, j \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$, the state $q_{i,j,k}$ is accessible if and only if $i + k \bmod p = j \bmod p$.*

Proof. Let us prove the result for the automaton $\mathcal{A}'_{\boldsymbol{\beta}}$. The reasoning for the automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is similar. Suppose that $i + k \bmod p = j \bmod p$. There exists a path from $q_{i,i,0}$ to $q_{i,j,k}$ labeled by $\ell_0^{(i)} \cdots \ell_k^{(i)}$. In fact, for all $k' \in [\![0, k-1]\!]$, we have

$$E'(q_{i,(i+k') \bmod p, k'}, \ell_{k'}^{(i)}) = q_{i,(i+k'+1) \bmod p, k'+1}. \tag{3.5}$$

Conversely, let $i, j \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$. Suppose that the state $q_{i,j,k}$ is accessible. Let c be an initial path ending in $q_{i,j,k}$. By definition of the transitions, if a path starts in $q_{i',i',0}$ with $i' \in [\![0, p-1]\!] \setminus \{i\}$ and ends in $q_{i,j,k}$ then it necessarily goes through $q_{i,i,0}$ by using a transition of the form (3.4). Hence, we may suppose that the path c only uses transitions of the form (3.3). The conclusion follows since for all $k' \in [\![0, k-1]\!]$, we have (3.5) and

$$E'(q_{i,(i+m_i+n_i-1) \bmod p, m_i+n_i-1}, \ell_{m_i+n_i-1}^{(i)}) = q_{i,(i+m_i+n_i) \bmod p, m_i}$$

where $n_i \bmod p = 0$ by assumption. ∎

By the previous lemma, from now on, we consider the automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ by preserving only the set

$$\left\{ q_{i,(i+k) \bmod p, k} : i \in [\![0, p-1]\!],\ k \in [\![0, m_i + n_i - 1]\!] \right\} \tag{3.6}$$

of accessible states and we keep the same notation. Moreover, for the sake of clarity, we now denote $q_{i,k}$ instead of $q_{i,(i+k) \bmod p,k}$ since the second index is completely determined by the other two.

Lemma 3.4.13. *Let $a \in A_{\boldsymbol{\beta}}$, $i_1, i_2 \in [\![0, p-1]\!]$ and $k_1 \in [\![0, m_{i_1}+n_{i_1}-1]\!]$, $k_2 \in [\![0, m_{i_2} + n_{i_2} - 1]\!]$. We have*

$$E(q_{i_1,k_1}, a) = q_{i_2,k_2}$$

if and only if

$$E'(q_{i_1,k_1}, \lceil \beta_{i_1+k_1} \rceil - 1 - a) = q_{i_2,k_2}.$$

Proof. Fix $a \in A_{\boldsymbol{\beta}}$, $i \in [\![0, p-1]\!]$ and $k \in [\![0, m_i + n_i - 1]\!]$. By definition of the automaton $\mathcal{A}_{\boldsymbol{\beta}}$, from $q_{i,k}$ we have the following transitions

$$E(q_{i,k}, a) = \begin{cases} q_{i,k+1} & \text{if } a = t_k^{(i)} \text{ and } k \neq m_i + n_i - 1 \\ q_{i,m_i} & \text{if } a = t_k^{(i)} \text{ and } k = m_i + n_i - 1 \\ q_{(i+k+1) \bmod p,0} & \text{if } a \in [\![0, t_k^{(i)} - 1]\!]. \end{cases}$$

Similarly, by definition of $\mathcal{A}'_{\boldsymbol{\beta}}$, we have

$$E'(q_{i,k}, a) = \begin{cases} q_{i,k+1} & \text{if } a = \ell_k^{(i)} \text{ and } k \neq m_i + n_i - 1 \\ q_{i,m_i} & \text{if } a = \ell_k^{(i)} \text{ and } k = m_i + n_i - 1 \\ q_{(i+k+1) \bmod p,0} & \text{if } a \in [\![\ell_k^{(i)} + 1, \lceil \beta_{i+k} \rceil - 1]\!]. \end{cases}$$

We get the conclusion since $\ell_k^{(i)} = \lceil \beta_{i+k} \rceil - 1 - t_k^{(i)}$, and hence $a \in [\![0, t_k^{(i)} - 1]\!]$ if and only if $\lceil \beta_{i+k} \rceil - 1 - a \in [\![\ell_k^{(i)} + 1, \lceil \beta_{i+k} \rceil - 1]\!]$. ■

Example 3.4.14. Let $\boldsymbol{\beta} = (\overline{\varphi^2, 2\varphi^2})$ from Example 3.4.5. We have $d_{\boldsymbol{\beta}}(1) = 2(30)^\omega$, $d_{\boldsymbol{\beta}^{(1)}}(1) = 5(03)^\omega$ and $\ell_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = 02^\omega$, $\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = 02^\omega$. The corresponding accessible automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ are depicted in Figure 3.4 with red and blue labels respectively. Note that the accessible automaton $\mathcal{A}_{\boldsymbol{\beta}}$ is already depicted in Figure 3.2, but we also depicted the (greedy) red labels in Figure 3.4 to illustrate Lemma 3.4.13.

Lemma 3.4.15. *Let $i \in [\![0, p-1]\!]$ and consider $w \in A_{\boldsymbol{\beta}}^{\mathbb{N}}$. The word w is accepted in $\mathcal{A}_{\boldsymbol{\beta}}$ from $q_{i,0}$ if and only if $\theta_{\boldsymbol{\beta}^{(i)}}(w)$ is accepted in $\mathcal{A}'_{\boldsymbol{\beta}}$ from $q_{i,0}$.*

Proof. This immediately follows from Lemma 3.4.13. ■

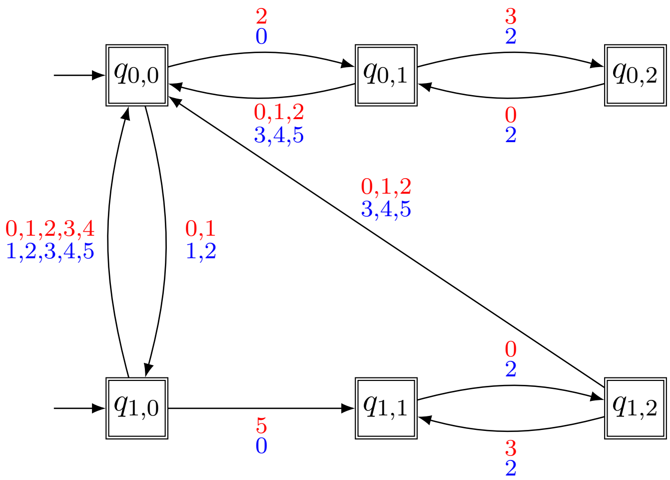

Figure 3.4: An accessible automaton accepting $\mathrm{Fac}(\Sigma_{\overline{(\varphi^2,2\varphi^2)}})$ (red labels) and $\mathrm{Fac}(\Sigma'_{\overline{(\varphi^2,2\varphi^2)}})$ (blue labels).

We are now ready to prove Theorem 3.4.10.

Proof of Theorem 3.4.10. Suppose that, for all $i \in [\![0, p-1]\!]$, $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1)$ is ultimately periodic. For all $i \in [\![0, p-1]\!]$, let

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^\omega$$

with n_i multiple of p. By Proposition 2.4.25, for all $i \in [\![0, p-1]\!]$, we obtain

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^\omega$$

with $t_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - \ell_n^{(i)}$ for all $n \in [\![0, m_i+n_i-1]\!]$. Let $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ be the automata associated with the greedy and lazy expansions respectively. By Theorem 3.4.6, for each $i \in [\![0, p-1]\!]$, the language accepted in $\mathcal{A}_{\boldsymbol{\beta}}$ from the initial state $q_{i,0}$ is precisely $\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}})$. Hence, by Lemma 3.4.15, in $\mathcal{A}'_{\boldsymbol{\beta}}$ the language accepted from the initial state $q_{i,0}$ is precisely $\theta_{\boldsymbol{\beta}^{(i)}}(\mathrm{Pref}(D_{\boldsymbol{\beta}^{(i)}}))$. We get the conclusion by Corollary 2.4.48.

Conversely, suppose that there exists $j \in [\![0, p-1]\!]$ such that $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)$ is not ultimately periodic. Then we prove that $\Sigma'_{\boldsymbol{\beta}}$ is not sofic. This follows the same lines as in the greedy case (see Theorem 3.4.6). Hence, in what follows, the main ideas of the proof are given. Let

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \ell_1^{(i)} \cdots \quad \text{for every } i \in [\![0, p-1]\!].$$

We define a partition $(G_1,\dots,G_q)$ of $[\![0,p-1]\!]$ as follows. Let $r = \mathrm{Card}\{\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) : i \in [\![0,p-1]\!]\}$ and let $i_1,\dots,i_r \in [\![0,p-1]\!]$ be such that $\ell^*_{\boldsymbol{\beta}^{(i_1)}}(x_{\boldsymbol{\beta}^{(i_1)}}-1),\dots,\ell^*_{\boldsymbol{\beta}^{(i_r)}}(x_{\boldsymbol{\beta}^{(i_r)}}-1)$ are pairwise distinct and $\ell^*_{\boldsymbol{\beta}^{(i_1)}}(x_{\boldsymbol{\beta}^{(i_1)}}-1) <_{\mathrm{lex}} \cdots <_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(i_r)}}(x_{\boldsymbol{\beta}^{(i_r)}}-1)$. Let $q \in [\![1,r]\!]$ be the unique index such that $\ell^*_{\boldsymbol{\beta}^{(i_q)}}(x_{\boldsymbol{\beta}^{(i_q)}}-1) = \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)$ where $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)$ is not ultimately periodic by assumption. We set, for $s \in [\![1,q-1]\!]$,

$$G_s = \{i \in [\![0,p-1]\!] : \ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) = \ell^*_{\boldsymbol{\beta}^{(i_s)}}(x_{\boldsymbol{\beta}^{(i_s)}}-1)\}$$

and

$$G_q = \{i \in [\![0,p-1]\!] : \ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) \geq_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)\}.$$

For each $s \in [\![1,q-1]\!]$, we write $G_s = \{i_{s,1},\dots,i_{s,\alpha_s}\}$ where $i_{s,1} < \dots < i_{s,\alpha_s}$ and we use the convention that $i_{s,\alpha_s+1} = i_{s+1,1}$ for $s \leq q-2$ and $i_{q-1,\alpha_{q-1}+1} = j$. Moreover, we let $g \in \mathbb{N}_{\geq 1}$ be such that for all $i,i' \in [\![0,p-1]\!]$ such that $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) \neq \ell^*_{\boldsymbol{\beta}^{(i')}}(x_{\boldsymbol{\beta}^{(i')}}-1)$, the length-$g$ prefixes of $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1)$ and $\ell^*_{\boldsymbol{\beta}^{(i')}}(x_{\boldsymbol{\beta}^{(i')}}-1)$ are distinct. Then, for $s \in [\![1,q-1]\!]$, we define C_s to be the least $c \in \mathbb{N}_{\geq 1}$ such that $\ell^{(i_s)}_{g-1+c} < \lceil \beta_{i_s+g-1+c} \rceil - 1$. Finally, let $N \in \mathbb{N}_{\geq 1}$ be such that $pN \geq \max\{g, C_1,\dots,C_{q-1}\}$.

For all $m \in \mathbb{N}$, we define the word $w^{(m)}$ by

$$\left(\prod_{s=1}^{q-1}\prod_{k=1}^{\alpha_s} \ell^{(i_s)}_0 \cdots \ell^{(i_s)}_{g-1}(\lceil \beta_{i_{s,k}+g} \rceil - 1)\cdots(\lceil \beta_{i_{s,k+1}+p(2N+1)-1} \rceil - 1)\right) \ell^{(j)}_0 \cdots \ell^{(j)}_{m-1}.$$

Now, let $m,n \in \mathbb{N}$ be distinct. Since $\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)$ is not ultimately periodic, $\sigma^m\big(\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)\big) \neq \sigma^n\big(\ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)\big)$. Thus, there exists $k \in \mathbb{N}_{\geq 1}$ such that $\ell^{(j)}_m \cdots \ell^{(j)}_{m+k-2} = \ell^{(j)}_n \cdots \ell^{(j)}_{n+k-2}$ and $\ell^{(j)}_{m+k-1} \neq \ell^{(j)}_{n+k-1}$. Without loss of generality, we suppose that $\ell^{(j)}_{m+k-1} < \ell^{(j)}_{n+k-1}$. Let $z = \ell^{(j)}_m \cdots \ell^{(j)}_{m+k-1}$. Similarly to the proof of Theorem 3.4.6 it can be shown that $w^{(m)}z \in \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}}) \cap \mathrm{Pref}(D'_{\boldsymbol{\beta}^{(i_{1,1})}})$ and $w^{(n)}z \notin \mathrm{Fac}(\Sigma'_{\boldsymbol{\beta}})$. ∎

Remark 3.4.16. In the proof of the necessary condition of Theorem 3.4.10, the parameters $\{r, i_1,\dots,i_r,q,G_1,\dots,G_q,\dots\}$ may not coincide with those in the necessary condition of Theorem 3.4.6. In fact, it may happen that there exist $i,j \in [\![0,p-1]\!]$ such that $d^*_{\boldsymbol{\beta}^{(i)}}(1) >_{\mathrm{lex}} d^*_{\boldsymbol{\beta}^{(j)}}(1)$ whereas $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}}-1) \leq_{\mathrm{lex}} \ell^*_{\boldsymbol{\beta}^{(j)}}(x_{\boldsymbol{\beta}^{(j)}}-1)$. For instance, this is illustrated in Examples 2.4.28 and 3.4.14.

CHAPTER

4

SPECTRUM AND NORMALIZATION IN ALTERNATE BASES

In this chapter, we study the algebraic properties of alternate base expansions and we generalize the normalization function in real bases to the setting of alternate bases. For this purpose, we generalize the spectrum in real bases to the complex base and alternate base frameworks.

In order to define the spectrum of numeration systems associated with alternate bases $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$, one needs to consider the spectrum of $\beta = \prod_{i=0}^{p-1} \beta_i$ with a more general alphabet of non-integer digits. Hence, we first study the spectrum $X^A(\delta)$ in the general framework of a complex base δ such that $|\delta| > 1$ with a finite alphabet $A \subset \mathbb{C}$. We prove that the set $Z(\delta, A)$ of δ-representations of zero over A is accepted by a finite Büchi automaton if and only if the spectrum $X^A(\delta)$ has no accumulation point. In doing so, we also define and study an associated zero Büchi automaton $\mathcal{Z}(\delta, A)$.

Second, we define the spectrum associated with an alternate base $\boldsymbol{\beta}$ as a particular case of the complex spectra. We then prove that the alternate base spectrum has no accumulation point if and only if the set of $\boldsymbol{\beta}$-representations of zero is accepted by a finite Büchi automaton, and furthermore, if and only if the alternate zero automaton is finite.

Third, using the spectra associated with alternate bases, we study the algebraic properties of the Parry alternate bases. In particular, we show that if $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ is a Parry alternate base, then the product $\beta = \prod_{i=0}^{p-1} \beta_i$ is an algebraic integer and all of the bases $\beta_0, \ldots, \beta_{p-1}$ belong to the algebraic field $\mathbb{Q}(\beta)$. On the other hand, we also give a sufficient condition: if β is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\beta)$, then $\boldsymbol{\beta}$ is a Parry alternate base.

Finally, we show that if β is a Pisot number and each of the bases β_i belongs to the algebraic field $\mathbb{Q}(\beta)$ then the greedy and lazy normalization functions in the alternate base $\boldsymbol{\beta}$ are computable by finite Büchi automata, and we effectively construct such automata.

The results presented in this chapter are from [CCMP22]. Since this chapter generalizes the spectrum and normalization in real base expansions to the alternate base framework, Sections 1.4.2 and 1.4.3 are related preliminaries.

Contents of the Chapter

4.1 Spectrum and representations of zero in complex bases

The spectrum of a real number $\delta > 1$ and a finite alphabet $A \subset \mathbb{Z}$ was introduced by Erdős et al [EJK90]. For our purposes, we use a generalized concept with $\delta \in \mathbb{C}$ and $A \subset \mathbb{C}$ and study its topological properties. In particular, in this section, we generalize Theorem 1.4.29 to the setting of complex bases and general alphabets of complex digits.

Definition 4.1.1. A *complex base* is a complex number δ such that $|\delta| > 1$. For a complex base δ and a finite alphabet A of complex numbers, we define the *set of δ-representations of zero over A* by

$$Z(\delta, A) = \{a \in A^{\mathbb{N}} : \sum_{n\in\mathbb{N}} \frac{a_n}{\delta^{n+1}} = 0\}$$

and the *spectrum of δ over the alphabet A* by

$$X^A(\delta) = \{\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n} : \ell \in \mathbb{N},\ a_0, a_1, \ldots, a_{\ell-1} \in A\}. \tag{4.1}$$

We say that a word $a_0 \cdots a_{\ell-1}$ over A *corresponds* to the element $\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n}$ in the spectrum $X^A(\delta)$.

For the remaining part of this section, we consider a fixed complex base δ and a fixed finite alphabet $A \subset \mathbb{C}$.

Consider the right congruence $\sim_{Z(\delta,A)}$ over A^* (see Definition 1.2.26). For the sake of simplicity, we simply write $\sim_{\delta,A}$. For $a, b \in A^*$, we have $a \sim_{\delta,A} b$ whenever for all $s \in A^{\mathbb{N}}$, we have

$$as \in Z(\delta, A) \iff bs \in Z(\delta, A).$$

Obviously, the language $A^* \setminus \mathrm{Pref}(Z(\delta, A))$ is one of the equivalence classes of $\sim_{\delta,A}$. In the context of real bases β and integer digits, this right congruence may be interpreted in terms of the remainders of the Euclidean division of polynomials in $\mathbb{Z}[x]$ by $x - \beta$; see [Fro92]. This interpretation is no longer possible in the present context of complex digits.

Lemma 4.1.2. *Let $a, b \in \mathrm{Pref}(Z(\delta, A))$ be such that $|a| = k$ and $|b| = \ell$. We have $a \sim_{\delta,A} b$ if and only if*

$$\sum_{n=0}^{k-1} a_n \delta^{k-1-n} = \sum_{n=0}^{\ell-1} b_n \delta^{\ell-1-n},$$

that is, the words a and b correspond to the same element in the spectrum $X^A(\delta)$.

Proof. Suppose that $a \sim_{\delta,A} b$. Since a and b belong to the set $\mathrm{Pref}(Z(\delta, A))$, there exists $s \in A^{\mathbb{N}}$ such that $as, bs \in Z(\delta, A)$. We get

$$-\sum_{n\in\mathbb{N}} \frac{s_n}{\delta^{n+1}} = \sum_{n=0}^{k-1} a_n \delta^{k-1-n} = \sum_{n=0}^{\ell-1} b_n \delta^{\ell-1-n}.$$

Conversely, suppose that $a \not\sim_{\delta,A} b$. Without loss of generality we can suppose that $as \in Z(\delta, A)$ and $bs \notin Z(\delta, A)$ for some $s \in A^{\mathbb{N}}$. Then

$$-\sum_{n\in\mathbb{N}} \frac{s_n}{\delta^{n+1}} = \sum_{n=0}^{k-1} a_n \delta^{k-1-n} \neq \sum_{n=0}^{\ell-1} b_n \delta^{\ell-1-n}.$$

■

Lemma 4.1.3. *If the spectrum $X^A(\delta)$ has an accumulation point in $\mathbb{C}$ then there exists an infinite word in $Z(\delta, A)$ with pairwise non-equivalent prefixes with respect to the right congruence $\sim_{\delta,A}$. In particular, the right congruence $\sim_{\delta,A}$ has infinitely many classes.*

Proof. Suppose that the spectrum $X^A(\delta)$ has a complex accumulation point. Then there exists an injective sequence $(x^{(j)})_{j\in\mathbb{N}}$ in $X^A(\delta)$ such that $\lim_{j\to+\infty} x^{(j)}$ is finite. For each $j \in \mathbb{N}$ we let $\rho(j)$ denote the minimal exponent such that there exists a representation of $x^{(j)}$ in the form $x_0^{(j)} \cdots x_{\rho(j)-1}^{(j)} \in A^*$, that is

$$x^{(j)} = \sum_{n=0}^{\rho(j)-1} x_n^{(j)} \delta^{\rho(j)-1-n}.$$

Obviously, the sequence $(\rho(j))_{j\in\mathbb{N}}$ is unbounded, and without loss of generality we can assume that $(\rho(j))_{j\in\mathbb{N}}$ is strictly increasing. Thus $\lim_{j\to+\infty} \rho(j) = +\infty$ and we get

$$\lim_{j\to+\infty} \frac{x^{(j)}}{\delta^{\rho(j)}} = \lim_{j\to+\infty} \sum_{n=0}^{\rho(j)-1} \frac{x_n^{(j)}}{\delta^{n+1}} = 0. \tag{4.2}$$

With this, we will show the existence of the desired δ-representation a of zero over A. Set a_0 as a digit in A which occurs infinitely many times among $x_0^{(j)}$ with $j \in \mathbb{N}$. Inductively, for $n \geq 1$, set a_n as a digit in A which occurs infinitely many times among $x_n^{(j)}$, where $j \in \mathbb{N}$ runs through the indices such that $x_0^{(j)} \cdots x_{n-1}^{(j)} = a_0 \cdots a_{n-1}$. By (4.2), we get that

$$\sum_{n\in\mathbb{N}} \frac{a_n}{\delta^{n+1}} = 0,$$

that is, that $a = a_0a_1a_2\cdots$ belongs to the set $Z(\delta, A)$.

We will show that no pair of distinct prefixes of the infinite word a belong to the same equivalence class. To show this by contradiction, we consider

$k, \ell \in \mathbb{N}$ such that $a_0 \cdots a_{k-1} \sim_{\delta,A} a_0 \cdots a_{\ell-1}$ with $k > \ell$. By construction, there exists $j \in \mathbb{N}$ such that $a_0 \cdots a_{k-1}$ is a prefix of $x_0^{(j)} \cdots x_{\rho(j)-1}^{(j)}$. Moreover, by Lemma 4.1.2, we get

$$\begin{aligned}
x^{(j)} &= \sum_{n=0}^{k-1} a_n \delta^{\rho(j)-1-n} + \sum_{n=k}^{\rho(j)-1} x_n^{(j)} \delta^{\rho(j)-1-n} \\
&= \delta^{\rho(j)-k} \sum_{n=0}^{k-1} a_n \delta^{k-1-n} + \sum_{n=k}^{\rho(j)-1} x_n^{(j)} \delta^{\rho(j)-1-n} \\
&= \delta^{\rho(j)-k} \sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n} + \sum_{n=k}^{\rho(j)-1} x_n^{(j)} \delta^{\rho(j)-1-n} \\
&= \sum_{n=0}^{\ell-1} a_n \delta^{\rho(j)-k+\ell-1-n} + \sum_{n=\ell}^{\rho(j)-k+\ell-1} x_{n+k-\ell}^{(j)} \delta^{\rho(j)-k+\ell-1-n}.
\end{aligned}$$

Thus, we have found a representation $a_0 \cdots a_{\ell-1} x_k^{(j)} \cdots x_{\rho(j)-1}^{(j)}$ of $x^{(j)}$ which is shorter than $x_0^{(j)} \cdots x_{\rho(j)-1}^{(j)}$. This contradicts the definition of $\rho(j)$. ∎

Similarly as what is done in [Fro92], we define a zero Büchi automaton.

Definition 4.1.4. The *zero automaton in base δ over the alphabet A* is the Büchi automaton $\mathcal{Z}(\delta, A) = (Q, 0, Q, A, E)$ where

$$Q = X^A(\delta) \cap \{z \in \mathbb{C} : |z| \leq \frac{M}{|\delta| - 1}\}$$

with $M = \max\{|a| : a \in A\}$, and the transitions are given by the triplets $(z, a, z\delta + a)$ in $Q \times A \times Q$.

Proposition 4.1.5. *The zero automaton $\mathcal{Z}(\delta, A)$ accepts the set $Z(\delta, A)$.*

Proof. Let a be an infinite word accepted by $\mathcal{Z}(\delta, A)$. For each $\ell \in \mathbb{N}$, the prefix $a_0 \cdots a_{\ell-1}$ labels a path in $\mathcal{Z}(\delta, A)$ from the initial state 0 to the state $\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n}$, that is, its corresponding element in the spectrum $X^A(\delta)$. By definition of the set of states Q, we get that the sequence $(\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n})_{\ell \in \mathbb{N}}$ is bounded. Hence, we obtain that

$$\sum_{n \in \mathbb{N}} \frac{a_n}{\delta^{n+1}} = \lim_{\ell \to +\infty} \frac{\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n}}{\delta^\ell} = 0.$$

Conversely, consider an infinite word a over A that is not accepted by $\mathcal{Z}(\delta, A)$. Then there exists $\ell \in \mathbb{N}$ such that $|\sum_{n=0}^{\ell-1} a_n \delta^{\ell-1-n}| > \frac{M}{|\delta|-1}$. Then

$$\left|\sum_{n\in\mathbb{N}} \frac{a_n}{\delta^{n+1}}\right| \geq \left|\sum_{n=0}^{\ell-1} \frac{a_n}{\delta^{n+1}}\right| - \sum_{n=\ell}^{+\infty} \frac{M}{|\delta|^{n+1}} = \frac{\left|\sum_{n=0}^{\ell-1} a_n \delta^{\ell-n-1}\right| - \frac{M}{|\delta|-1}}{|\delta|^{\ell}} > 0.$$

■

We are now ready to state and prove the main theorem of this section, which is a generalization of Theorem 1.4.29. This solves a problem that was left open in [FP18].

Theorem 4.1.6. *Let δ be a complex number such that $|\delta| > 1$ and let A be an alphabet of complex numbers. Then the following assertions are equivalent.*

1. *The set $Z(\delta, A)$ is accepted by a finite Büchi automaton.*
2. *The right congruence $\sim_{\delta,A}$ has finitely many classes.*
3. *The spectrum $X^A(\delta)$ has no accumulation point in $\mathbb{C}$.*
4. *The zero automaton $\mathcal{Z}(\delta, A)$ is finite.*

Proof. Suppose that $Z(\delta, A)$ is accepted by a finite Büchi automaton. By Theorem 1.2.27, we get that the right congruence $\sim_{\delta,A}$ has only finitely many classes. Hence (1) $\implies$ (2). The implication (2) $\implies$ (3) is given by Lemma 4.1.3. The implication (3) $\implies$ (4) follows directly from the definition of the zero automaton. Finally, the implication (4) $\implies$ (1) follows from Proposition 4.1.5. ■

Note that the zero automaton is deterministic. Therefore, the previous result shows in particular that if the set $Z(\delta, A)$ is accepted by an arbitrary Büchi automaton, possibly non-deterministic, then it must be also accepted by a deterministic one.

4.2 Spectrum and representations of zero in alternate bases

From now on, we consider a fixed positive integer p and an alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}})$, and we set $\beta = \prod_{i=0}^{p-1} \beta_i$.

For the purpose of this chapter, we extend the definition of $\boldsymbol{\beta}$-representations of real numbers (see Definition 2.1.4) in order to allow negative digits.

That is, we say that a $\boldsymbol{\beta}$-representation of a real number x is an infinite sequence a of integers such that $\mathrm{val}_{\boldsymbol{\beta}}(a) = x$.

Definition 4.2.1. An *alternate alphabet* of *length* p is a sequence

$$\boldsymbol{D} = (D_0, \ldots, D_{p-1}, D_0, \ldots, D_{p-1}, \ldots)$$

where $D_0, \ldots, D_{p-1}$ are finite alphabets of integers containing 0. We write

$$\boldsymbol{D} = (\overline{D_0, \ldots, D_{p-1}}).$$

As for alternate bases, we use the convention that for all $n \in \mathbb{Z}$, $D_n = D_{n \bmod p}$ and $\boldsymbol{D}^{(n)} = (\overline{D_n, \ldots, D_{n+p-1}})$.

From now on, we consider an alternate alphabet $\boldsymbol{D} = (\overline{D, \ldots, D_{p-1}})$ of length p.

Definition 4.2.2. Let $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$ denote the digit set defined by

$$\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D}) = \{\sum_{i=0}^{p-1} a_i \beta_{i+1} \cdots \beta_{p-1} : \forall i \in [\![0, p-1]\!],\ a_i \in D_i\}. \tag{4.3}$$

Grouping terms p by p, Equality (3.1) can be written as

$$x = \sum_{m \in \mathbb{N}} \frac{\sum_{i=0}^{p-1} a_{mp+i} \beta_{i+1} \cdots \beta_{p-1}}{\beta^{m+1}}.$$

If we add the constraint that each letter a_n belongs to D_n, then we obtain a β-representation of x over the alphabet $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$.

Let $\otimes_{n \in \mathbb{N}} D_n$ denote the set of infinite words over the alphabet $\cup_{i=0}^{p-1} D_i$ such that for all $n \in \mathbb{N}$, the $(n+1)^{\text{st}}$ letter belongs to the alphabet D_n:

$$\bigotimes_{n \in \mathbb{N}} D_n = \{a \in (\cup_{i=0}^{p-1} D_i)^{\mathbb{N}} : \forall n \in \mathbb{N},\ a_n \in D_n\}.$$

Definition 4.2.3. Let $Z(\boldsymbol{\beta}, \boldsymbol{D})$ denote the set of $\boldsymbol{\beta}$-representations of zero the $(n+1)^{\text{th}}$ digit of which belongs to the alphabet D_n:

$$Z(\boldsymbol{\beta}, \boldsymbol{D}) = \{a \in \bigotimes_{n \in \mathbb{N}} D_n : \sum_{n \in \mathbb{N}} \frac{a_n}{\prod_{k=0}^{n} \beta_k} = 0\}.$$

The set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ can be seen as a subset of $(\cup_{i=0}^{p-1} D_i)^{\mathbb{N}}$.

For $\beta = \prod_{i=0}^{p-1} \beta_i$ and the alphabet $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$, the corresponding spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ defined in (4.1) can be rewritten as

$$X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta) = \sum_{i=0}^{p-1} X^{D_i}(\beta) \cdot \beta_{i+1} \cdots \beta_{p-1}.$$

For the sake of simplicity, for each $i \in [\![0, p-1]\!]$, we let $X(i)$ denote the spectrum built from the shifted base $\boldsymbol{\beta}^{(i)}$ and the shifted alternate alphabet $\boldsymbol{D}^{(i)}$. In particular, we have $X(0) = X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$.

Lemma 4.2.4. *For each $i \in [\![0, p-1]\!]$, we have $X(i) \cdot \beta_i + D_i = X(i+1)$ where it is understood that $X(p) = X(0)$.*

Proof. For each $i \in [\![0, p-1]\!]$, we have

$$X(i) = \sum_{j=0}^{p-1} X^{D_{i+j}}(\beta) \cdot \beta_{i+j+1} \cdots \beta_{i+p-1}.$$

Since

$$(X^{D_i}(\beta) \cdot \beta_{i+1} \cdots \beta_{i+p-1}) \cdot \beta_i + D_i = X^{D_i}(\beta) \cdot \beta + D_i = X^{D_i}(\beta),$$

the conclusion follows. ∎

Lemma 4.2.5. *For all $\ell \in \mathbb{N}$, we have*

$$\sum_{n=0}^{\ell-1} D_n \cdot \beta_{n+1} \cdots \beta_{\ell-1} \subset X(\ell \bmod p).$$

Proof. We prove the inclusion by induction. If $\ell = 0$, it is immediate. Suppose the result true for $\ell \in \mathbb{N}$. We have

$$\sum_{n=0}^{\ell} D_n \cdot \beta_{n+1} \cdots \beta_\ell = \Big(\sum_{n=0}^{\ell-1} D_n \cdot \beta_{n+1} \cdots \beta_{\ell-1}\Big)\beta_\ell + D_\ell.$$

By induction, we get

$$\sum_{n=0}^{\ell} D_n \cdot \beta_{n+1} \cdots \beta_\ell \subset X(\ell \bmod p) \cdot \beta_\ell + D_\ell.$$

The conclusion follows by Lemma 4.2.4. ∎

In view of the previous lemma, if for each $n \in [\![0, \ell-1]\!]$, the digit a_n belongs to the alphabet D_n, then we say that the finite word $a_0 \dots a_{\ell-1}$ *corresponds* to the element $\sum_{n=0}^{\ell-1} a_n \beta_{n+1} \cdots \beta_{\ell-1}$ of the spectrum $X(\ell \bmod p)$.

Let us now generalize the notion of zero automaton to the context of alternate bases.

Definition 4.2.6. For each $i \in [\![0, p-1]\!]$, we define

$$M^{(i)} = \sum_{n=i}^{+\infty} \frac{\max(D_n)}{\prod_{k=i}^{n} \beta_k} \qquad \text{and} \qquad m^{(i)} = \sum_{n=i}^{+\infty} \frac{\min(D_n)}{\prod_{k=i}^{n} \beta_k}$$

where $\max(D_n)$ and $\min(D_n)$ respectively denote the maximal and minimal digit in the alphabet D_n.

As usual, for $n \in \mathbb{Z}$, we set $M^{(n)} = M^{(n \bmod p)}$ and $m^{(n)} = m^{(n \bmod p)}$.

Definition 4.2.7. The *zero automaton* associated with the alternate base $\boldsymbol{\beta}$ and the alternate alphabet $\boldsymbol{D}$ is the Büchi automaton

$$\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D}) = \big(Q_{\boldsymbol{\beta},\boldsymbol{D}}, (0,0), Q_{\boldsymbol{\beta},\boldsymbol{D}}, \cup_{i=0}^{p-1} D_i, E\big)$$

where

- $Q_{\boldsymbol{\beta},\boldsymbol{D}} = \bigcup_{i=0}^{p-1} \big(\{i\} \times (X(i) \cap [-M^{(i)}, -m^{(i)}])\big)$
- E is the set of transitions defined as follows: for $(i,s), (j,t) \in Q_{\boldsymbol{\beta},\boldsymbol{D}}$ and $a \in \cup_{i=0}^{p-1} D_i$, there is a transition $(i,s) \xrightarrow{a} (j,t)$ if and only if $j \equiv i+1 \pmod{p}$, $a \in D_i$ and $t = \beta_i s + a$.

Observe that since we have assumed that all the alphabets D_i contain the digit 0, the initial state $(0,0)$ is indeed an element of $Q_{\boldsymbol{\beta},\boldsymbol{D}}$. Moreover, if $s \in X(i)$ and $a \in D_i$ then $\beta_i s + a \in X(i+1)$ by Lemma 4.2.4.

Proposition 4.2.8. *The zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ accepts the set $Z(\boldsymbol{\beta}, \boldsymbol{D})$.*

Proof. Let a be an infinite word accepted by $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$. For each $\ell \in \mathbb{N}$, the prefix $a_0 \cdots a_{\ell-1}$ labels a path in $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ from the initial state $(0,0)$ to the state

$$(\ell \bmod p, \sum_{n=0}^{\ell-1} a_n \beta_{n+1} \cdots \beta_{\ell-1}).$$

Therefore, the sequence $\left(\sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1}\right)_{\ell\in\mathbb{N}}$ is bounded. Hence, we get

$$\sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{k=0}^{n}\beta_k} = \lim_{\ell\to+\infty} \frac{\sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1}}{\prod_{n=0}^{\ell-1}\beta_n} = 0.$$

Conversely, consider an infinite word a such that $a_n \in D_n$ for all $n \in \mathbb{N}$ and that is not accepted by $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D})$. Then, there exists $\ell \in \mathbb{N}$ such that

$$(\ell \bmod p, \sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1}) \notin Q_{\boldsymbol{\beta},\boldsymbol{D}}.$$

In view of Lemma 4.2.5, we get

$$\sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1} \notin [-M^{(\ell)}, -m^{(\ell)}].$$

Suppose that

$$\sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1} > -m^{(\ell)}$$

(the other case is symmetric). We have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{a_n}{\prod_{k=0}^{n}\beta_k} &\geq \sum_{n=0}^{\ell-1} \frac{a_n}{\prod_{k=0}^{n}\beta_k} + \sum_{n=\ell}^{+\infty} \frac{\min(D_n)}{\prod_{k=0}^{n}\beta_k} \\ &= \frac{\sum_{n=0}^{\ell-1} a_n\beta_{n+1}\cdots\beta_{\ell-1} + m^{(\ell)}}{\prod_{n=0}^{\ell-1}\beta_n} \\ &> 0.\end{aligned}$$

■

Example 4.2.9. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ and the pair of alphabets $\boldsymbol{D} = (\overline{[\![-2,2]\!], [\![-1,1]\!]})$. Then

$$M^{(0)} = \mathrm{val}_{\boldsymbol{\beta}}((21)^\omega) \simeq 1.67994$$

and

$$M^{(1)} = \mathrm{val}_{\boldsymbol{\beta}^{(1)}}((12)^\omega) \simeq 1.86852.$$

The zero automaton $\mathcal{Z}(\boldsymbol{\beta},\boldsymbol{D})$ is depicted in Figure 4.1 where the states with first components 0 and 1 are colored in pink and purple respectively, and where the edges labeled by $-2, -1, 0, 1$ and 2 are colored in dark blue, dark green, red, light green and light blue respectively. For instance, the infinite words $1(\overline{1}0)^\omega$ and $(0\overline{1}21\overline{2}\overline{1})^\omega$ have value 0 in base $\boldsymbol{\beta}$ (where $\overline{1}$ and $\overline{2}$ designate the digits -1 and -2 respectively).

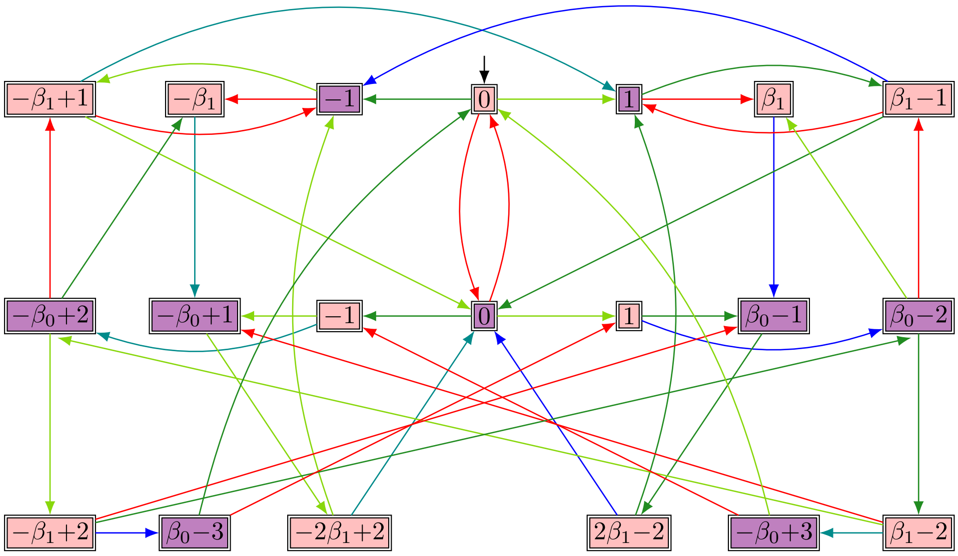

Figure 4.1: The zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ for $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$ and $\boldsymbol{D} = \overline{(\llbracket -2, 2\rrbracket, \llbracket -1, 1\rrbracket)}$. The conventions for colors are described within Example 4.2.9.

Theorem 4.2.10. *Let $\boldsymbol{\beta}$ be an alternate base of length p and let $\boldsymbol{D}$ be an alternate alphabet. Then the following assertions are equivalent.*

1. *The set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ is accepted by a finite Büchi automaton.*
2. *The spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$.*
3. *The zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ is finite.*

Proof. By Lemma 4.2.4, if the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$ then for all $i \in \llbracket 0, p-1\rrbracket$, the spectrum $X(i)$ based on the cyclic shift $\boldsymbol{\beta}^{(i)}$ of the base and the corresponding shifted alternate alphabet $\boldsymbol{D}^{(i)}$ has no accumulation point in $\mathbb{R}$ either. The implication (2) $\implies$ (3) then follows directly from the definition of the set of states of the zero automaton. The implication (3) $\implies$ (1) follows from Proposition 4.2.8.

Let us show that (1) $\implies$ (2). Suppose that the set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ is accepted by a finite Büchi automaton $\mathcal{A} = (Q, q_0, F, \cup_{i=0}^{p-1} D_i, E)$. In view of Theorem 4.1.6, it suffices to construct a finite Büchi $\mathcal{B}$ automaton accepting the set $Z(\beta, \mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D}))$ in order to obtain that $X^{\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$. Consider the finite Büchi automaton

$$\mathcal{B} = \big(Q \times \{f, \overline{f}\}, (q_0, f_0), Q \times \{f\}, \mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D}), E'\big)$$

where $f_0 = f$ if the initial state q_0 is final and $f_0 = \overline{f}$ otherwise, and the transitions in E' are defined as follows. For $q, q' \in Q$, $x, x' \in \{f, \overline{f}\}$ and $a_0 \in D_0, \ldots, a_{p-1} \in D_{p-1}$, there is a transition

$$\Big((q, x), \sum_{i=0}^{p-1} a_i \beta_{i+1} \cdots \beta_{p-1}, (q', x')\Big)$$

in E' if there is a path labeled by $a_0 \cdots a_{p-1}$ from q to q' in $\mathcal{A}$ and $x' = f$ if the path in $\mathcal{A}$ goes through a final state and $x' = \overline{f}$ otherwise.

We prove that $\mathcal{B}$ accepts $Z(\beta, \mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D}))$. Consider

$$b \in Z(\beta, \mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})).$$

For all $n \in \mathbb{N}$, there exists $a_{n,0} \in D_0, \ldots, a_{n,p-1} \in D_{p-1}$ such that

$$b_n = \sum_{i=0}^{p-1} a_{n,i} \beta_{i+1} \cdots \beta_{p-1}.$$

Clearly, the infinite word $a = (a_{0,0} \cdots a_{0,p-1})(a_{1,0} \cdots a_{1,p-1}) \cdots$ belongs to $Z(\boldsymbol{\beta}, \boldsymbol{D})$. Hence, there exists an accepting path labeled by a in $\mathcal{A}$. Let $(q_n)_{n \in \mathbb{N}}$ be the sequence of states of this path. Then there is a path labeled by b in $\mathcal{B}$ and going through the sequence of states $((q_{np}, f_n))_{n \in \mathbb{N}}$ where for $n \in \mathbb{N}_{\geq 1}$, $f_n = f$ if there exists $i \in [\![1, p]\!]$ such that $q_{(n-1)p+i} \in F$ and $f_n = \overline{f}$ otherwise. Since there are infinitely many n such that $q_n \in F$, we obtain that there also are infinitely many n such that $f_n = f$. Thus, the path in $\mathcal{B}$ labeled by b going through the states $((q_{np}, f_n))_{n \in \mathbb{N}}$ is accepting.

Conversely, consider an infinite word b over $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$ accepted by $\mathcal{B}$. Let $((q_n, f_n))_{n \in \mathbb{N}}$ be the sequence of states of an accepting path labeled by b in $\mathcal{B}$. By definition of the automaton $\mathcal{B}$, for all $n \in \mathbb{N}$, there exists $a_{n,0} \in D_0, \ldots, a_{n,p-1} \in D_{p-1}$ such that

$$b_n = \sum_{i=0}^{p-1} a_{n,i} \beta_{i+1} \cdots \beta_{p-1}$$

and a path from q_n to q_{n+1} in $\mathcal{A}$ labeled by $a_{n,0} \cdots a_{n,p-1}$, and moreover, there is such path going through a final state in $\mathcal{A}$ if and only if $f_n = f$. Hence, since there exist infinitely many n such that $f_n = f$, there is an accepting path labeled by $a = (a_{0,0} \cdots a_{0,p-1})(a_{1,0} \cdots a_{1,p-1}) \cdots$ in $\mathcal{A}$. Since $\mathcal{A}$ accepts the set $Z(\boldsymbol{\beta}, \boldsymbol{D})$, we get that

$$\sum_{n \in \mathbb{N}} \frac{b_n}{\beta^{n+1}} = \mathrm{val}_{\boldsymbol{\beta}}(a) = 0.$$

■

Remark 4.2.11. In the proof of Theorem 4.2.10, if the Büchi automaton $\mathcal{A}$ is deterministic, it is possible that the Büchi automaton $\mathcal{B}$ is not. This is not problematic since we do not require that the set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ is accepted by a deterministic finite Büchi automaton. However, if the map

$$f_{\boldsymbol{\beta},\boldsymbol{D}} \colon \boldsymbol{D} \to \mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D}),\ (a_0, \dots, a_{p-1}) \mapsto \sum_{i=0}^{p-1} a_i \beta_{i+1} \cdots \beta_{p-1}$$

is injective and $\mathcal{A}$ is deterministic then $\mathcal{B}$ is deterministic as well.

4.3 Algebraic properties of Parry alternate bases

An algebraic description of Parry numbers $\beta > 1$ is not obvious. However, we have links with algebraic and Pisot numbers (see Remark 1.4.12): any Parry number is an algebraic integer and any Pisot number is a Parry number. The aim of this section is to give such algebraic properties for Parry alternate bases (see Definition 3.3.1).

Recall that we fixed an alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}})$ of length p, we set $\beta = \prod_{i=0}^{p-1} \beta_i$ and we fixed an alternate alphabet $\boldsymbol{D} = (\overline{D_0, \dots, D_{p-1}})$.

4.3.1 A necessary condition to be a Parry alternate base

The following theorem gives a necessary condition on $\boldsymbol{\beta}$ to be a Parry alternate base. By Section 3.3, we know that the definition of a Parry alternate base can be equivalently stated by using the periodicity of the greedy, quasi-greedy or quasi-lazy $\boldsymbol{\beta}^{(i)}$-expansions. In this section, we use the periodicity of the greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1.

Theorem 4.3.1. *If $\boldsymbol{\beta}$ is a Parry alternate base, then β is an algebraic integer and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$.*

In order to give intuition on the algebraic techniques that will be used in the proof, we start with an example.

Example 4.3.2. Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$ be an alternate base such that the expansions of 1 are given by

$$d_{\boldsymbol{\beta}}(1) = 30^\omega, \quad d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega, \quad d_{\boldsymbol{\beta}^{(2)}}(1) = 1(110)^\omega. \tag{4.4}$$

We easily derive that $\beta_0, \beta_1, \beta_2$ satisfy the following set of equations

$$\frac{3}{\beta_0} = 1, \quad \frac{1}{\beta_1} + \frac{1}{\beta_1\beta_2} = 1, \quad \frac{1}{\beta_2} + \left(\frac{1}{\beta_2\beta_0} + \frac{1}{\beta}\right)\frac{\beta}{\beta - 1} = 1,$$

where $\beta = \beta_0\beta_1\beta_2$. Multiplying the first equation by β, the second one by $\beta_1\beta_2$ and the third one by $(\beta - 1)\beta_2$, we obtain identities

$$3\beta_1\beta_2 - \beta = 0, \quad -\beta_1\beta_2 + \beta_2 + 1 = 0, \quad \beta_1\beta_2 + (2 - \beta)\beta_2 + \beta - 1 = 0.$$

In a matrix formalism, we have

$$\begin{pmatrix} 3 & 0 & -\beta \\ -1 & 1 & 1 \\ 1 & 2-\beta & \beta-1 \end{pmatrix} \begin{pmatrix} \beta_1\beta_2 \\ \beta_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.5}$$

The existence of a non-zero vector $(\beta_1\beta_2, \beta_2, 1)^T$ as a solution of this equation forces that the determinant of the coefficient matrix is zero, that is, that $\beta^2 - 9\beta + 9 = 0$. Hence we must have $\beta = \frac{9+3\sqrt{5}}{2} = 3\varphi^2$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Solving (4.5) for this β, we obtain $\beta_1\beta_2 = \frac{\beta}{3} = \varphi^2$ and $\beta_2 = \beta_1\beta_2 - 1 = \varphi^2 - 1 = \varphi$, and finally $\beta_1 = 1 + \frac{1}{\varphi} = \varphi$. Consequently, $\beta_0 = \frac{\beta}{\beta_1\beta_2} = 3$. Indeed, the triple $\boldsymbol{\beta} = (3, \varphi, \varphi)$ is an alternate base giving precisely (4.4) as the expansions of 1, as already observed in Example 2.3.8.

In the previous example, for obtaining the values $\beta_0, \beta_1, \beta_2$ from the known ultimately periodic expansions we have used the fact that $\beta_0, \beta_1, \beta_2$ and $\beta = \beta_0\beta_1\beta_2$ are solutions of a system of polynomial equations in four unknowns x_0, x_1, x_2, y, in our case

$$\begin{cases} 3x_1x_2 - y &= 0 \\ -x_1x_2 + x_2 + 1 &= 0 \\ x_1x_2 + (2 - y)x_2 + y - 1 &= 0 \\ x_1x_2x_3 &= y. \end{cases}$$

The solution of the system yielded that β is a root of a monic polynomial with integer coefficient, that is, is an algebraic integer. The same strategy can be applied to any Parry alternate basis, that is, to any alternate base where all the expansions $d_{\boldsymbol{\beta}^{(i)}}(1)$, with $i \in [\![0, p-1]\!]$, are ultimately periodic.

In the proof of Theorem 4.3.1, we will work with formal power series whose coefficients are given by ultimately periodic sequences. Let us prepare explicit form of these sums.

Definition 4.3.3. For given $m \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, we define $P_{m,k}$ as the set of polynomials in $\mathbb{Z}[y]$ of degree at most $m+k-1$ of the form

$$(y^k - 1)(\sum_{n=0}^{m-1} a_n y^{m-1-n}) + \sum_{n=0}^{k-1} a_{m+n} y^{k-1-n} \tag{4.6}$$

where $a_0, \ldots, a_{m+k-1} \in \mathbb{Z}$. We say that the polynomial (4.6) is *associated with the integers* $a_0, \ldots, a_{m+k-1}$. Note that this polynomial has maximal degree $m+k-1$ if $a_0 \neq 0$.

Lemma 4.3.4. *Let a be an ultimately periodic sequence of integers with preperiod $m \in \mathbb{N}$ and period $k \in \mathbb{N}_{\geq 1}$, that is,*

$$a = a_0 a_1 \cdots a_{m-1} (a_m a_{m+1} \cdots a_{m+k-1})^\omega.$$

Then, we have

$$\sum_{n \in \mathbb{N}} \frac{a_n}{y^{n+1}} = \frac{g}{y^m (y^k - 1)}$$

where g is the polynomial in $P_{m,k}$ associated with the integers $a_0, \ldots, a_{m+k-1}$.

Proof. We have

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \frac{a_n}{y^{n+1}} &= \sum_{n=0}^{m-1} \frac{a_n}{y^{n+1}} + \sum_{n=m}^{+\infty} \frac{a_n}{y^{n+1}} \\
&= \sum_{n=0}^{m-1} \frac{a_n}{y^{n+1}} + \frac{1}{y^m}\Big(\sum_{n=0}^{k-1} \frac{a_{m+n}}{y^{n+1}}\Big)(1 + \frac{1}{y^k} + \frac{1}{y^{2k}} + \cdots) \\
&= \sum_{n=0}^{m-1} \frac{a_n}{y^{n+1}} + \frac{1}{y^m}\Big(\sum_{n=0}^{k-1} \frac{a_{m+n}}{y^{n+1}}\Big)\frac{1}{1 - \frac{1}{y^k}} \\
&= \frac{y^m (y^k - 1)\big(\sum_{n=0}^{m-1} \frac{a_n}{y^{n+1}}\big) + y^k \big(\sum_{n=0}^{k-1} \frac{a_{m+n}}{y^{n+1}}\big)}{y^m (y^k - 1)} \\
&= \frac{(y^k - 1)\big(\sum_{n=0}^{m-1} a_n y^{m-1-n}\big) + \sum_{n=0}^{k-1} a_{m+n} y^{k-1-n}}{y^m (y^k - 1)} \\
&= \frac{g}{y^m (y^k - 1)}
\end{aligned}$$

where g is the polynomial in $P_{m,k}$ associated with the integers $a_0, \ldots, a_{m+k-1}$. ∎

Lemma 4.3.5. *Suppose that* 1 *has an ultimately periodic* $\boldsymbol{\beta}$*-representation* a *of preperiod* mp *and period* kp *with* $m \in \mathbb{N}$ *and* $k \in \mathbb{N}_{\geq 1}$*. Then*

$$\beta^m(\beta^k - 1) = \sum_{j=0}^{p-1} g_j(\beta)\beta_{j+1}\cdots\beta_{p-1}$$

where for each $j \in [\![0, p-1]\!]$*,* g_j *is the polynomial in* $P_{m,k}$ *associated with the integers* $a_j, a_{j+p} \ldots, a_{j+(m+k-1)p}$.

Proof. Rewrite (3.1) as

$$1 = \sum_{j=0}^{p-1}\sum_{n\in\mathbb{N}} \frac{a_{np+j}}{\beta^{n+1}}\beta_{j+1}\cdots\beta_{p-1}.$$

Since for every $j \in [\![0, p-1]\!]$, the sequence $(a_{np+j})_{n\in\mathbb{N}}$ is ultimately periodic with preperiod m and period k, the result follows from Lemma 4.3.4. ∎

Whenever all p expansions $d_{\boldsymbol{\beta}^{(i)}}(1)$ are ultimately periodic, for $i \in [\![0, p-1]\!]$, we associate a system of polynomial equations, which we call the $\boldsymbol{\beta}$-*polynomial system* by analogy to the β-*polynomial* for real bases β [Par60], as follows.

Without loss of generality, we suppose that for all $i \in [\![0, p-1]\!]$, the expansion $d_{\boldsymbol{\beta}^{(i)}}(1)$ has a preperiod $m_i p$ and a period $k_i p$ with $m_i \in \mathbb{N}$ and $k_i \in \mathbb{N}_0$. Then, for all $i \in [\![0, p-1]\!]$, we let $g_{i,0}, g_{i,1}, \ldots, g_{i,p-1}$ be the associated polynomials in P_{m_i,k_i} as in Lemma 4.3.5, so that

$$\beta^{m_i}(\beta^{k_i} - 1) = \sum_{j=0}^{p-1} g_{i,j}(\beta)\beta_{i+j+1}\cdots\beta_{i+p-1}.$$

For each $i \in [\![0, p-1]\!]$, since the first digit of $d_{\boldsymbol{\beta}^{(i)}}(1)$ is $\lfloor \beta_i \rfloor \geq 1$, the degree of $g_{i,0}$ is $k_i + m_i - 1$.

Definition 4.3.6. The $\boldsymbol{\beta}$-*polynomial system* is the system of $p+1$ polynomial equations in $p+1$ variables $x_0, x_1, \ldots, x_{p-1}, y$ given by

$$\begin{cases} y^{m_i}(y^{k_i} - 1) = \displaystyle\sum_{j=0}^{p-1} g_{i,j}x_{i+j+1}\cdots x_{i+p-1}, & \text{for } i \in [\![0, p-1]\!] \\ y = \displaystyle\prod_{i=0}^{p-1} x_i \end{cases} \tag{4.7}$$

where, as usual, we use the convention $x_n = x_{n \bmod p}$ for $n \in \mathbb{Z}$.

By construction, the p-tuple $(\beta_0, \ldots, \beta_{p-1}, \beta)$ is a solution of the associated $\boldsymbol{\beta}$-polynomial system.

Example 4.3.7. We resume Example 4.3.2. By writing each of the expansions from (4.4) with a preperiod 3 and a period 3, that is,

$$d_{\boldsymbol{\beta}^{(0)}}(1) = 300(000)^\omega, \quad d_{\boldsymbol{\beta}^{(1)}}(1) = 110(000)^\omega \quad \text{and} \quad d_{\boldsymbol{\beta}^{(2)}}(1) = 111(011)^\omega,$$

we get $g_{0,0} = 3(y-1)$, $g_{1,0} = g_{1,1} = g_{2,0} = y-1$, $g_{2,1} = g_{2,2} = y$ and $g_{0,1} = g_{0,2} = g_{1,2} = 0$. The associated $\boldsymbol{\beta}$-polynomial system is

$$\begin{cases} y(y-1) = 3(y-1)x_1x_2 \\ y(y-1) = (y-1)x_2x_0 + (y-1)x_0 \\ y(y-1) = (y-1)x_0x_1 + yx_1 + y \\ y = x_0x_1x_2. \end{cases}$$

By multiplying the second equation by x_1x_2 and the third one by x_2 and by substituting $x_0x_1x_2$ by y, we get the three equations

$$\begin{cases} y(y-1) = 3(y-1)x_1x_2 \\ y(y-1)x_1x_2 = y(y-1)x_2 + y(y-1) \\ y(y-1)x_2 = (y-1)y + yx_1x_2 + yx_2. \end{cases}$$

Placing the first equation in the last line, this can be rewritten as

$$\begin{pmatrix} -y(y-1) & y(y-1) & y(y-1) \\ y & y - y(y-1) & y(y-1) \\ 3(y-1) & 0 & -y(y-1) \end{pmatrix} \begin{pmatrix} x_1x_2 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix of this system is equal to $M(y) - y(y-1)I_3$ where

$$M(y) = \begin{pmatrix} g_{1,2} & yg_{1,0} & yg_{1,1} \\ g_{2,1} & g_{2,2} & yg_{2,0} \\ g_{0,0} & g_{0,1} & g_{0,2} \end{pmatrix}$$

and I_3 is the identity matrix of size 3.

Proof of Theorem 4.3.1. Let $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$ be such that the expansions $d_{\boldsymbol{\beta}^{(i)}}(1)$ all have preperiod mp and period kp, for $i \in [\![0, p-1]\!]$. Then we consider the associated polynomial system (4.7), where $m_i = m$ and $k_i = k$

for all $i \in [\![0, p-1]\!]$. We index the equations of this system from 0 to p. For each $i \in [\![1, p-1]\!]$, we multiply the i^{th} equation by $\prod_{k=i}^{p-1} x_k$, which becomes

$$y^m(y^k - 1)\prod_{k=i}^{p-1} x_k = \sum_{j=0}^{p-1}(g_{i,j} \prod_{k=i+j+1}^{2p-1} x_k).$$

By substituting $x_0 \cdots x_{p-1}$ by y, the latter equation can be rewritten as

$$y^m(y^k - 1)\prod_{k=i}^{p-1} x_k = \sum_{j=0}^{p-i-1}(y g_{i,j} \prod_{k=i+j+1}^{p-1} x_k) + \sum_{j=p-i}^{p-1}(g_{i,j} \prod_{k=i+j+1-p}^{p-1} x_k).$$

Now, the first p equations of the system can be written in the matrix form

$$(M(y) - y^m(y^k-1)I_p)\overrightarrow{v}(x_1, \ldots, x_{p-1}) = \vec{0} \tag{4.8}$$

where I_p is the identity matrix of size p, $\vec{0}$ is the zero column vector of size p,

$$\overrightarrow{v}(x_1, \ldots, x_{p-1}) = \begin{pmatrix} x_1 x_2 \cdots x_{p-1} \\ x_2 \cdots x_{p-1} \\ \vdots \\ x_{p-1} \\ 1 \end{pmatrix}$$

and

$$M(y) = \begin{pmatrix} g_{1,p-1} & y g_{1,0} & \cdots & y g_{1,p-3} & y g_{1,p-2} \\ g_{2,p-2} & g_{2,p-1} & \cdots & y g_{2,p-4} & y g_{2,p-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{p-1,1} & g_{p-1,2} & \cdots & g_{p-1,p-1} & y g_{p-1,0} \\ g_{0,0} & g_{0,1} & \cdots & g_{0,p-2} & g_{0,p-1} \end{pmatrix}$$

Since $(\beta_0, \ldots, \beta_{p-1}, \beta)$ is a non-trivial solution of the original system, we get that β is a root of the polynomial

$$h = \det(M(y) - y^m(y^k - 1)I_p)$$

of $\mathbb{Z}[y]$. By construction, for every $i, j \in [\![0, p-1]\!]$, the polynomial $g_{i,j}$ has degree at most $m + k - 1$. Therefore, the highest degree of h is obtained from the product $\prod_{i=0}^{p-1}(g_{i,p-1} - y^m(y^k - 1))$. This shows that h has leading coefficient $(-1)^p$. Since β is a root of h, we get that β is an algebraic integer.

It remains to prove that $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$. To that purpose, we will apply the famous Perron-Frobenius theorem (see for example [Rig14, Theorem 2.67]). First, thanks to Lemma 4.3.4, we know that the matrix

$M(\beta)$ has non-negative entries. Then, by Lemma 4.3.5 and since any $\boldsymbol{\beta}$-expansion starts with a non-zero digit, the entries

$$\beta g_{1,0}(\beta),\ \beta g_{2,0}(\beta),\ \ldots,\ \beta g_{p-1,0}(\beta),\ g_{0,0}(\beta)$$

of $M(\beta)$ in respective positions

$$(0,1),\ (1,2),\ \ldots,\ (p-2,p-1),\ (p-1,0)$$

are positive. Therefore, the matrix $M(\beta)$ is irreducible. By the Perron-Frobenius theorem, the vector $\overrightarrow{v}(\beta_1,\ldots,\beta_{p-1})$ is the unique positive eigenvector of $M(\beta)$ having 1 as its last entry and the corresponding eigenvalue $\beta^m(\beta^k-1)$ is the Perron-Frobenius eigenvalue of $M(\beta)$. Moreover, the rank of the matrix $M(\beta)-\beta^m(\beta^k-1)I$ is $p-1$. Thus, the corresponding linear system in the unknowns

$$c_1 = x_1x_2\cdots x_{p-1},\ c_2 = x_2\cdots x_{p-1},\ \ldots,\ c_{p-1} = x_{p-1}$$

is equivalent to that obtained by deleting one its p equations. The obtained system has full rank $p-1$. Since all entries of $M(\beta)-\beta^m(\beta^k-1)I$ belong to the field $\mathbb{Q}(\beta)$, any solution vector of the latter system has components c_i in $\mathbb{Q}(\beta)$. Hence, the products $\beta_1\beta_2\cdots\beta_{p-1},\beta_2\cdots\beta_{p-1},\ldots,\beta_{p-1}$ all belong to $\mathbb{Q}(\beta)$. We obtain in turn that $\beta_1,\ldots,\beta_{p-1}\in\mathbb{Q}(\beta)$. Since moreover $\beta_0=\beta/(\beta_1\cdots\beta_{p-1})$, we also get that $\beta_0\in\mathbb{Q}(\beta)$. ∎

Let us emphasize that the greediness of the representations was not necessary in the proof of Theorem 4.3.1. We only need that each $\boldsymbol{\beta}^{(i)}$-representation of 1 starts with a non-zero digit. Therefore, we have actually proved the following stronger result.

Theorem 4.3.8. *If 1 has ultimately periodic $\boldsymbol{\beta}^{(i)}$-representations for all $i\in[\![0,p-1]\!]$, then β is an algebraic integer. If moreover these p representations have non-negative digits and they all start with a non-zero digit, then $\beta_0,\ldots,\beta_{p-1}\in\mathbb{Q}(\beta)$.*

From the proof of Theorem 4.3.1, we deduce the following result about the uniqueness of the base.

Proposition 4.3.9. *Suppose that $\boldsymbol{\alpha}=(\overline{\alpha_0,\ldots,\alpha_{p-1}})$ and $\boldsymbol{\beta}=(\overline{\beta_0,\ldots,\beta_{p-1}})$ are two alternate bases such that $\prod_{i=0}^{p-1}\alpha_i=\prod_{i=0}^{p-1}\beta_i$, and suppose that there exists p ultimately periodic sequences $a^{(0)},\ldots,a^{(p-1)}$ of non-negative integers such that $a_0^{(i)}\geq 1$ and $\mathrm{val}_{\boldsymbol{\alpha}^{(i)}}(a^{(i)})=\mathrm{val}_{\boldsymbol{\beta}^{(i)}}(a^{(i)})=1$ for every $i\in[\![0,p-1]\!]$. Then $\boldsymbol{\alpha}=\boldsymbol{\beta}$.*

Proof. Using the same notation as in the proof of Theorem 4.3.1, given the product $\beta = \prod_{i=0}^{p-1} \beta_i$, the vector $\overrightarrow{v}(\beta_1, \ldots, \beta_{p-1})$ is the unique positive eigenvector of $M(\beta)$ having 1 as its last entry. Therefore, we must have $\overrightarrow{v}(\alpha_1, \ldots, \alpha_{p-1}) = \overrightarrow{v}(\beta_1, \ldots, \beta_{p-1})$, hence $\alpha_i = \beta_i$ for all $i \in [\![1, p]\!]$. Moreover, we have $\alpha_0 = \beta/(\alpha_1 \cdots \alpha_{p-1}) = \beta/(\beta_1 \cdots \beta_{p-1}) = \beta_0$. ■

In particular, we get the following two corollaries.

Corollary 4.3.10. *Let $\boldsymbol{\alpha} = (\overline{\alpha_0, \ldots, \alpha_{p-1}})$ and $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$ be two alternate bases such that $\prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i$ and suppose that for every $i \in [\![0, p-1]\!]$, the $\boldsymbol{\alpha}^{(i)}$-expansion of 1 and $\boldsymbol{\beta}^{(i)}$-expansions of 1 coincide and are ultimately periodic. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$.*

Corollary 4.3.11. *If $d_{\boldsymbol{\beta}^{(i)}}(1) = d_{\boldsymbol{\beta}}(1)$ for all $i \in [\![0, p-1]\!]$ and $d_{\boldsymbol{\beta}}(1)$ is ultimately periodic, then $\beta_i = \beta_0$ for all $i \in [\![0, p-1]\!]$.*

Proof. Apply Corollary 4.3.10 to $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{(1)}$. ■

4.3.2 Spectrum and a sufficient condition to be a Parry alternate base

In Section 4.3.1, we have derived a necessary condition for an alternate base to be Parry. Namely that the product β of the bases is an algebraic integer and all β_j with $j \in [\![0, p-1]\!]$ belong to the field $\mathbb{Q}(\beta)$. In this section, we give a sufficient condition.

We adopt the same notation and convention as in Section 4.2: we fix an alternate base $\boldsymbol{\beta} = (\overline{\beta_0, \ldots, \beta_{p-1}})$, we set $\beta = \prod_{i=0}^{p-1} \beta_i$, we consider an alternate alphabet $\boldsymbol{D} = (\overline{D_0, \ldots, D_{p-1}})$ and we let $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$ be the corresponding alphabet of real numbers as defined in (4.3).

Proposition 4.3.12. *If $D_i \supseteq [\![-\lfloor \beta_i \rfloor, \lfloor \beta_i \rfloor]\!]$ for all $i \in [\![0, p-1]\!]$ and if the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$, then $d_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic for all $i \in [\![0, p-1]\!]$.*

Proof. Suppose that $d_{\boldsymbol{\beta}}(1)$ is not ultimately periodic. Then the sequence of remainders $(r_{\ell p-1}(1))_{\ell \in \mathbb{N}}$ of the greedy algorithm (see Definition 2.3.1) is injective. For all $x \in [0, 1]$ and $\ell \in \mathbb{N}$, we have

$$r_{\ell p-1}(x) = \beta^{\ell} x - \sum_{n=0}^{\ell-1} d_n \beta^{\ell-1-n} \tag{4.9}$$

where

$$d_n = \sum_{i=0}^{p-1} \varepsilon_{np+i}(x)\beta_{i+1}\cdots\beta_{p-1}.$$

Since $D_i \supseteq [\![-\lfloor\beta_i\rfloor, \lfloor\beta_i\rfloor]\!]$ for each $i \in [\![0, p-1]\!]$, we get that for all $\ell \in \mathbb{N}$, the remainder $r_{\ell p-1}(1)$ is an element of $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$. Since the remainders all belong to the interval $[0, 1)$, the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has an accumulation point in $\mathbb{R}$. By Lemma 4.2.4, either all the spectra $X(i)$ based on the cyclic shifts $\boldsymbol{\beta}^{(i)}$ of the alternate base and the corresponding shifted alternate alphabet $\boldsymbol{D}^{(i)}$ for $i \in [\![0, p-1]\!]$ have an accumulation point or none of them has. The result follows. ■

Proposition 4.3.13. *If β is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$ then the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$.*

Proof. The set $\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})$ is a finite subset of $\mathbb{Q}(\beta)$ where β is an algebraic integer. Hence, since every integer is an algebraic integer and since the set of all algebraic integers is a ring (see Proposition 1.1.3), there exist a positive integer q and a finite subset A of the ring of algebraic integers in $\mathbb{Q}(\beta)$ such that

$$\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D}) \cup (\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D}) - \mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})) = \frac{1}{q}A.$$

Let $x, y \in X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ such that $x \neq y$. There exists $\ell \in \mathbb{N}$ and $a_0, \ldots, a_{\ell-1} \in A$ such that

$$x - y = \frac{1}{q}\sum_{n=0}^{\ell-1} a_n\beta^n.$$

We obtain that $q(x-y)$ is an algebraic integer. Let d denote the (algebraic) degree of β and let $\beta_2, \ldots, \beta_d$ be the Galois conjugates of β. Moreover, set $\beta_1 = \beta$. Then, by Proposition 1.1.8, by using the isomorphisms from Definition 1.1.7, we get

$$1 \leq \Big|\prod_{k=1}^{d} \psi_k(q(x-y))\Big| = q|x-y| \prod_{k=2}^{d} |\psi_k(q(x-y))|.$$

Since β is a Pisot number, for all $k \in [\![2, d]\!]$, we have $|\beta_k| < 1$ and hence

$$|\psi_k(q(x-y))| \leq M \sum_{n=0}^{\ell-1} |\beta_k|^n \leq \frac{M}{1-|\beta_k|}$$

where $M = \max\{|\psi_k(a)| : k \in [\![2, d]\!],\ a \in A\}$. We get that

$$|x - y| \geq \frac{1}{q} \prod_{k=2}^{d} \frac{1 - |\beta_k|}{M}.$$

The latter inequality states that the distance between distinct elements x, y of the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})}(\beta)$ is bounded from below by a constant uniformly for all pairs x, y. ∎

As a consequence, we get the following theorem.

Theorem 4.3.14. *If β is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$ then $\boldsymbol{\beta}$ is a Parry alternate base.*

Proof. First apply Proposition 4.3.13 with

$$\boldsymbol{D} = (\overline{[\![- \lfloor \beta_0 \rfloor, \lfloor \beta_0 \rfloor]\!], \ldots, [\![- \lfloor \beta_{p-1} \rfloor, \lfloor \beta_{p-1} \rfloor]\!]})$$

and then apply Proposition 4.3.12. ∎

Let us make several remarks concerning the previous result. First, the following example shows that the condition of β being a Pisot number is neither sufficient nor necessary for $\boldsymbol{\beta}$ to be a Parry alternate base

Example 4.3.15. Being a Pisot number is not necessary to be a Parry number even for $p = 1$ since there exist Parry numbers which are not Pisot (see Remark 1.4.12 and Example 1.4.13). To see that it is not sufficient for $p \geq 2$, consider the alternate base $\boldsymbol{\beta} = (\overline{\sqrt{\beta}, \sqrt{\beta}})$ where β is the smallest Pisot number. The product β is the Pisot number β. However, the $\boldsymbol{\beta}$-expansion of 1 is equal to $d_{\sqrt{\beta}}(1)$, which is known to be aperiodic. This follows from the fact that the only Galois conjugate of $\sqrt{\beta}$ is $-\sqrt{\beta}$, and thus $\sqrt{\beta}$ is not a Perron number, hence not a Parry number either.

Furthermore, the bases $\beta_0, \ldots, \beta_{p-1}$ need not be algebraic integers in order to have the property that $d_{\boldsymbol{\beta}^{(i)}}(1)$ is ultimately periodic for all $i \in [\![0, p-1]\!]$.

Example 4.3.16. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. We have $d_{\boldsymbol{\beta}^{(0)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$. However, $\frac{5+\sqrt{13}}{6}$ is not an algebraic integer.

As illustrated in the following example, for the same non-Pisot algebraic integer β, there may exist two length-p alternate bases $\boldsymbol{\alpha} = (\overline{\alpha_0, \cdots, \alpha_{p-1}})$ and $\boldsymbol{\beta} = (\overline{\beta_0 \cdots \beta_{p-1}})$ such that $\prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i = \beta$, $\alpha_0, \ldots, \alpha_{p-1} \in \mathbb{Q}(\beta)$, $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\beta)$ and for all $i \in [\![0, p-1]\!]$, the expansion $d_{\boldsymbol{\alpha}^{(i)}}(1)$ is ultimately periodic whereas there exists $i \in [\![0, p-1]\!]$ such that $d_{\boldsymbol{\beta}^{(i)}}(1)$ is not. The technique used for showing aperiodicity is inspired by the work [LS12].

Example 4.3.17. Consider the real root $\beta > 1$ of the polynomial $x^6 - x^5 - 1$. This number is an algebraic integer but it is not a Pisot number since two of its Galois conjugates have modulus greater than 1 (see Example 1.1.12). Consider the alternate base $\boldsymbol{\alpha} = (\overline{\frac{1+\beta^7}{\beta^7}, \frac{\beta^8}{1+\beta^7}})$. We can compute that

$$d_{\boldsymbol{\alpha}}(1) = 10^{13}10^{\omega} \quad \text{and} \quad d_{\boldsymbol{\alpha}^{(1)}}(1) = 10^{18}10^{20}(10^{27})^{\omega}.$$

Now consider $\boldsymbol{\beta} = (\overline{\frac{6}{5}, \frac{5}{6}\beta})$. We prove that $d_{\boldsymbol{\beta}}(1)$ is not ultimately periodic. Let γ be a Galois conjugate of β such that $|\gamma| > 1$ and let $\psi \colon \mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$ be the corresponding field isomorphism induced by $\psi(\beta) = \gamma$. We prove that $(r_{12n-1}(1))_{n\in\mathbb{N}}$ is not ultimately periodic, where we set $r_{-1}(1) = 1$. To do so, it is enough to prove that $\big(|\psi(r_{12n-1}(1))|\big)_{n\in\mathbb{N}}$ is ultimately strictly increasing. It can be computed that the word 10^{12} is a prefix of $d^*_{\boldsymbol{\beta}}(1)$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1)$. Therefore, by Theorem 2.3.33 and Equality (4.9), for all $x \in [0, 1]$, we get

$$r_{11}(x) \in \{\beta^6 x\} \cup \{\beta^6 x - \beta_1 \beta^k : k \in [\![0, 5]\!]\} \cup \{\beta^6 x - \beta^k : k \in [\![0, 5]\!]\}.$$

Hence, for all $x \in [0, 1] \cap \mathbb{Q}(\beta)$, we have

$$\psi(r_{11}(x)) \in \{\gamma^6 \psi(x)\} \cup \{\gamma^6 \psi(x) - \frac{5}{6}\gamma^{k+1} : k \in [\![0, 5]\!]\} \\ \cup \{\gamma^6 \psi(x) - \gamma^k : k \in [\![0, 5]\!]\}.$$

Since $|\gamma| \leq \frac{6}{5}$, we get

$$|\psi(r_{11}(x))| \geq |\gamma|^6 |\psi(x)| - |\gamma|^5.$$

Thus, if we have

$$|\psi(x)| > \frac{|\gamma|^5}{|\gamma|^6 - 1} \simeq 5.49$$

then we obtain $|\psi(r_{11}(x))| > |\psi(x)|$. It can be computed that

$$10^{13}10^{15}10^{13}10^{27}10^{11}$$

is the prefix of $d_{\boldsymbol{\beta}}(1)$ of length 84. Hence, by using (4.9) again, we get

$$r_{83}(1) = \beta^{42} - \beta_1\beta^{41} - \beta_1\beta^{34} - \beta_1\beta^{26} - \beta_1\beta^{19} - \beta_1\beta^{5}.$$

This implies

$$\psi(r_{83}(1)) = \gamma^{42} - \frac{5}{6}\gamma^{42} - \frac{5}{6}\gamma^{35} - \frac{5}{6}\gamma^{27} - \frac{5}{6}\gamma^{20} - \frac{5}{6}\gamma^{6}.$$

Now for $x = r_{83}(1)$, we have $|\psi(x)| \simeq 6.23 > 5.49$. We get

$$|\psi(r_{11}(x))| > |\psi(x)|$$

where $r_{11}(x) = r_{95}(1)$. Iterating the argument, we obtain that the sequence $(|\psi(r_{12n-1}(1))|)_{n\geq 7}$ is strictly increasing.

4.4 Alternate bases whose set of zero representations is accepted by a finite Büchi automaton

Once again, we use the notation introduced in Section 4.2: namely we use fixed $\boldsymbol{\beta}, \beta, \boldsymbol{D}$ and then we work with the corresponding digit set $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$, set of representations of zero $Z(\boldsymbol{\beta}, \boldsymbol{D})$ and spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$. We combine the previously established results in order to characterize for which alternate bases the set $Z(\boldsymbol{\beta}, \boldsymbol{D})$ is accepted by a finite Büchi automaton. In doing so, we generalize Theorem 1.4.31 to alternate bases. We need one more lemma.

Lemma 4.4.1. *If the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$ and if there exists $j \in [\![0, p-1]\!]$ such that $[\![-\lceil\beta\rceil + 1, \lceil\beta\rceil - 1]\!] \subseteq D_j$, then β is a Pisot number.*

Proof. Suppose that $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$ and let j be an index as in the statement. Since

$$X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta) = \sum_{i=0}^{p-1} X^{D_i}(\beta)\beta_{i+1}\cdots\beta_{p-1},$$

the spectrum $X^{D_j}(\beta)$ has no accumulation point in $\mathbb{R}$. By hypothesis on j, the spectrum $X^{\lceil\beta\rceil-1}(\beta)$ has no accumulation point in $\mathbb{R}$ either. By Theorem 1.4.30, we get that β is a Pisot number. ■

Theorem 4.4.2. *The following assertions are equivalent.*

1. *The set* $Z(\boldsymbol{\beta}, \boldsymbol{D})$ *is accepted by a finite Büchi automaton for all alternate alphabet* $\boldsymbol{D} = (\overline{D_0, \dots, D_{p-1}})$.

2. *The set* $Z(\boldsymbol{\beta}, \boldsymbol{D})$ *is accepted by a finite Büchi automaton for one alternate alphabet* $\boldsymbol{D} = (\overline{D_0, \dots, D_{p-1}})$ *such that* $D_i \supseteq [\![- \lfloor \beta_i \rfloor, \lfloor \beta_i \rfloor]\!]$ *for all* $i \in [\![0, p-1]\!]$ *and* $\lfloor \beta_j \rfloor \geq \lceil \beta \rceil - 1$ *for some* $j \in [\![0, p-1]\!]$.

3. β *is a Pisot number and* $\beta_i \in \mathbb{Q}(\beta)$ *for all* $i \in [\![0, p-1]\!]$.

Proof. The implication (1) $\implies$ (2) is straightforward. Now, suppose that (2) holds. By Theorem 4.2.10 and Proposition 4.3.12, the greedy expansions $d_{\boldsymbol{\beta}^{(i)}}(1)$ are ultimately periodic for all $i \in [\![0, p-1]\!]$. Then, by Theorem 4.3.1, we get that β is an algebraic integer and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$. Moreover, since there exists $j \in [\![0, p-1]\!]$ such that $\lfloor \beta_j \rfloor \geq \lceil \beta \rceil - 1$, we obtain from Theorem 4.2.10 and Lemma 4.4.1 that β is a Pisot number. Hence, we have shown that (2) $\implies$ (3). Finally, the implication (3) $\implies$ (1) is obtained by combining Proposition 4.3.13 and Theorem 4.2.10. ■

4.5 Greedy and lazy normalizations in alternate bases

In this section, we apply our results in order to show that the greedy and lazy normalizations in alternate base are computable by finite Büchi automata under certain hypotheses, in which case we construct such automata.

Definition 4.5.1. The *greedy normalization function*

$$\nu_{\boldsymbol{\beta}, \boldsymbol{D}} \colon (\cup_{i=0}^{p-1} D_i)^{\mathbb{N}} \to (\cup_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!])^{\mathbb{N}}$$

is the partial function mapping any $\boldsymbol{\beta}$-representation $a \in \otimes_{n \in \mathbb{N}} D_n$ of a real number $x \in [0, 1)$ to the greedy $\boldsymbol{\beta}$-expansion of x. We say that $\nu_{\boldsymbol{\beta}, \boldsymbol{D}}$ is *computable by a finite Büchi automaton* if there exists a finite Büchi automaton accepting the set

$$\{(u, v) \in \bigotimes_{n \in \mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \ \mathrm{val}_{\boldsymbol{\beta}}(u) \in [0, 1) \text{ and } v = \nu_{\boldsymbol{\beta}, \boldsymbol{D}}(u)\}.$$

Such a Büchi automaton is called a *greedy normalizer* in base $\boldsymbol{\beta}$ over $\boldsymbol{D}$.

Similarly, the *lazy normalization function*

$$\nu'_{\boldsymbol{\beta}, \boldsymbol{D}} \colon (\cup_{i=0}^{p-1} D_i)^{\mathbb{N}} \to (\cup_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!])^{\mathbb{N}}$$

is the partial function mapping any $\boldsymbol{\beta}$-representation $a \in \otimes_{n\in\mathbb{N}} D_n$ of a real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ to the lazy $\boldsymbol{\beta}$-expansion of x. We say that $\nu'_{\boldsymbol{\beta},\boldsymbol{D}}$ is *computable by a finite Büchi automaton* if there exists a finite Büchi automaton accepting the set

$$\{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil\beta_n\rceil - 1]\!]) : \mathrm{val}_{\boldsymbol{\beta}}(u) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}] \text{ and } v = \nu'_{\boldsymbol{\beta},\boldsymbol{D}}(u)\}.$$

Such a Büchi automaton is called a *lazy normalizer* in base $\boldsymbol{\beta}$ over $\boldsymbol{D}$.

4.5.1 Alternate base converter

Following the same lines as in the real base case, we start by constructing a *converter* by using the zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D})$ defined in Section 4.2.

Consider two alternate alphabets

$$\boldsymbol{D} = (\overline{D_0, \ldots, D_{p-1}}) \quad \text{and} \quad \boldsymbol{D'} = (\overline{D'_0, \ldots, D'_{p-1}}).$$

We let $\boldsymbol{D} - \boldsymbol{D'}$ denote the alternate alphabet

$$(\overline{D_0 - D'_0, \ldots, D_{p-1} - D'_{p-1}}).$$

Definition 4.5.2. The *converter from $\boldsymbol{D}$ to $\boldsymbol{D'}$* is the Büchi automaton

$$\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D} \times \boldsymbol{D'}) = (Q_{\boldsymbol{\beta},\boldsymbol{D}-\boldsymbol{D'}}, (0,0), Q_{\boldsymbol{\beta},\boldsymbol{D}-\boldsymbol{D'}}, \cup_{i=0}^{p-1}(D_i \times D'_i), E')$$

where E' is the set of transitions defined as follows: for $(i,s), (j,t) \in Q_{\boldsymbol{\beta},\boldsymbol{D}-\boldsymbol{D'}}$ and for $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \cup_{i=0}^{p-1}(D_i \times D'_i)$, there is a transition

$$(i,s) \xrightarrow{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]} (j,t)$$

if and only if $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in D_i \times D'_i$ and there is a transition $(i,s) \xrightarrow{a-b} (j,t)$ in $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D} - \boldsymbol{D'})$.

Proposition 4.5.3. *The converter $\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D} \times \boldsymbol{D'})$ accepts the set*

$$\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \bigotimes_{n\in\mathbb{N}} (D_n \times D'_n) : \mathrm{val}_{\boldsymbol{\beta}}(u) = \mathrm{val}_{\boldsymbol{\beta}}(v)\}.$$

Proof. This is a direct consequence of Proposition 4.2.8. ∎

Proposition 4.5.4. *If β is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$, then the converter $\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{D}\times\boldsymbol{D'}}$ is finite.*

Proof. By Theorems 4.4.2 and 4.2.10, the zero automaton $\mathcal{Z}(\boldsymbol{\beta}, \boldsymbol{D} - \boldsymbol{D'})$ is finite. Hence, so is the converter $\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{D}\times\boldsymbol{D'}}$. ∎

4.5.2 Büchi automata accepting $D_{\boldsymbol{\beta}}$ and $D'_{\boldsymbol{\beta}}$

In Chapter 3, we proved that when $\boldsymbol{\beta}$ is a Parry alternate base, the associated deterministic finite automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ accept $\mathrm{Fac}(D_{\boldsymbol{\beta}})$ and $\mathrm{Fac}(D'_{\boldsymbol{\beta}})$ respectively.

As in Remark 1.4.24, we consider a modification of these automata in order to get Büchi automata accepting $D_{\boldsymbol{\beta}}$ and $D'_{\boldsymbol{\beta}}$.

Suppose that $\boldsymbol{\beta}$ is a Parry alternate base and write

$$d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} \cdots t_{m_i-1}^{(i)} \big(t_{m_i}^{(i)} \cdots t_{m_i+n_i-1}^{(i)}\big)^\omega$$

for all $i \in [\![0, p-1]\!]$. Without loss of generality, we suppose that $d_{\boldsymbol{\beta}^{(i)}}(1)$ has a non-zero preperiod for all $i \in [\![0, p-1]\!]$, that is, in the case of a purely periodic expansion $(t_0 \cdots t_{n-1})^\omega$, we work with the writing $t_0(t_1 \cdots t_{n-1} t_0)^\omega$ instead. Moreover, we suppose that n_i is a multiple of p. By Proposition 2.4.25, we get

$$\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \cdots \ell_{m_i-1}^{(i)} \big(\ell_{m_i}^{(i)} \cdots \ell_{m_i+n_i-1}^{(i)}\big)^\omega$$

where $\ell_n^{(i)} = \lceil \beta_{i+n} \rceil - 1 - t_n^{(i)}$ for all $n \in [\![0, m_i + n_i - 1]\!]$.

Since we supposed that n_i is a multiple of p, consider the automata

$$\mathcal{A}_{\boldsymbol{\beta}} = (Q, I, F, [\![0, \max_{0 \le i < p} \lceil \beta_i \rceil - 1]\!], E)$$

and

$$\mathcal{A}'_{\boldsymbol{\beta}} = (Q, I, F, [\![0, \max_{0 \le i < p} \lceil \beta_i \rceil - 1]\!], E')$$

from Definitions 3.4.4 and 3.4.11 obtained by only preserving the set

$$\big\{ q_{i,(i+k) \bmod p, k} : i \in [\![0, p-1]\!],\ k \in [\![0, m_i + n_i - 1]\!] \big\}$$

of accessible states (see Lemma 3.4.12 and (3.6)). Moreover, as in Chapter 3, for the sake of clarity, we now denote $q_{i,k}$ instead of $q_{i,(i+k) \bmod p,k}$ since the second index is completely determined by the other two.

We define associated Büchi automata as follows.

Definition 4.5.5. Let $\mathcal{B}_{\boldsymbol{\beta}}$ and $\mathcal{B}'_{\boldsymbol{\beta}}$ denote the Büchi automata defined by

$$\mathcal{B}_{\boldsymbol{\beta}} = (Q, q_{0,0}, F_{\mathcal{B}}, [\![0, \max_{0 \le i < p} \lceil \beta_i \rceil - 1]\!], E)$$

and

$$\mathcal{B}'_{\boldsymbol{\beta}} = (Q, q_{0,0}, F_{\mathcal{B}}, [\![0, \max_{0 \le i < p} \lceil \beta_i \rceil - 1]\!], E').$$

where the set of states Q is the one of the (accessible) automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$, the transition functions E and E' are the same as the ones of the automata $\mathcal{A}_{\boldsymbol{\beta}}$ and $\mathcal{A}'_{\boldsymbol{\beta}}$ respectively and the set of final states $F_{\mathcal{B}}$ is given by

$$F_{\mathcal{B}} = \big\{q_{i,0} : i \in [\![0, p-1]\!]\big\}.$$

Proposition 4.5.6. *If $\boldsymbol{\beta}$ is a Parry alternate base then the Büchi automaton $\mathcal{B}_{\boldsymbol{\beta}}$ (resp., $\mathcal{B}'_{\boldsymbol{\beta}}$) accepts the set $D_{\boldsymbol{\beta}}$ (resp., $D'_{\boldsymbol{\beta}}$).*

Proof. An infinite word is accepted by $\mathcal{B}_{\boldsymbol{\beta}}$ if and only if it can be factored as $u_0u_1u_2\cdots$ where each factor u_n corresponds to a first return to a final state, that is, for all $n \in \mathbb{N}$, there is a path labeled by u_n from a state of the form $q_{i,0}$ to a state of the form $q_{j,0}$ and u_n is the shortest next factor with this property. Since we have built $\mathcal{B}_{\boldsymbol{\beta}}$ by using a non-zero preperiod for each $d^*_{\boldsymbol{\beta}^{(i)}}(1)$, each such factor u_n must belong to the set $Y(\boldsymbol{\beta}^{(|u_0|+\cdots+|u_{n-1}|)}, |u_n|)$ from Definition 3.4.1. The conclusion for the greedy case follows from Proposition 3.4.2. Moreover, as in Lemma 3.4.15, by Lemma 3.4.13, a word $w \in A^{\mathbb{N}}_{\boldsymbol{\beta}}$ is accepted in $\mathcal{B}_{\boldsymbol{\beta}}$ if and only if $\theta_{\boldsymbol{\beta}}(w)$ is accepted in $\mathcal{B}'_{\boldsymbol{\beta}}$. The conclusion follows since $\theta_{\boldsymbol{\beta}}(D_{\boldsymbol{\beta}}) = D'_{\boldsymbol{\beta}}$ by Proposition 2.4.39. ■

Example 4.5.7. Consider again the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. We have $d_{\boldsymbol{\beta}^{(0)}}(1) = 2010^\omega$ and $d_{\boldsymbol{\beta}^{(1)}}(1) = 110^\omega$, hence $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$ and $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$. As explained above, since $d^*_{\boldsymbol{\beta}^{(1)}}(1)$ is purely periodic, we consider the writing $1(01)^\omega$ instead of $(10)^\omega$. Moreover, we have $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}}-1) = 012(02)^\omega$ and $\ell^*_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = 0(20)^\omega$. We obtain the Büchi automata $\mathcal{B}_{\boldsymbol{\beta}}$ and $\mathcal{B}'_{\boldsymbol{\beta}}$ depicted in Figure 4.2.

4.5.3 A sufficient condition for the greedy and lazy normalizations to be computable by finite Büchi automata

We are now able to state a generalization of Theorem 1.4.42.

Theorem 4.5.8. *If β is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$, then the greedy and lazy normalization functions $\nu_{\boldsymbol{\beta},\boldsymbol{D}}$ and $\nu'_{\boldsymbol{\beta},\boldsymbol{D}}$ are computable by finite Büchi automata.*

Proof. If β is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$, then by Theorem 4.3.14, the alternate base $\boldsymbol{\beta}$ is a Parry alternate base. First, consider the greedy case. By Proposition 4.5.6, the finite Büchi automaton

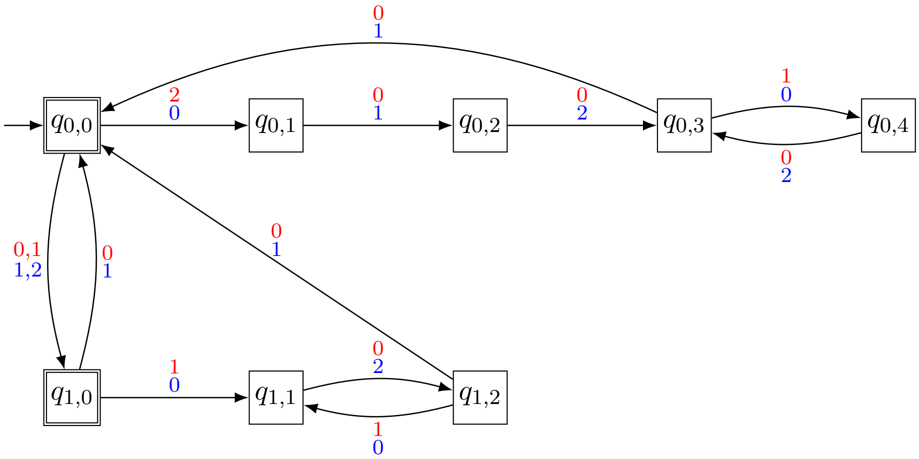

Figure 4.2: A Büchi automaton accepting $D_{\boldsymbol{\beta}}$ (red labels) and $D'_{\boldsymbol{\beta}}$ (blue labels) for $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$.

$\mathcal{B}_{\boldsymbol{\beta}}$ accepts the set $D_{\boldsymbol{\beta}}$. Thanks to this automaton, we construct a finite Büchi automaton accepting the set

$$\Big\{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \exists x \in [0,1),\ v = d_{\boldsymbol{\beta}}(x)\Big\}.$$

By computing the product of the latter Büchi automaton and the converter $\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D} \times \boldsymbol{D}')$ where

$$\boldsymbol{D}' = (\overline{[\![0, \lceil \beta_0 \rceil - 1]\!], \ldots, [\![0, \lceil \beta_{p-1} \rceil - 1]\!]}),$$

which is finite by Proposition 4.5.4, we get a finite Büchi automaton accepting the set

$$\begin{aligned}
&\{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \mathrm{val}_{\boldsymbol{\beta}}(u) = \mathrm{val}_{\boldsymbol{\beta}}(v) \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \exists x \in [0,1),\ v = d_{\boldsymbol{\beta}}(x)\} \\
&= \{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \mathrm{val}_{\boldsymbol{\beta}}(u) \in [0,1) \text{ and } v = \nu_{\boldsymbol{\beta},\boldsymbol{D}}(u)\}.
\end{aligned}$$

Therefore, the so-constructed finite Büchi automaton is a finite greedy normalizer in base $\boldsymbol{\beta}$ over $\boldsymbol{D}$. Similarly, by Proposition 4.5.6, the finite Büchi automaton $\mathcal{B}'_{\boldsymbol{\beta}}$ accepts the set $D'_{\boldsymbol{\beta}}$. Thanks to this automaton, we construct a finite Büchi automaton accepting the set

$$\Big\{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \exists x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}],\ v = \ell_{\boldsymbol{\beta}}(x)\Big\}.$$

By computing the product of the latter Büchi automaton and the finite converter $\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{D} \times \boldsymbol{D}')$, we get a finite Büchi automaton accepting the set

$$\begin{aligned}
&\{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \mathrm{val}_{\boldsymbol{\beta}}(u) = \mathrm{val}_{\boldsymbol{\beta}}(v) \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \exists x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}],\ v = \ell_{\boldsymbol{\beta}}(x)\} \\
&= \{(u,v) \in \bigotimes_{n\in\mathbb{N}} (D_n \times [\![0, \lceil \beta_n \rceil - 1]\!]) : \mathrm{val}_{\boldsymbol{\beta}}(u) \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}] \text{ and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad v = \nu'_{\boldsymbol{\beta},\boldsymbol{D}}(u)\}.
\end{aligned}$$

The so-constructed finite Büchi automaton is a finite lazy normalizer in base $\boldsymbol{\beta}$ over $\boldsymbol{D}$. ■

Example 4.5.9. Consider again the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. Following the same steps as described in the proof of Theorem 4.5.8, from the automata depicted in Figures 4.1 and 4.2, we obtain a finite Büchi automaton computing the greedy normalization function in base $\boldsymbol{\beta}$ over the pair of alphabets $\boldsymbol{D} = (\overline{[\![-2, 2]\!], [\![-1, 1]\!]})$.

CHAPTER

5 DYNAMICAL PROPERTIES OF ALTERNATE BASE EXPANSIONS

In this chapter, we generalize the greedy and lazy β-transformations for a real base β to the setting of alternate bases $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_{p-1}})$. As in the real base case, these new transformations, denoted $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$ respectively, can be iterated in order to generate the digits of the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers. The aim of this chapter is to describe the measure theoretical dynamical behaviors of $T_{\boldsymbol{\beta}}$ and $L_{\boldsymbol{\beta}}$.

We first prove the existence of a unique absolutely continuous (with respect to an extended Lebesgue measure, called the p-Lebesgue measure) $T_{\boldsymbol{\beta}}$-invariant measure. We then show that this unique measure is in fact equivalent to the p-Lebesgue measure and that the corresponding dynamical system is ergodic and has entropy $\frac{1}{p}\log(\beta)$ with $\beta = \prod_{i=0}^{p-1} \beta_i$.

Then, we express the density function of this measure and compute the frequencies of letters in the greedy $\boldsymbol{\beta}$-expansions. We also obtain the dynamical properties of $L_{\boldsymbol{\beta}}$ by showing that the lazy dynamical system is isomorphic to the greedy one. We also provide an isomorphism with suitable extensions of the real base shift.

Finally, we show that the $\boldsymbol{\beta}$-expansions can be seen as β-representations over general digit sets with $\beta = \prod_{i=0}^{p-1} \beta_i$ and we compare both frameworks.

The results presented in this chapter are from [CCD21]. Since this chapter generalizes the dynamical properties of real base expansions to the alternate base framework, Sections 1.3, 1.4.4 and 1.4.5 are needed preliminaries for the good understanding of the contents of this chapter.

Contents of the Chapter

5.1 Definition of the $\boldsymbol{\beta}$-transformations

In this chapter, we let $\boldsymbol{\beta}$ be a fixed alternate base, we let p be its length and we let β be the product $\prod_{i=0}^{p-1} \beta_i$. In this case, recall that $x_{\boldsymbol{\beta}}$ from (2.9) satisfies $x_{\boldsymbol{\beta}} < +\infty$.

5.1.1 The greedy $\boldsymbol{\beta}$-transformation

As said in Chapter 2 the greedy $\boldsymbol{\beta}$-expansion can be obtained by alternating the β_i-transformations: for all $x \in [0,1)$ and $n \in \mathbb{N}$,

$$\varepsilon_n(x) = \left\lfloor \beta_n \big(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) \big) \right\rfloor .$$

This can symbolically be seen in the subsequent example.

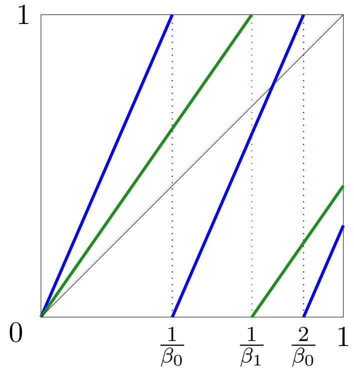

Figure 5.1: The transformations $T_{\frac{1+\sqrt{13}}{2}}$ (blue) and $T_{\frac{5+\sqrt{13}}{6}}$ (green).

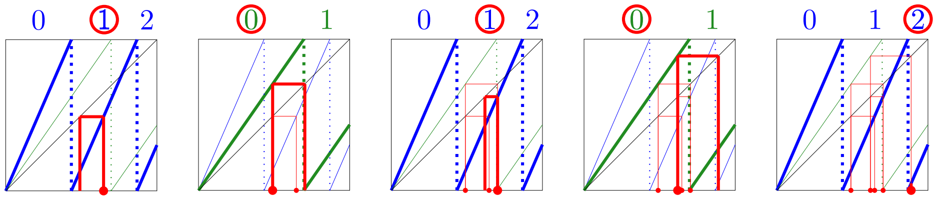

Figure 5.2: The first five digits of the greedy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$ are 10102 for $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$.

Example 5.1.1. Consider the alternate base $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$ already studied in Chapters 2, 3 and 4. The greedy $\boldsymbol{\beta}$-expansions are obtained by alternating the transformations $T_{\frac{1+\sqrt{13}}{2}}$ and $T_{\frac{5+\sqrt{13}}{6}}$, which are both depicted in Figure 5.1. Moreover, in Figure 5.2, we see the computation of the first five digits of the greedy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$.

We define the transformation associated with the greedy $\boldsymbol{\beta}$-expansions.

Definition 5.1.2. The *greedy $\boldsymbol{\beta}$-transformation* is the transformation defined by

$$\begin{aligned} T_{\boldsymbol{\beta}} \colon [\![0, p-1]\!] \times [0,1) &\to [\![0, p-1]\!] \times [0,1), \\ (i, x) &\mapsto \big((i+1) \bmod p, T_{\beta_i}(x)\big). \end{aligned} \tag{5.1}$$

In order to see that the greedy $\boldsymbol{\beta}$-transformation generates the digits of the greedy $\boldsymbol{\beta}$-expansions, we define $p+1$ maps.

Definition 5.1.3. Define the maps

$$\pi_2 \colon \mathbb{N} \times \mathbb{R} \to \mathbb{R},\ (n,x) \mapsto x$$

and

$$\delta_i \colon \mathbb{R} \to \{i\} \times \mathbb{R},\ x \mapsto (i,x)$$

with $i \in [\![0, p-1]\!]$.

Therefore, for all $x \in [0,1)$ and $n \in \mathbb{N}$, we have

$$\varepsilon_n(x) = \lfloor \beta_n \big(\pi_2 \circ T_{\boldsymbol{\beta}}^n \circ \delta_0(x)\big) \rfloor$$

and

$$r_n(x) = \pi_2 \circ T_{\boldsymbol{\beta}}^{n+1} \circ \delta_0(x).$$

That is, the greedy $\boldsymbol{\beta}$-expansions of real numbers in $[0,1)$ can be obtained by alternating the p maps

$$\pi_2 \circ T_{\boldsymbol{\beta}} \circ \delta_i\big|_{[0,1)} \colon [0,1) \to [0,1)$$

with $i \in [\![0, p-1]\!]$.

As in Section 1.4.4, the greedy $\boldsymbol{\beta}$-transformation can be extended to intervals of real numbers bigger than $[0,1)$ thanks to the definition of $x_{\boldsymbol{\beta}}$. Recall le link between the values $x_{\boldsymbol{\beta}^{(n)}}$ and $x_{\boldsymbol{\beta}^{(n+1)}}$, for all $n \in \mathbb{N}$, given in Proposition 2.4.5: we have

$$x_{\boldsymbol{\beta}^{(n)}} = \frac{x_{\boldsymbol{\beta}^{(n+1)}} + \lceil \beta_n \rceil - 1}{\beta_n}.$$

Definition 5.1.4. The *extended greedy $\boldsymbol{\beta}$-transformation*, denoted $T_{\boldsymbol{\beta}}^{\mathrm{ext}}$, is defined by

$$T_{\boldsymbol{\beta}}^{\mathrm{ext}} \colon \bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big) \to \bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big), \tag{5.2}$$

$$(i,x) \mapsto \begin{cases} \big((i+1) \bmod p, \beta_i x - \lfloor \beta_i x \rfloor\big) & \text{if } x \in [0,1) \\ \big((i+1) \bmod p, \beta_i x - (\lceil \beta_i \rceil - 1)\big) & \text{if } x \in [1, x_{\boldsymbol{\beta}^{(i)}}). \end{cases}$$

We extend the definition of the greedy $\boldsymbol{\beta}$-expansions of real numbers to the interval of real numbers $[0, x_{\boldsymbol{\beta}})$. The (extended) greedy $\boldsymbol{\beta}$-expansion of $x \in [0, x_{\boldsymbol{\beta}})$ is defined as the concatenation of the digits obtained thanks to the remainders defined by alternating the p maps

$$\pi_2 \circ T_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_i\big|_{[0, x_{\boldsymbol{\beta}^{(i)}})} \colon [0, x_{\boldsymbol{\beta}^{(i)}}) \to [0, x_{\boldsymbol{\beta}^{(i+1)}})$$

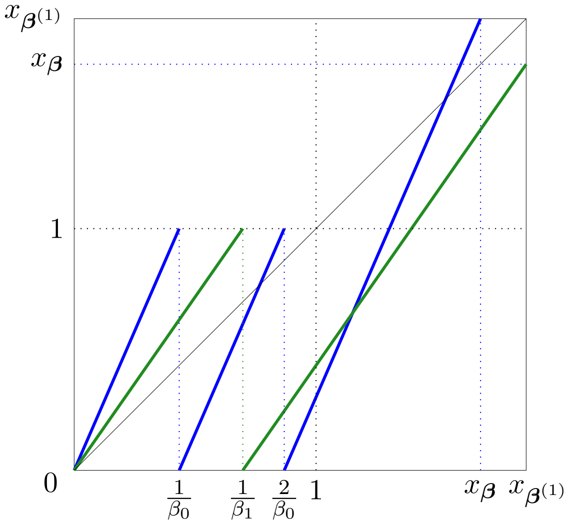

Figure 5.3: The maps $\pi_2 \circ T_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_0|_{[0,x_{\boldsymbol{\beta}})}$ (blue) and $\pi_2 \circ T_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_1|_{[0,x_{\boldsymbol{\beta}^{(1)}})}$ (green) with $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$.

with $i \in [\![0, p-1]\!]$

Example 5.1.5. Let $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$ be the alternate base of Example 5.1.1. The maps

$$\pi_2 \circ T_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_0|_{[0,x_{\boldsymbol{\beta}})} \colon [0, x_{\boldsymbol{\beta}}) \to [0, x_{\boldsymbol{\beta}^{(1)}})$$

and

$$\pi_2 \circ T_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_1|_{[0,x_{\boldsymbol{\beta}^{(1)}})} \colon [0, x_{\boldsymbol{\beta}^{(1)}}) \to [0, x_{\boldsymbol{\beta}})$$

are depicted in Figure 5.3.

Remark 5.1.6. Following Remark 2.3.16, it is important to note that here, when $\beta_0 \in \mathbb{N}_{\geq 2}$, the greedy $\boldsymbol{\beta}$-expansion of 1 is $(\beta_0 - 1)d_{\boldsymbol{\beta}^{(1)}}(1)$ instead of $\beta_0 0^\omega$ as in Chapter 2. Note that, as said in Remark 2.3.26, the quasi-greedy $\boldsymbol{\beta}$-expansion built on this greedy $\boldsymbol{\beta}$-expansion of 1 coincides with the one defined and used in Chapter 2. Hence, this chapter can make use of results from Sections 2.3.3, 2.3.4, 2.3.5 and 3.2.

The restriction of the extended greedy $\boldsymbol{\beta}$-transformation to the domain $[\![0, p-1]\!] \times [0, 1)$ gives back the greedy $\boldsymbol{\beta}$-transformation initially defined in (5.1). Moreover, the subspace $[\![0,p-1]\!] \times [0,1)$ is an attractor of $T_{\boldsymbol{\beta}}^{\mathrm{ext}}$ in the sense given by the following proposition.

Proposition 5.1.7. *For each $(i,x) \in \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n(i,x) \in [\![0,p-1]\!] \times [0,1). \tag{5.3}$$

Proof. Let $(i,x) \in \bigcup_{i=0}^{p-1} \left(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\right)$. On the one hand, if

$$(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^N(i,x) \in [\![0,p-1]\!] \times [0,1)$$

for some $N \in \mathbb{N}$, then clearly (5.3) occurs for all $n \geq N$. On the other hand, if

$$(T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n(i,x) \notin [\![0,p-1]\!] \times [0,1)$$

for all $n \in \mathbb{N}$, then we would get that $x = x_{\boldsymbol{\beta}^{(i)}}$ since at each step n, the greedy algorithm would pick the maximal digit $\lceil \beta_{i+n} \rceil - 1$. ∎

We now prove a result linking the iterations of the extended greedy maps and the lexicographic order on n-tuples. In what follows, we suppose that, for all $n \in \mathbb{N}$, the set of n-tuples $\prod_{i=0}^{n-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ is equipped with the lexicographic order: $(c_0,\ldots,c_{n-1}) <_{\mathrm{lex}} (c_0',\ldots,c_{n-1}')$ if there exists $i \in [\![0,n-1]\!]$ such that $c_0 = c_0', \ldots, c_{i-1} = c_{i-1}'$ and $c_i < c_i'$.

Proposition 5.1.8. *For all $x \in [0, x_{\boldsymbol{\beta}})$ and $n \in \mathbb{N}$, we have*

$$\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^n \circ \delta_0(x) = x\beta_0 \cdots \beta_{n-1} - \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}$$

where $(c_0,\ldots,c_{n-1})$ is the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}}{\beta_0 \cdots \beta_{n-1}} \leq x$.

Proof. We proceed by induction on n. The base case $n = 0$ is immediate: both members of the equality are equal to x. Now, suppose that the result is satisfied for some $n \in \mathbb{N}$. Let $x \in [0, x_{\boldsymbol{\beta}})$. Let $(c_0,\ldots,c_{n-1})$ be the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}}{\beta_0 \cdots \beta_{n-1}} \leq x$. Then it is easily seen that for all $m < n$, $(c_0,\ldots,c_m)$ is the lexicographically greatest $(m+1)$-tuple in $\prod_{k=0}^{m} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{m} c_k \beta_{k+1} \cdots \beta_{m}}{\beta_0 \cdots \beta_{m}} \leq x$. In fact, otherwise they exists a $(m+1)$-tuple $(c_0',\ldots,c_m')$ in $\prod_{k=0}^{m} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $(c_0',\ldots,c_m') >_{\mathrm{lex}} (c_0,\ldots,c_m)$

and $\frac{\sum_{k=0}^{m} c'_k \beta_{k+1}\cdots\beta_m}{\beta_0\cdots\beta_m} \le x$. Therefore, by setting $c'_{m+1} = \cdots = c'_{n-1} = 0$, the n-tuple $(c'_0, \dots, c'_{n-1})$ in $\prod_{k=0}^{n-1} [\![0, \lceil\beta_k\rceil - 1]\!]$ is lexicographically greater than $(c_0, \dots, c_{n-1})$ and satisfies

$$\frac{\sum_{k=0}^{n-1} c'_k \beta_{k+1}\cdots\beta_{n-1}}{\beta_0\cdots\beta_{n-1}} = \frac{\sum_{k=0}^{m} c'_k \beta_{k+1}\cdots\beta_m}{\beta_0\cdots\beta_m} \le x,$$

which is absurd. Now, set $y = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^n \circ \delta_0(x)$. Then $y \in [0, x_{\boldsymbol{\beta}^{(n)}})$ and by induction hypothesis, we obtain that $y = x\beta_0\cdots\beta_{n-1} - \sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}$. Then, by setting

$$c_n = \begin{cases} \lfloor y\beta_n \rfloor & \text{if } y \in [0,1) \\ \lceil \beta_n \rceil - 1 & \text{if } y \in [1, x_{\boldsymbol{\beta}^{(n)}}) \end{cases}$$

we obtain that

$$\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^{n+1} \circ \delta_0(x) = x\beta_0\cdots\beta_n - \sum_{k=0}^{n} c_k\beta_{k+1}\cdots\beta_n.$$

In order to conclude, we have to show that

a) $\frac{\sum_{k=0}^{n} c_k\beta_{k+1}\cdots\beta_n}{\beta_0\cdots\beta_n} \le x$

b) $(c_0, \dots, c_n)$ is the lexicographically greatest $(n+1)$-tuple in $\prod_{k=0}^{n} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that a) holds.

By definition of c_n, we have $c_n \le y\beta_n$. Therefore,

$$\begin{aligned} \sum_{k=0}^{n} c_k\beta_{k+1}\cdots\beta_n &= \big(\sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}\big)\beta_n + c_n \\ &= (x\beta_0\cdots\beta_{n-1} - y)\beta_n + c_n \\ &\le x\beta_0\cdots\beta_n. \end{aligned}$$

This shows that a) holds.

Let us show b) by contradiction. Suppose that there exists $(c'_0, \dots, c'_n) \in \prod_{k=0}^{n} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that $(c'_0, \dots, c'_n) >_{\text{lex}} (c_0, \dots, c_n)$ and $\frac{\sum_{k=0}^{n} c'_k\beta_{k+1}\cdots\beta_n}{\beta_0\cdots\beta_n} \le x$. Then there exists $m \le n$ such that $c'_0 = c_0, \dots, c'_{m-1} = c_{m-1}$ and $c'_m \ge c_m + 1$. We again consider two cases. First, suppose that $m < n$. Since $(c'_0, \dots, c'_m) >_{\text{lex}} (c_0, \dots, c_m)$, we get $\frac{\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m}{\beta_0\cdots\beta_m} > x$. But then

$$\sum_{k=0}^{n} c'_k\beta_{k+1}\cdots\beta_n \ge \big(\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m\big)\beta_{m+1}\cdots\beta_n > x\beta_0\cdots\beta_n,$$

a contradiction. Second, suppose that $m = n$. Then

$$x\beta_0 \cdots \beta_n \geq \sum_{k=0}^{n} c'_k \beta_{k+1} \cdots \beta_n \geq \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_n + c_n + 1,$$

hence $y\beta_n \geq c_n + 1$. If $y \in [0,1)$ then $c_n + 1 = \lfloor y\beta_n \rfloor + 1 > y\beta_n$, a contradiction. Otherwise, $y \in [1, x_{\boldsymbol{\beta}^{(n)}})$ and $c_n + 1 = \lceil \beta_n \rceil$. But then $c'_n \geq \lceil \beta_n \rceil$, which is impossible since $c'_n \in [\![0, \lceil \beta_n \rceil - 1]\!]$. This shows b) and ends the proof. ■

5.1.2 The lazy $\boldsymbol{\beta}$-transformation

Let us now define the lazy $\boldsymbol{\beta}$-transformation.

Definition 5.1.9. The *lazy $\boldsymbol{\beta}$-transformation* is the transformation defined by

$$\begin{aligned} L_{\boldsymbol{\beta}} \colon \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big) &\to \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big), \\ (i,x) &\mapsto \big((i+1) \bmod p, \beta_i x - \lceil \beta_i x - x_{\boldsymbol{\beta}^{(i+1)}} \rceil\big). \end{aligned} \tag{5.4}$$

The lazy $\boldsymbol{\beta}$-transformation $L_{\boldsymbol{\beta}}$ generates the digits of the lazy $\boldsymbol{\beta}$-expansions of real numbers in the interval $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ as follows. For all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and $n \in \mathbb{N}$, we have

$$\xi_n(x) = \left\lceil \beta_n\big(\pi_2 \circ L_{\boldsymbol{\beta}}^n \circ \delta_0(x)\big) - x_{\boldsymbol{\beta}^{(n+1)}} \right\rceil$$

and

$$s_n(x) = \pi_2 \circ L_{\boldsymbol{\beta}}^{n+1} \circ \delta_0(x)$$

As for the greedy $\boldsymbol{\beta}$-transformation, the lazy $\boldsymbol{\beta}$-transformation $L_{\boldsymbol{\beta}}$ can be extended to a bigger interval.

Definition 5.1.10. The *extended lazy $\boldsymbol{\beta}$-transformation*, denoted $L_{\boldsymbol{\beta}}^{\text{ext}}$, is the transformation defined by

$$\begin{aligned} &L_{\boldsymbol{\beta}}^{\text{ext}} \colon \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big) \to \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big), \\ &(i,x) \mapsto \begin{cases} \big((i+1) \bmod p, \beta_i x\big) & \text{if } x \in (0, x_{\boldsymbol{\beta}^{(i)}} - 1] \\ \big((i+1) \bmod p, \beta_i x - \left\lceil \beta_i x - x_{\boldsymbol{\beta}^{(i+1)}} \right\rceil\big) & \text{if } x \in (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]. \end{cases} \end{aligned}$$

Therefore, we define the (extended) lazy $\boldsymbol{\beta}$-expansion of $x \in (0, x_{\boldsymbol{\beta}}]$ as the concatenations of the digits obtained thanks to the remainders defined by alternating the p maps

$$\pi_2 \circ L^{\text{ext}}_{\boldsymbol{\beta}} \circ \delta_i\big|_{(0,x_{\boldsymbol{\beta}^{(i)}}]} \colon (0, x_{\boldsymbol{\beta}^{(i)}}] \to (0, x_{\boldsymbol{\beta}^{(i+1)}}]$$

for $i \in [\![0, p-1]\!]$.

Example 5.1.11. Consider again the length-2 alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$ from Examples 5.1.1 and 5.1.5. We have $x_{\boldsymbol{\beta}} = \frac{5+7\sqrt{13}}{18} \simeq 1.67$ and $x_{\boldsymbol{\beta}^{(1)}} = \frac{2+\sqrt{13}}{3} \simeq 1.86$. The maps

$$\pi_2 \circ L^{\text{ext}}_{\boldsymbol{\beta}} \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]} \colon (0, x_{\boldsymbol{\beta}}] \to (0, x_{\boldsymbol{\beta}^{(1)}}]$$

and

$$\pi_2 \circ L^{\text{ext}}_{\boldsymbol{\beta}} \circ \delta_1\big|_{(0,x_{\boldsymbol{\beta}^{(1)}}]} \colon (0, x_{\boldsymbol{\beta}^{(1)}}] \to (0, x_{\boldsymbol{\beta}}]$$

are depicted in Figure 5.4. In Figure 5.5, we see the computation of the first five digits of the lazy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$.

Note that for each $i \in [\![0, p-1]\!]$,

$$L^{\text{ext}}_{\boldsymbol{\beta}}\big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big) \subseteq \{(i+1) \bmod p\} \times (x_{\boldsymbol{\beta}^{(i+1)}} - 1, x_{\boldsymbol{\beta}^{(i+1)}}].$$

Therefore, the restriction of the extended lazy $\boldsymbol{\beta}$-transformation $L^{\text{ext}}_{\boldsymbol{\beta}}$ to the domain $\bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big)$ gives back the lazy $\boldsymbol{\beta}$-transformation $L_{\boldsymbol{\beta}}$ initially defined in (5.4). Similarly to the greedy case, we obtain that the subspace $\bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big)$ is an attractor of $L^{\text{ext}}_{\boldsymbol{\beta}}$.

Proposition 5.1.12. *For each $(i, x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,*

$$(L^{\text{ext}}_{\boldsymbol{\beta}})^n(i, x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big). \tag{5.5}$$

Proof. Let $(i, x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big)$. On the one hand, if

$$(L^{\text{ext}}_{\boldsymbol{\beta}})^N(i, x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big)$$

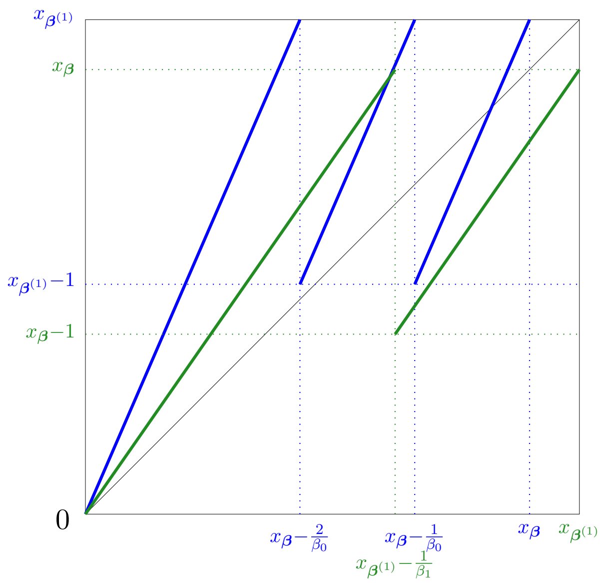

Figure 5.4: The maps $\pi_2 \circ L^{\mathrm{ext}}_{\boldsymbol{\beta}} \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ (blue) and $\pi_2 \circ L^{\mathrm{ext}}_{\boldsymbol{\beta}} \circ \delta_1\big|_{(0,x_{\boldsymbol{\beta}^{(1)}}]}$ (green) with $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$.

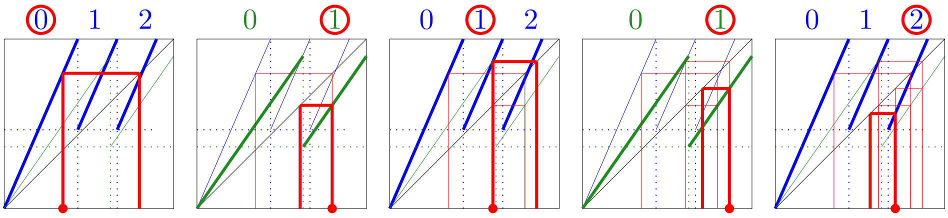

Figure 5.5: The first five digits of the lazy $\boldsymbol{\beta}$-expansion of $\frac{1+\sqrt{5}}{5}$ are 01112 for $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$.

for some $N \in \mathbb{N}$, then clearly (5.5) occurs for all $n \geq N$. On the other hand, if

$$(L^{\mathrm{ext}}_{\boldsymbol{\beta}})^n(i,x) \notin \bigcup_{i=0}^{p-1} \left(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\right)$$

for all $n \in \mathbb{N}$, then we would get that $x = 0$ since at each step, the lazy algorithm would pick the minimal digit, which is always 0. ■

The following proposition is the analogue of Proposition 5.1.8 for the lazy $\boldsymbol{\beta}$-transformation.

Proposition 5.1.13. *For all $x \in (0, x_{\boldsymbol{\beta}}]$ and $n \in \mathbb{N}$, we have*

$$\pi_2 \circ (L_{\boldsymbol{\beta}}^{\text{ext}})^n \circ \delta_0(x) = x\beta_0 \cdots \beta_{n-1} - \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}$$

where $(c_0, \ldots, c_{n-1})$ is the lexicographically least n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}}{\beta_0\cdots\beta_{n-1}} + \sum_{m=n}^{+\infty} \frac{\lceil\beta_m\rceil-1}{\prod_{k=0}^{m}\beta_k} \geq x$.

Proof. We proceed by induction on n. The base case $n = 0$ is immediate: both members of the equality are equal to x. Now, suppose that the result is satisfied for some $n \in \mathbb{N}$. Let $x \in (0, x_{\boldsymbol{\beta}}]$. Let $(c_0, \ldots, c_{n-1})$ be the lexicographically least n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that

$$\frac{\sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}}{\beta_0\cdots\beta_{n-1}} + \sum_{m=n}^{+\infty} \frac{\lceil\beta_m\rceil - 1}{\prod_{k=0}^{m}\beta_k} \geq x. \tag{5.6}$$

Note that inequality (5.6) wan be rewritten as

$$\frac{\sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1} + x_{\boldsymbol{\beta}^{(n)}}}{\beta_0\cdots\beta_{n-1}} \geq x. \tag{5.7}$$

Then it is easily seen that for all $m < n$, $(c_0, \ldots, c_m)$ is the lexicographically least $(m+1)$-tuple in $\prod_{k=0}^{m} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{m} c_k\beta_{k+1}\cdots\beta_m + x_{\boldsymbol{\beta}^{(m+1)}}}{\beta_0\cdots\beta_m} \geq x$. In fact, otherwise they exists a $(m+1)$-tuple $(c'_0, \ldots, c'_m)$ in $\prod_{k=0}^{m} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that $(c'_0, \ldots, c'_m) <_{\text{lex}} (c_0, \ldots, c_m)$ and $\frac{\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m + x_{\boldsymbol{\beta}^{(m+1)}}}{\beta_0\cdots\beta_m} \geq x$. Therefore, by setting $c'_{m+1} = \lceil\beta_{m+1}\rceil - 1, \ldots, c'_{n-1} = \lceil\beta_{n-1}\rceil - 1$, the n-tuple $(c'_0, \ldots, c'_{n-1})$ in $\prod_{k=0}^{n-1} [\![0, \lceil\beta_k\rceil - 1]\!]$ is lexicographically smaller than $(c_0, \ldots, c_{n-1})$ and satisfies

$$\frac{\sum_{k=0}^{n-1} c'_k\beta_{k+1}\cdots\beta_{n-1} + x_{\boldsymbol{\beta}^{(n)}}}{\beta_0\cdots\beta_{n-1}} = \frac{\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m + x_{\boldsymbol{\beta}^{(m+1)}}}{\beta_0\cdots\beta_m} \geq x,$$

which is absurd. Now, set $y = \pi_2 \circ (L_{\boldsymbol{\beta}}^{\text{ext}})^n \circ \delta_0(x)$. Then $y \in (0, x_{\boldsymbol{\beta}^{(n)}}]$ and by induction hypothesis, we obtain that $y = x\beta_0\cdots\beta_{n-1} - \sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}$. Then, by setting

$$c_n = \begin{cases} 0 & \text{if } y \in (0, x_{\boldsymbol{\beta}^{(n)}} - 1] \\ \lceil y\beta_n - x_{\boldsymbol{\beta}^{(n+1)}}\rceil & \text{if } y \in (x_{\boldsymbol{\beta}^{(n)}} - 1, x_{\boldsymbol{\beta}^{(n)}}] \end{cases}$$

we obtain that

$$\pi_2 \circ (L_{\boldsymbol{\beta}}^{\text{ext}})^{n+1} \circ \delta_0(x) = \pi_2 \circ L_{\boldsymbol{\beta}}^{\text{ext}} \circ \delta_n(y)$$

$$
\begin{aligned}
&= y\beta_n - c_n \\
&= x\beta_0 \cdots \beta_n - \sum_{k=0}^{n} c_k \beta_{k+1} \cdots \beta_n.
\end{aligned}
$$

In order to conclude, we have to show that

a) $\frac{\sum_{k=0}^{n} c_k\beta_{k+1}\cdots\beta_n + x_{\boldsymbol{\beta}^{(n+1)}}}{\beta_0\cdots\beta_n} \geq x$

b) $(c_0, \ldots, c_n)$ is the lexicographically least $(n+1)$-tuple in $\prod_{k=0}^{n} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that a) holds.

By definition of c_n, we have $c_n \geq y\beta_n - x_{\boldsymbol{\beta}^{(n+1)}}$. Therefore,

$$
\begin{aligned}
\sum_{k=0}^{n} c_k\beta_{k+1}\cdots\beta_n &= \big(\sum_{k=0}^{n-1} c_k\beta_{k+1}\cdots\beta_{n-1}\big)\beta_n + c_n \\
&= (x\beta_0\cdots\beta_{n-1} - y)\beta_n + c_n \\
&\geq x\beta_0\cdots\beta_n - x_{\boldsymbol{\beta}^{(n+1)}}.
\end{aligned}
$$

This shows that a) holds.

Let us show b) by contradiction. Suppose that there exists $(c'_0, \ldots, c'_n) \in \prod_{k=0}^{n} [\![0, \lceil\beta_k\rceil - 1]\!]$ such that $(c'_0, \ldots, c'_n) <_{\text{lex}} (c_0, \ldots, c_n)$ and

$$
\frac{\sum_{k=0}^{n} c'_k\beta_{k+1}\cdots\beta_n + x_{\boldsymbol{\beta}^{(n+1)}}}{\beta_0\cdots\beta_n} \geq x.
$$

Then there exists $m \leq n$ such that $c'_0 = c_0, \ldots, c'_{m-1} = c_{m-1}$ and $c'_m + 1 \leq c_m$. We again consider two cases. First, suppose that $m < n$. Since $(c'_0, \ldots, c'_m) <_{\text{lex}} (c_0, \ldots, c_m)$, we get $\frac{\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m + x_{\boldsymbol{\beta}^{(m+1)}}}{\beta_0\cdots\beta_m} < x$. But then

$$
\begin{aligned}
&\sum_{k=0}^{n} c'_k\beta_{k+1}\cdots\beta_n + x_{\boldsymbol{\beta}^{(n+1)}} \\
&\leq \big(\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m\big)\beta_{m+1}\cdots\beta_n + \sum_{k=m+1}^{n} (\lceil\beta_k\rceil - 1)\beta_{k+1}\cdots\beta_m + x_{\boldsymbol{\beta}^{(n+1)}} \\
&= \big(\sum_{k=0}^{m} c'_k\beta_{k+1}\cdots\beta_m\big)\beta_{m+1}\cdots\beta_n + x_{\boldsymbol{\beta}^{(m+1)}}\beta_{m+1}\cdots\beta_n \\
&< \big(x\beta_0\cdots\beta_m - x_{\boldsymbol{\beta}^{(m+1)}}\big)\beta_{m+1}\cdots\beta_n + x_{\boldsymbol{\beta}^{(m+1)}}\beta_{m+1}\cdots\beta_n \\
&= x\beta_0\cdots\beta_n,
\end{aligned}
$$

a contradiction, where the first equality is obtained by iterating (2.10). Second, suppose that $m = n$. Then

$$\begin{aligned} x\beta_0 \cdots \beta_n &\leq \sum_{k=0}^{n} c'_k \beta_{k+1} \cdots \beta_n + x_{\boldsymbol{\beta}^{(n+1)}} \\ &= \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_n + c'_n + x_{\boldsymbol{\beta}^{(n+1)}} \\ &\leq \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_n + c_n - 1 + x_{\boldsymbol{\beta}^{(n+1)}}, \end{aligned}$$

hence $y\beta_n \leq c_n - 1 + x_{\boldsymbol{\beta}^{(n+1)}}$. If $y \in (0, x_{\boldsymbol{\beta}^{(n)}} - 1]$ then $c_n = 0$. But then $c'_n + 1 \leq 0$ which is impossible since $c'_n \in [\![0, \lceil \beta_n \rceil - 1]\!]$. Otherwise, $y \in (x_{\boldsymbol{\beta}^{(n)}} - 1, x_{\boldsymbol{\beta}^{(n)}}]$ and we have

$$\begin{aligned} c_n - 1 + x_{\boldsymbol{\beta}^{(n+1)}} &= \lceil y\beta_n - x_{\boldsymbol{\beta}^{(n+1)}} \rceil - 1 + x_{\boldsymbol{\beta}^{(n+1)}} \\ &< y\beta_n - x_{\boldsymbol{\beta}^{(n+1)}} + x_{\boldsymbol{\beta}^{(n+1)}} \\ &= y\beta_n, \end{aligned}$$

a contradiction. This shows b) and ends the proof. ■

5.1.3 A note on Cantor bases

Since the greedy algorithm described in Section 2.3 is well defined in the context of Cantor bases, a natural question is to ask if the notion of iteration of a greedy $\boldsymbol{\beta}$-transformation can be extended to this framework. The following proposition is a generalization of Proposition 5.1.8 when restricted to $[0, 1)$ but in the general framework of a Cantor base $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$.

Proposition 5.1.14. *For all $x \in [0, 1)$, $n \in \mathbb{N}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, we have*

$$T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x) = x\beta_0 \cdots \beta_{n-1} - \sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}$$

where $(c_0, \ldots, c_{n-1})$ is the lexicographically greatest n-tuple in $\prod_{k=0}^{n-1} [\![0, \lceil \beta_k \rceil - 1]\!]$ such that $\frac{\sum_{k=0}^{n-1} c_k \beta_{k+1} \cdots \beta_{n-1}}{\beta_0 \cdots \beta_{n-1}} \leq x$.

For all $k \in [\![0, n-1]\!]$, the β_k-transformation L_{β_k} is defined on $(x_{\beta_k} - 1, x_{\beta_k}]$. So, the transformations $L_{\beta_0}, \ldots, L_{\beta_{n-1}}$ cannot be composed to one another in general. Therefore, even if the lazy algorithm can be defined for Cantor bases, provided that $x_{\boldsymbol{\beta}} < +\infty$, we cannot state an analogue of Proposition 5.1.14 in terms of the lazy transformations for Cantor bases.

Even though this chapter is mostly concerned with alternate bases, let us emphasize that some results are indeed valid for any sequence $(\beta_n)_{n\in\mathbb{N}} \in (\mathbb{R}_{>1})^{\mathbb{N}}$, and hence for any Cantor base. This is the case of Proposition 5.1.14, Theorem 5.2.3, Corollary 5.2.4 and Proposition 5.2.19.

5.2 Dynamical properties of $T_{\boldsymbol{\beta}}$

In this section, we study the dynamics of the greedy $\boldsymbol{\beta}$-transformation. First, we generalize Theorem 1.4.46 to the transformation $T_{\boldsymbol{\beta}}$ on $[\![0, p-1]\!] \times [0, 1)$. Second, we extend the obtained result to the extended transformation $T_{\boldsymbol{\beta}}^{\text{ext}}$. Third, we provide a formula for the density functions of the measures found in the first two parts. Finally, we compute the frequencies of the digits in the greedy $\boldsymbol{\beta}$-expansions.

5.2.1 Unique absolutely continuous $T_{\boldsymbol{\beta}}$-invariant measure

In order to generalize Theorem 1.4.46 to the alternate base framework, we start by recalling a result of Lasota and Yorke [LY82, Theorem 4].

Theorem 5.2.1. *Let $T\colon [0, 1) \to [0, 1)$ be a transformation for which there exists a partition $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ of the interval $[0, 1)$ with $a_0 < \cdots < a_K$ such that for each $k \in [\![0, K-1]\!]$, $T_{|[a_k,a_{k+1})}$ is convex, $T(a_k) = 0$, $T'(a_k) > 0$ and $T'(0) > 1$. Then there exists a unique T-invariant absolutely continuous probability measure. Furthermore, its density function is bounded and decreasing, and the corresponding dynamical system is exact.*

We then prove a stability lemma.

Lemma 5.2.2. *Let $\mathcal{I}$ be the family of transformations $T\colon [0, 1) \to [0, 1)$ for which there exist a partition $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ of the interval $[0, 1)$ with $a_0 < \cdots < a_K$ and a slope $s > 1$ such that for all $k \in [\![0, K-1]\!]$, $a_{k+1} - a_k \le \frac{1}{s}$ and for all $x \in [a_k, a_{k+1})$, $T(x) = s(x - a_k)$. Then $\mathcal{I}$ is closed under composition.*

Proof. Let $S, T \in \mathcal{I}$. Let $[a_0, a_1), \ldots, [a_{K-1}, a_K)$ and $[b_0, b_1), \ldots, [b_{L-1}, b_L)$ be partitions of the interval $[0, 1)$ with $a_0 < \cdots < a_K$, $b_0 < \cdots < b_L$, and let $s, t > 1$ such that for all $k \in [\![0, K-1]\!]$, $a_{k+1} - a_k \le \frac{1}{s}$, for all $\ell \in [\![0, L-1]\!]$, $b_{\ell+1} - b_\ell \le \frac{1}{t}$ and for all $x \in [0, 1)$, $S(x) = s(x - a_k)$ if $x \in [a_k, a_{k+1})$ and

$T(x) = t(x - b_\ell)$ if $x \in [b_\ell, b_{\ell+1})$. For each $k \in [\![0, K-1]\!]$, define L_k to be the greatest $\ell \in [\![0, L-1]\!]$ such that $a_k + \frac{b_\ell}{s} < a_{k+1}$. Consider the partition

$$\begin{array}{l} \Big[a_0 + \frac{b_0}{s}, a_0 + \frac{b_1}{s}\Big), \ldots, \Big[a_0 + \frac{b_{L_0-1}}{s}, a_0 + \frac{b_{L_0}}{s}\Big), \Big[a_0 + \frac{b_{L_0}}{s}, a_1\Big) \\ \vdots \\ \Big[a_{K-1} + \frac{b_0}{s}, a_{K-1} + \frac{b_1}{s}\Big), \ldots, \Big[a_{K-1} + \frac{b_{L_{K-1}-1}}{s}, a_{K-1} + \frac{b_{L_{K-1}}}{s}\Big), \Big[a_{K-1} + \frac{b_{L_{K-1}}}{s}, a_K\Big) \end{array}$$

of the interval $[0,1)$. For each $k \in [\![0, K-1]\!]$ and $\ell \in [\![0, L_k - 1]\!]$, $a_k + \frac{b_{\ell+1}}{s} - a_k - \frac{b_\ell}{s} \le \frac{1}{ts}$ and $a_{k+1} - a_k - \frac{b_{L_k}}{s} = (a_{k+1} - a_k - \frac{b_{L_k+1}}{s}) + \frac{b_{L_k+1} - b_{L_k}}{s} \le \frac{1}{ts}$. Now, let $x \in [0,1)$ and $k \in [\![0, K-1]\!]$ be such that $x \in [a_k, a_{k+1})$. Then $S(x) = s(x - a_k) \in [0,1)$. We distinguish two cases: either there exists $\ell \in [\![0, L_k - 1]\!]$ such that $x \in [a_k + \frac{b_\ell}{s}, a_k + \frac{b_{\ell+1}}{s})$, or $x \in [a_k + \frac{b_{L_k}}{s}, a_{k+1})$. In the former case, $S(x) \in [b_\ell, b_{\ell+1})$ and $T \circ S(x) = t(S(x) - b_\ell) = ts(x - (a_k + \frac{b_\ell}{s}))$. In the latter case, since $a_{k+1} - a_k \le \frac{b_{L_k+1}}{s}$, we get that $S(x) \in [b_{L_k}, b_{L_k+1})$ and hence that $T \circ S(x) = t(S(x) - b_{L_k}) = ts(x - (a_k + \frac{b_{L_k}}{s}))$. This shows that the composition $T \circ S$ belongs to $\mathcal{I}$. ∎

The following theorem provides us with the main tool for the construction of a $T_{\boldsymbol{\beta}}$-invariant measure.

Theorem 5.2.3. *For all $n \in \mathbb{N}_{\ge 1}$ and all $\beta_0, \ldots, \beta_{n-1} > 1$, there exists a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$. Furthermore, the measure μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$, its density function is bounded and decreasing, and the dynamical system*

$$([0,1), \mathcal{B}([0,1)), \mu, T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$$

is exact and has entropy $\log(\beta_0 \cdots \beta_{n-1})$.

Proof. The existence of a unique $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant absolutely continuous probability measure μ on $\mathcal{B}([0,1))$, the fact that its density function is bounded and decreasing, and the exactness of the corresponding dynamical system follow from Theorem 5.2.1 and Lemma 5.2.2. With a similar argument as in [DK10, Lemma 2.6], we can conclude that $\frac{d\mu}{d\lambda} > 0$ λ-a.e. on $[0,1)$. It follows that μ is equivalent to the Lebesgue measure on $\mathcal{B}([0,1))$. Moreover, the entropy equals $\log(\beta_0 \cdots \beta_{n-1})$ since $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$ is a piecewise linear transformation of constant slope $\beta_0 \cdots \beta_{n-1}$ [DK21, Roh61]. ∎

The following consequence of Theorem 5.2.3 will be useful for proving our generalization of Theorem 1.4.46.

Corollary 5.2.4. *Let $n \in \mathbb{N}_{\geq 1}$ and $\beta_0, \ldots, \beta_{n-1} > 1$. Then for all $B \in \mathcal{B}([0,1))$ such that $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})^{-1}(B) = B$, we have $\lambda(B) \in \{0,1\}$.*

Definition 5.2.5. For each $i \in [\![0, p-1]\!]$, we let $\mu_{\boldsymbol{\beta},i}$ denote the unique $(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})$-invariant absolutely continuous probability measure given by Theorem 5.2.3.

We use the convention that for all $n \in \mathbb{Z}$, $\mu_{\boldsymbol{\beta},n} = \mu_{\boldsymbol{\beta},n \bmod p}$.

Note that if $p = 1$, the measure $\mu_{\boldsymbol{\beta},0}$ is the unique invariant measure found independently by Gel'fond in 1959 [Gel59] and Parry in 1960 [Par60] (see Theorem 1.4.46).

Lemma 5.2.6. [1] *For $i \in [\![0, p-1]\!]$, we have $\mu_{\boldsymbol{\beta},i} = \mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}$.*

Proof. Let $i \in [\![0, p-1]\!]$. By definition of $\mu_{\boldsymbol{\beta},i}$, it suffices to show that $\mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}$ is a $(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})$-invariant absolutely continuous probability measure on $\mathcal{B}([0,1))$. First, we have $\mu_{\boldsymbol{\beta},i-1}\big(T_{\beta_{i-1}}^{-1}([0,1))\big) = \mu_{\boldsymbol{\beta},i-1}([0,1)) = 1$. Second, for all $B \in \mathcal{B}([0,1))$, we have

$$\begin{aligned}
&\mu_{\boldsymbol{\beta},i-1} \circ T_{\beta_{i-1}}^{-1}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})^{-1}(B)\big) \\
&= \mu_{\boldsymbol{\beta},i-1}\big((T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}} \circ T_{\beta_{i-p-1}})^{-1}(B)\big) \\
&= \mu_{\boldsymbol{\beta},i-1}\big((T_{\beta_{i-2}} \circ \cdots \circ T_{\beta_{i-p-1}})^{-1}(T_{\beta_{i-1}}^{-1}(B))\big) \\
&= \mu_{\boldsymbol{\beta},i-1}\big(T_{\beta_{i-1}}^{-1}(B)\big).
\end{aligned}$$

Third, for all $B \in \mathcal{B}([0,1))$ such that $\lambda(B) = 0$, we get that $\lambda(T_{\beta_{i-1}}^{-1}(B)) = 0$ by Remark 1.4.47, and hence that $\mu_{\boldsymbol{\beta},i-1}(T_{\beta_{i-1}}^{-1}(B)) = 0$ since $\mu_{\boldsymbol{\beta},i-1}$ is absolutely continuous. ■

Definition 5.2.7. Consider the σ-algebra

$$\mathcal{T}_p = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}([0,1)) \right\} \tag{5.8}$$

[1] I thank Julien Leroy for suggesting this lemma, which allowed me and my co-authors to simplify several proofs.

over $[\![0, p-1]\!] \times [0,1)$. We define a probability measure $\mu_{\boldsymbol{\beta}}$ on the σ-algebra $\mathcal{T}_p$ as follows: for all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right) = \frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}(B_i). \tag{5.9}$$

We now study the properties of the probability measure $\mu_{\boldsymbol{\beta}}$.

Proposition 5.2.8. *The measure $\mu_{\boldsymbol{\beta}}$ is $T_{\boldsymbol{\beta}}$-invariant.*

Proof. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$,

$$\begin{aligned}
\mu_{\boldsymbol{\beta}}\left(T_{\boldsymbol{\beta}}^{-1}\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right)\right) &= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1} T_{\boldsymbol{\beta}}^{-1}(\{i\} \times B_i)\right) \\
&= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}\big(\{(i-1) \bmod p\} \times T_{\beta_{i-1}}^{-1}(B_i)\big)\right) \\
&= \frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i-1}(T_{\beta_{i-1}}^{-1}(B_i)) \\
&= \frac{1}{p}\sum_{i=0}^{p-1}\mu_{\boldsymbol{\beta},i}(B_i) \\
&= \mu_{\boldsymbol{\beta}}\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right)
\end{aligned}$$

where we applied Lemma 5.2.6 for the fourth equality. ∎

Corollary 5.2.9. *The quadruple $\big([\![0,p-1]\!]\times[0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}\big)$ is a dynamical system.*

Let us define a new measure over the σ-algebra $\mathcal{T}_p$, which extends to the "p-dimensional setting" the Lebesgue measure.

Definition 5.2.10. For all $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$, we set

$$\lambda_p\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right) = \frac{1}{p}\sum_{i=0}^{p-1}\lambda(B_i). \tag{5.10}$$

We call this measure the *p-Lebesgue measure* on $\mathcal{T}_p$.

Proposition 5.2.11. *The measure $\mu_{\boldsymbol{\beta}}$ is equivalent to the p-Lebesgue measure on $\mathcal{T}_p$.*

Proof. This follows from the fact that the p measures $\mu_{\boldsymbol{\beta},0},\dots,\mu_{\boldsymbol{\beta},p-1}$ are equivalent to the Lebesgue measure λ on $\mathcal{B}([0,1))$. ■

Next, we compute the entropy of the dynamical system $\big([\![0,p-1]\!]\times[0,1),\mathcal{T}_p,\mu_{\boldsymbol{\beta}},T_{\boldsymbol{\beta}}\big)$. To do so, we consider the p induced transformations

$$T_{\boldsymbol{\beta},i}\colon \{i\}\times[0,1)\to\{i\}\times[0,1),\ (i,x)\mapsto T_{\boldsymbol{\beta}}^p(i,x)$$

for $i\in[\![0,p-1]\!]$. Note that indeed, for all $(i,x)\in[\![0,p-1]\!]\times[0,1)$, the first return of (i,x) to $\{i\}\times[0,1)$ is equal to p. Thus $T_{\boldsymbol{\beta},i}=T_{\boldsymbol{\beta}}^p\big|_{\{i\}\times[0,1)}$. For each $i\in[\![0,p-1]\!]$, the induced transformation $T_{\boldsymbol{\beta},i}$ is measure preserving with respect to the measure $\gamma_{\boldsymbol{\beta},i}$ on the σ-algebra $\{\{i\}\times B : B\in\mathcal{B}([0,1))\}$ defined as follows: for all $B\in\mathcal{B}([0,1))$,

$$\gamma_{\boldsymbol{\beta},i}(\{i\}\times B)=p\mu_{\boldsymbol{\beta}}(\{i\}\times B).$$

Lemma 5.2.12. *For every $i\in[\![0,p-1]\!]$, the map $\delta_i\big|_{[0,1)}\colon[0,1)\to\{i\}\times[0,1),\ x\mapsto(i,x)$ defines an isomorphism between the dynamical systems*

$$\big([0,1),\mathcal{B}([0,1)),\mu_{\boldsymbol{\beta},i},T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-p}}\big)$$

and

$$\big(\{i\}\times[0,1),\{\{i\}\times B : B\in\mathcal{B}([0,1))\},\gamma_{\boldsymbol{\beta},i},T_{\boldsymbol{\beta},i}\big)$$

Proof. Let $i\in[\![0,p-1]\!]$. For the sake of clarity, in this proof, we simply denote the map $\delta_i\big|_{[0,1)}$ by δ_i. Clearly, δ_i is measurable, bijective and

$$\delta_i\circ(T_{\beta_{i-1}}\circ\cdots\circ T_{\beta_{i-p}})=T_{\boldsymbol{\beta},i}\circ\delta_i.$$

Moreover, for all $B\in\mathcal{B}([0,1))$, we have

$$\gamma_{\boldsymbol{\beta},i}(\{i\}\times B)=p\mu_{\boldsymbol{\beta}}(\{i\}\times B)=\mu_{\boldsymbol{\beta},i}(B)=\mu_{\boldsymbol{\beta},i}\circ(\delta_i)^{-1}(\{i\}\times B).$$

■

We are now ready to calculate the entropy of the greedy dynamical system. Recall that $\beta=\prod_{i=0}^{p-1}\beta_i$.

Proposition 5.2.13. *The entropy of the dynamical system*

$$\left([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}\right)$$

is $\frac{1}{p}\log(\beta)$.

Proof. Let $i \in [\![0,p-1]\!]$. By Abramov's formula (see Theorem 1.3.33), we have

$$h_{\mu_{\boldsymbol{\beta}}}(T_{\boldsymbol{\beta}}) = \mu_{\boldsymbol{\beta}}(\{i\} \times [0,1))\, h_{\gamma_{\boldsymbol{\beta},i}}(T_{\boldsymbol{\beta},i}) = \frac{1}{p}\, h_{\gamma_{\boldsymbol{\beta},i}}(T_{\boldsymbol{\beta},i}).$$

Since the entropy is an isomorphic invariant, it follows from Theorem 5.2.3 and Lemma 5.2.12 that $h_{\gamma_{\boldsymbol{\beta},i}}(T_{\boldsymbol{\beta},i}) = \log(\beta)$. ■

Finally, we prove that any $T_{\boldsymbol{\beta}}$-invariant set has p-Lebesgue measure 0 or 1.

Proposition 5.2.14. *For all $A \in \mathcal{T}_p$ such that $T_{\boldsymbol{\beta}}^{-1}(A) = A$, we have $\lambda_p(A) \in \{0,1\}$.*

Proof. Let $B_0, \ldots, B_{p-1}$ be sets in $\mathcal{B}([0,1))$ such that

$$T_{\boldsymbol{\beta}}^{-1}\left(\bigcup_{i=0}^{p-1}(\{i\} \times B_i)\right) = \bigcup_{i=0}^{p-1}(\{i\} \times B_i).$$

This implies that

$$T_{\beta_{i-1}}^{-1}(B_i) = B_{(i-1) \bmod p} \quad \text{for all } i \in [\![0,p-1]\!]. \tag{5.11}$$

We use the convention that $B_n = B_{n \bmod p}$ for all $n \in \mathbb{Z}$. An easy induction yields that for all $i \in [\![0,p-1]\!]$ and $n \in \mathbb{N}$,

$$(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-n}})^{-1}(B_i) = B_{i-n}.$$

In particular, for $n = p$, we get that for each $i \in [\![0,p-1]\!]$,

$$(T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})^{-1}(B_i) = B_i.$$

By Corollary 5.2.4, for each $i \in [\![0,p-1]\!]$, $\lambda(B_i) \in \{0,1\}$. By definition of λ_p, in order to conclude, it suffices to show that either $\lambda(B_i) = 0$ for all $i \in [\![0,p-1]\!]$, or $\lambda(B_i) = 1$ for all $i \in [\![0,p-1]\!]$. From (5.11) and Remark 1.4.47, we get that for each $i \in [\![0,p-1]\!]$, $\lambda(B_i) = 0$ if and only if $\lambda(B_{i+1}) = 0$. The conclusion follows. ■

We are now able to state the announced generalization of Theorem 1.4.46 to alternate bases.

Theorem 5.2.15. *The measure $\mu_{\boldsymbol{\beta}}$ is the unique $T_{\boldsymbol{\beta}}$-invariant probability measure on $\mathcal{T}_p$ that is absolutely continuous with respect to λ_p. Furthermore, $\mu_{\boldsymbol{\beta}}$ is equivalent to λ_p on $\mathcal{T}_p$ and the dynamical system*

$$([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$$

is ergodic and has entropy $\frac{1}{p}\log(\beta)$.

Proof. By Propositions 5.2.8 and 5.2.11, $\mu_{\boldsymbol{\beta}}$ is a $T_{\boldsymbol{\beta}}$-invariant probability measure that is absolutely continuous with respect to λ_p on $\mathcal{B}([0,1))$. Then we get from Proposition 5.2.14 that for all $A \in \mathcal{T}_p$ such that $T_{\boldsymbol{\beta}}^{-1}(A) = A$, we have $\mu_{\boldsymbol{\beta}}(A) \in \{0,1\}$. Therefore, the dynamical system $([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$ is ergodic. Now, we obtain that the measure $\mu_{\boldsymbol{\beta}}$ is unique by Theorem 1.3.28. The equivalence between $\mu_{\boldsymbol{\beta}}$ and λ_p and the entropy of the system were already obtained in Propositions 5.2.11 and 5.2.13. ■

Remark 5.2.16. For p greater than 1, the dynamical system

$$([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$$

is not exact even though for all $i \in [\![0,p-1]\!]$, the dynamical systems

$$([0,1), \mathcal{B}([0,1)), \mu_{\boldsymbol{\beta},i}, T_{\beta_{i-1}} \circ \cdots \circ T_{\beta_{i-p}})$$

are exact. It suffices to note that the dynamical system

$$([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}^p)$$

is not ergodic for $p > 1$. Indeed, $T_{\boldsymbol{\beta}}^{-p}(\{0\} \times [0,1)) = \{0\} \times [0,1)$ whereas $\mu_{\boldsymbol{\beta}}(\{0\} \times [0,1)) = \frac{1}{p}$.

5.2.2 Extended measure

In order to study the dynamics of the extended greedy $\boldsymbol{\beta}$-transformation defined in (5.2), we first define an *extended σ-algebra* $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and *extended measures* $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}}$ by extending the domain of the measures $\mu_{\boldsymbol{\beta}}$ and λ_p defined in (5.9) and (5.10) respectively.

Definition 5.2.17. Define a σ-algebra $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ on $\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big)$ as follows:

$$\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}=\left\{\bigcup_{i=0}^{p-1}(\{i\}\times B_i) : \forall i\in[\![0,p-1]\!],\ B_i\in\mathcal{B}([0,x_{\boldsymbol{\beta}^{(i)}}))\right\}.$$

For $A\in\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$, we set

$$\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}(A)=\mu_{\boldsymbol{\beta}}\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big)$$

and

$$\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}}(A)=\lambda_p\big(A\cap\big([\![0,p-1]\!]\times[0,1)\big)\big).$$

Note that, in the previous section, we could have denoted the σ-algebra $\mathcal{T}_p$ by $\mathcal{T}_{\boldsymbol{\beta}}$ and similarly, the measure λ_p by $\lambda_{\boldsymbol{\beta}}$. We chose to only emphasize the dependence in p since the definitions of $\mathcal{T}_p$ and λ_p indeed only depend on the length p of the corresponding alternate base $\boldsymbol{\beta}$.

Theorem 5.2.18. *The measure $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ is the unique $T_{\boldsymbol{\beta}}^{\mathrm{ext}}$-invariant probability measure on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ that is absolutely continuous with respect to $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}}$. Furthermore, $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ is equivalent to $\lambda_{\boldsymbol{\beta}}^{\mathrm{ext}}$ on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and the dynamical system*

$$\Big(\bigcup_{i=0}^{p-1}\big(\{i\}\times[0,x_{\boldsymbol{\beta}^{(i)}})\big),\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}},\mu_{\boldsymbol{\beta}}^{\mathrm{ext}},T_{\boldsymbol{\beta}}^{\mathrm{ext}}\Big)$$

is ergodic and has entropy $\frac{1}{p}\log(\beta)$.

Proof. Clearly, $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ is a probability measure on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$. For all $A\in\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$, we have

$$\begin{aligned}
&\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}((T_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}(A))\\
&=\mu_{\boldsymbol{\beta}}\big((T_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}(A)\cap([\![0,p-1]\!]\times[0,1))\big)\\
&=\mu_{\boldsymbol{\beta}}\big((T_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\cap([\![0,p-1]\!]\times[0,1))\big)\\
&=\mu_{\boldsymbol{\beta}}\big(T_{\boldsymbol{\beta}}^{-1}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\big)\\
&=\mu_{\boldsymbol{\beta}}\big(A\cap([\![0,p-1]\!]\times[0,1))\big)\\
&=\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}(A)
\end{aligned}$$

where we used Proposition 5.2.8 for the fourth equality. This shows that $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ is $T_{\boldsymbol{\beta}}^{\mathrm{ext}}$-invariant on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$. The conclusion then follows from the fact that the identity map from $[\![0, p-1]\!] \times [0,1)$ to $\bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big)$ defines an isomorphism between the dynamical systems

$$([\![0, p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}})$$

and

$$(\bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big), \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}, \mu_{\boldsymbol{\beta}}^{\text{ext}}, T_{\boldsymbol{\beta}}^{\text{ext}}).$$

■

5.2.3 Density functions

In the next proposition, we express the density function of the unique measure given in Theorem 5.2.3.

Proposition 5.2.19. *Consider $n \in \mathbb{N}_{\geq 1}$ and $\beta_0, \ldots, \beta_{n-1} > 1$. Suppose that*

- *K is the number of not onto branches of $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$*
- *for $j \in [\![1, K]\!]$, c_j is the right-hand side endpoint of the domain of the j^{th} not onto branch of $T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}$*
- *$T\colon [0,1) \to [0,1)$ is the transformation defined by*

$$T(x) = T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)$$

for $x \notin \{c_1, \ldots, c_K\}$ and

$$T(c_j) = \lim_{x \to c_j^-} T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0}(x)$$

for $j \in [\![1, K]\!]$

- *S is the matrix defined by $S = (S_{i,j})_{1 \leq i,j, \leq K}$ where*

$$S_{i,j} = \sum_{m \in \mathbb{N}_{\geq 1}} \frac{\delta(T^m(c_i) > c_j)}{(\beta_0 \cdots \beta_{n-1})^m},$$

where $\delta(P)$ equals 1 when the property P is satisfied and 0 otherwise

- *1 is not an eigenvalue of S*
- *$d_0 = 1$ and $\big(d_1 \cdots d_K\big) = \big(1 \cdots 1\big)\,(-S + Id_K)^{-1}$*

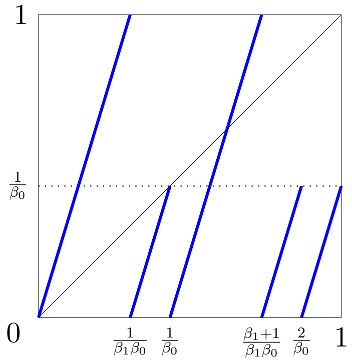

Figure 5.6: The composition $T_{\beta_1} \circ T_{\beta_0}$ with $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$.

- $C = \int_0^1 \left(d_0 + \sum_{j=1}^K d_j \sum_{m\in\mathbb{N}_{\ge 1}} \frac{\chi_{[0,T^m(c_j)]}}{(\beta_0\cdots\beta_{n-1})^m} \right) d\lambda$ *is the normalization constant.*

Then the density function of the $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$*-invariant measure given by Theorem 5.2.3 with respect to the Lebesgue measure is*

$$\frac{1}{C}\left(d_0 + \sum_{j=1}^K d_j \sum_{m\in\mathbb{N}_{\ge 1}} \frac{\chi_{[0,T^m(c_j)]}}{(\beta_0\cdots\beta_{n-1})^m} \right). \tag{5.12}$$

Proof. This is an application of the formula given in [Gór09, Theorem 2]. ■

In [Gór09], Góra conjectured that 1 is not an eigenvalue of the matrix S if and only if the dynamical system is exact. Thus, if Góra's conjecture were true, thanks to Theorem 5.2.3, the hypothesis that 1 is not an eigenvalue of the matrix S could be removed from the statement of Proposition 5.2.19. In particular, Proposition 5.2.19 would then provide the density function of the $(T_{\beta_{n-1}} \circ \cdots \circ T_{\beta_0})$-invariant measure given by Theorem 5.2.3 without any further conditions.

Example 5.2.20. Consider again the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. The composition $T_{\beta_1} \circ T_{\beta_0}$ is depicted in Figure 5.6. Since $\frac{1}{\beta_0} = \beta_1 - 1$, keeping the notation of Proposition 5.2.19, we have $K = 3$, $c_1 = \frac{1}{\beta_0}$, $c_2 = \frac{2}{\beta_0}$ and $c_3 = 1$. We have $T(c_1) = T(c_2) = T(c_3) = c_1$. Therefore, all elements in S equal 0, $d_0 = d_1 = d_2 = d_3 = 1$ and $C = 1 + \frac{3}{\beta_0(\beta_1\beta_0 - 1)} = 1 + \frac{3}{\beta_0^2}$. The density function of the unique absolutely continuous $(T_{\beta_1} \circ T_{\beta_0})$-invariant probability measure $\mu_{\boldsymbol{\beta},0}$ is

$$\frac{1}{C}\Big(1+\frac{3}{\beta_0}\chi_{[0,\frac{1}{\beta_0}]}\Big).$$

For example, $\mu_{\boldsymbol{\beta},0}\big([0,\frac{1}{\beta_0})\big) = \frac{13+\sqrt{13}}{26}$. Moreover, it can be checked that $\mu_{\boldsymbol{\beta},0}\big((T_{\beta_1}\circ T_{\beta_0})^{-1}[0,\frac{1}{\beta_0})\big) = \mu_{\boldsymbol{\beta},0}\big([0,\frac{1}{\beta_0})\big)$.

We obtain a formula for the density function $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ by using the density functions $\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}$ for $i\in[\![0,p-1]\!]$ given in Proposition 5.2.19. We first need a lemma.

Lemma 5.2.21. *For all $i\in[\![0,p-1]\!]$, all sets $B\in\mathcal{B}([0,1))$ and all $\mathcal{B}([0,1))$-measurable functions $f\colon[0,1)\to[0,+\infty)$, the map $f\circ\pi_2\colon[\![0,p-1]\!]\times[0,1)\to[0,+\infty)$ is $\mathcal{T}_p$-measurable and*

$$\int_{\{i\}\times B} f\circ\pi_2\ d\lambda_p = \frac{1}{p}\int_B f d\lambda.$$

Proof. First, consider a $\mathcal{B}([0,1))$-measurable function $f\colon[0,1)\to[0,+\infty)$. The map $f\circ\pi_2$ is measurable. In fact, it is sufficient to check Definition 1.3.16 for intervals of the form $[0,y)$ with $y>0$ and we have

$$\begin{aligned}(f\circ\pi_2)^{-1}\big([0,y)\big) &= \{(i,x)\in[\![0,p-1]\!]\times[0,1) : f\circ\pi_2(i,x)<y\}\\ &= \{(i,x)\in[\![0,p-1]\!]\times[0,1) : f(x)<y\}\\ &= \bigcup_{i=0}^{p-1}\big(\{i\}\times\{x\in[0,1) : f(x)<y\}\big)\\ &= \bigcup_{i=0}^{p-1}\big(\{i\}\times f^{-1}\big([0,y)\big)\big)\end{aligned}$$

where the set $f^{-1}\big([0,y)\big)$ belongs to the σ-algebra $\mathcal{B}([0,1))$ since f is measurable. Hence, we get $(f\circ\pi_2)^{-1}\big([0,y)\big)\in\mathcal{T}_p$.

Second, consider $i\in[\![0,p-1]\!]$ and $B\in\mathcal{B}([0,1))$. The integral equality follows from standard arguments by using the definition of the integral via simple functions (see Definition 1.3.17). In fact, we have

$$\begin{aligned}&\int_{\{i\}\times B} f\circ\pi_2\ d\lambda_p\\ =&\int (f\circ\pi_2)\ \chi_{\{i\}\times B}\ d\lambda_p\end{aligned}$$

$$= \sup\left\{\int g\ d\lambda_p : g \in S^+(\llbracket 0,p-1\rrbracket \times [0,1), \mathcal{T}_p), g \leq (f\circ\pi_2)\ \chi_{\{i\}\times B}\right\} \tag{5.13}$$

and

$$\begin{aligned}\frac{1}{p}\int_B f d\lambda &= \int \big(\frac{1}{p}f\big)\ \chi_B d\lambda \\ &= \sup\left\{\int h\ d\lambda : h \in S^+([0,1), \mathcal{B}([0,1))), h \leq \big(\frac{1}{p}f\big)\ \chi_B\right\}. \end{aligned} \tag{5.14}$$

We then prove the desired equality by double inequality between the values of the suprema (5.13) and (5.14). We show that the value (5.13) is smaller than or equal to (5.14), the other inequality can be done in a similar fashion. Consider a simple function $g \in S^+(\llbracket 0,p-1\rrbracket \times [0,1), \mathcal{T}_p)$ such that $g \leq (f\circ\pi_2)\ \chi_{\{i\}\times B}$. There exist $n \in \mathbb{N}_{\geq 1}$, $B_1,\ldots,B_n \in \mathcal{B}([0,1))$, $a_1,\ldots,a_n > 0$ such that $\cup_{k=1}^{n-1} B_k = B$, $B_n = [0,1)\setminus B$, $B_k \cap B_{k'} = \emptyset$ if $k,k' \in \llbracket 1,n\rrbracket$ and $k \neq k'$, $a_n = 0$ and for all $(j,x) \in \llbracket 0,p-1\rrbracket \times [0,1)$,

$$g(j,x) = \begin{cases} a_k & \text{if } j = i \text{ and } x \in B_k,\ k \in \llbracket 1,n-1\rrbracket \\ 0 & \text{otherwise.}\end{cases}$$

We set $h \in S^+([0,1), \mathcal{B}([0,1)))$ defined by

$$h(x) = \begin{cases} \frac{1}{p}a_k & \text{if } x \in B_k,\ k \in \llbracket 1,n-1\rrbracket \\ 0 & \text{otherwise.}\end{cases}$$

We obtain $h \leq (\frac{1}{p}f)\ \chi_B$ and, by the definition of the p-Lebesgue measure and the definition of integral of simple functions, we get

$$\begin{aligned}\int g\ d\lambda_p &= \sum_{k=1}^{n-1} a_k \lambda_p\big(\{i\}\times B_k\big) \\ &= \sum_{k=1}^{n-1} a_k \frac{1}{p}\lambda(B_k) \\ &= \int h\ d\lambda.\end{aligned}$$

■

Proposition 5.2.22. *The density function $\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}$ of $\mu_{\boldsymbol{\beta}}$ with respect to the p-Lebesgue measure λ_p on $\mathcal{T}_p$ is*

$$\sum_{i=0}^{p-1}\left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda}\circ\pi_2\right)\cdot\chi_{\{i\}\times[0,1)}. \tag{5.15}$$

Proof. Let $A \in \mathcal{T}_p$ and let $B_0, \ldots, B_{p-1} \in \mathcal{B}([0,1))$ such that $A = \bigcup_{i=0}^{p-1}(\{i\} \times B_i)$. It follows from Lemma 5.2.21 that

$$\begin{aligned}\int_A \sum_{i=0}^{p-1} \left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda} \circ \pi_2\right) \cdot \chi_{\{i\}\times[0,1)} \, d\lambda_p &= \sum_{i=0}^{p-1} \int_{\{i\}\times B_i} \frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda} \circ \pi_2 \, d\lambda_p \\ &= \frac{1}{p}\sum_{i=0}^{p-1} \int_{B_i} \frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda} \, d\lambda \\ &= \frac{1}{p}\sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}(B_i) \\ &= \mu_{\boldsymbol{\beta}}(A).\end{aligned}$$

■

Note that the formula (5.15) also holds for the extended measures $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and λ_p^{ext} on $\mathcal{T}_{\boldsymbol{\beta}}^{\mathrm{ext}}$.

5.2.4 Frequencies

We now turn to the frequencies of the digits in the greedy $\boldsymbol{\beta}$-expansions of real numbers in the interval $[0,1)$. Recall that the frequency of a digit d occurring in the greedy $\boldsymbol{\beta}$-expansion $a_0a_1a_2\cdots$ of a real number x in $[0,1)$ is equal to

$$\lim_{n\to+\infty} \frac{1}{n}\#\{0 \le k < n : a_k = d\},$$

provided that this limit exists.

Proposition 5.2.23. *For λ-almost all $x \in [0,1)$, the frequency of any digit d occurring in the greedy $\boldsymbol{\beta}$-expansion of x exists and is equal to*

$$\frac{1}{p}\sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}\Big(\big[\tfrac{d}{\beta_i}, \tfrac{d+1}{\beta_i}\big) \cap [0,1)\Big).$$

Proof. Let $x \in [0,1)$ and let d be a digit occurring in $d_{\boldsymbol{\beta}}(x) = a_0a_1a_2\cdots$. Then for all $k \in \mathbb{N}$, $a_k = d$ if and only if $\pi_2(T_{\boldsymbol{\beta}}^k(0,x)) \in [\frac{d}{\beta_k}, \frac{d+1}{\beta_k}) \cap [0,1)$. Moreover, since for all $k \in \mathbb{N}$, $T_{\boldsymbol{\beta}}^k(0,x) \in \{k \bmod p\} \times [0,1)$, we have

$$\chi_{[\frac{d}{\beta_k}, \frac{d+1}{\beta_k})\cap[0,1)}\big(\pi_2\big(T_{\boldsymbol{\beta}}^k(0,x)\big)\big) = \chi_{\{k \bmod p\}\times\big([\frac{d}{\beta_k}, \frac{d+1}{\beta_k})\cap[0,1)\big)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big)$$

$$= \sum_{i=0}^{p-1} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big).$$

Therefore, if it exists, the frequency of d in $d_{\boldsymbol{\beta}}(x)$ is equal to

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i=0}^{p-1} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(0,x)\big).$$

Yet, for each $i \in [\![0,p-1]\!]$ and for $\mu_{\boldsymbol{\beta}}$-almost all $y \in [\![0,p-1]\!] \times [0,1)$, we have

$$\begin{aligned}
&\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\big(T_{\boldsymbol{\beta}}^k(y)\big)\\
&= \int_{[\![0,p-1]\!]\times[0,1)} \chi_{\{i\}\times\left([\frac{d}{\beta_i},\frac{d+1}{\beta_i})\cap[0,1)\right)}\, d\mu_{\boldsymbol{\beta}}\\
&= \mu_{\boldsymbol{\beta}}\Big(\{i\}\times\big([\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i})\cap[0,1)\big)\Big)\\
&= \frac{1}{p}\mu_{\boldsymbol{\beta},i}\Big(\big[\tfrac{d}{\beta_i},\tfrac{d+1}{\beta_i}\big)\cap[0,1)\Big)
\end{aligned}$$

where we used Theorem 5.2.15 and the Ergodic theorem for the first equality. The conclusion now follows from Proposition 5.2.11. ■

Note that, when $p = 1$, Proposition 5.2.23 gives back the classical formula

$$\mu_\beta\big([\tfrac{d}{\beta},\tfrac{d+1}{\beta})\cap[0,1)\big)$$

for the frequency of the digit d, where μ_β is the measure given in Theorem 1.4.46.

5.3 Dynamical properties of $L_{\boldsymbol{\beta}}$

We now turn to the dynamical study of the lazy $\boldsymbol{\beta}$-transformation. To do so, we first prove that the greedy and lazy dynamical systems are isomorphic and then we deduce the dynamical properties of the lazy dynamical system thanks to the ones of the greedy dynamical system studied in the previous section.

5.3.1 Isomorphism between greedy and lazy $\boldsymbol{\beta}$-transformations

We first define a map which will then be proved to be an isomorphism between the greedy $\boldsymbol{\beta}$-transformation and the lazy $\boldsymbol{\beta}$-transformation.

Definition 5.3.1. Consider the map

$$\begin{aligned} \phi_{\boldsymbol{\beta}} \colon \bigcup_{i=0}^{p-1} \big(\{i\} \times [0,1)\big) &\to \bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big), \\ (i,x) &\mapsto \big(i, x_{\boldsymbol{\beta}^{(i)}} - x\big) \end{aligned} \tag{5.16}$$

and the σ-algebra

$$\mathcal{L}_{\boldsymbol{\beta}} = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}\big((x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big) \right\}$$

on $\bigcup_{i=0}^{p-1} \big(\{i\} \times \big(x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}\big]\big)$.

Remark that we let $\mathcal{L}_{\boldsymbol{\beta}}$ denote the lazy σ-algebra since there is a dependence on the alternate base $\boldsymbol{\beta}$ and not only on its length p as in the greedy case.

Theorem 5.3.2. *The map $\phi_{\boldsymbol{\beta}}$ is an isomorphism between the dynamical systems*

$$\big([\![0, p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}\big)$$

and

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}, L_{\boldsymbol{\beta}}\Big).$$

Proof. Clearly, $\phi_{\boldsymbol{\beta}}$ is a bimeasurable bijective map. Hence, we only have to show that $\phi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}} = L_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}$. Let $(i,x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times [0,1)\big)$. Then

$$\phi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}}(i,x) = \big((i+1) \bmod p, x_{\boldsymbol{\beta}^{(i+1)}} - \beta_i x + \lfloor \beta_i x \rfloor\big)$$

and

$$L_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}(i,x) = \Big((i+1) \bmod p, \beta_i(x_{\boldsymbol{\beta}^{(i)}} - x) - \Big\lceil \beta_i(x_{\boldsymbol{\beta}^{(i)}} - x) - x_{\boldsymbol{\beta}^{(i+1)}} \Big\rceil\Big).$$

We easily get that $\phi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}}(i,x) = L_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}(i,x)$ by using (2.10) linking the values $x_{\boldsymbol{\beta}^{(i)}}$ and $x_{\boldsymbol{\beta}^{(i+1)}}$. ■

Remark 5.3.3. We deduce from Theorem 5.3.2 that if the greedy $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1)$ is $a = a_0 a_1 a_2 \cdots$, then the lazy $\boldsymbol{\beta}$-expansion of $x_{\boldsymbol{\beta}} - x$ is

$$\theta_{\boldsymbol{\beta}}(a) = (\lceil \beta_0 \rceil - 1 - a_0)(\lceil \beta_1 \rceil - 1 - a_1)(\lceil \beta_2 \rceil - 1 - a_2) \cdots$$

as already shown in the wider context of Cantor bases in Proposition 2.4.12.

Moreover, if we extend the lazy σ-algebra and the map $\phi_{\boldsymbol{\beta}}$ as follows, we can similarly prove that the extended greedy and lazy dynamical systems are also isomorphic.

Definition 5.3.4. Consider the extended σ-algebra

$$\mathcal{L}_{\boldsymbol{\beta}}^{\text{ext}} = \left\{ \bigcup_{i=0}^{p-1} (\{i\} \times B_i) : \forall i \in [\![0, p-1]\!],\ B_i \in \mathcal{B}\big((0, x_{\boldsymbol{\beta}^{(i)}}]\big) \right\}.$$

We also set

$$\begin{aligned}\phi_{\boldsymbol{\beta}}^{\text{ext}} \colon \bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big) &\to \bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big), \\ (i, x) &\mapsto \big(i, x_{\boldsymbol{\beta}^{(i)}} - x\big).\end{aligned}$$

Theorem 5.3.5. *The map $\phi_{\boldsymbol{\beta}}^{\text{ext}}$ is an isomorphism between the dynamical systems*

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times [0, x_{\boldsymbol{\beta}^{(i)}})\big), \mathcal{T}_{\boldsymbol{\beta}}^{\text{ext}}, \mu_{\boldsymbol{\beta}}^{\text{ext}}, T_{\boldsymbol{\beta}}^{\text{ext}}\Big)$$

and

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}^{\text{ext}}, \mu_{\boldsymbol{\beta}}^{\text{ext}} \circ (\phi_{\boldsymbol{\beta}}^{\text{ext}})^{-1}, L_{\boldsymbol{\beta}}^{\text{ext}}\Big).$$

5.3.2 Unique absolutely continuous $L_{\boldsymbol{\beta}}$-invariant measure

Thanks to Theorems 5.3.2 and 5.3.5, we obtain two analogues of Theorems 5.2.15 and 5.2.18 for the lazy $\boldsymbol{\beta}$-transformation. Recall that $\beta = \prod_{i=0}^{p-1} \beta_i$.

Theorem 5.3.6. *The measure $\mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ is the unique $L_{\boldsymbol{\beta}}$-invariant probability measure on $\mathcal{L}_{\boldsymbol{\beta}}$ that is absolutely continuous with respect to $\lambda_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$. Furthermore, $\mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ is equivalent to $\lambda_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ on $\mathcal{L}_{\boldsymbol{\beta}}$ and the dynamical system*

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}, L_{\boldsymbol{\beta}}\Big)$$

is ergodic and has entropy $\frac{1}{p}\log(\beta)$.

Theorem 5.3.7. *The measure $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}$ is the unique $L_{\boldsymbol{\beta}}^{\mathrm{ext}}$-invariant probability measure on $\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ that is absolutely continuous with respect to $\lambda_p \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}$. Furthermore, $\mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}$ is equivalent to $\lambda_p \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}$ on $\mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{ext}}$ and the dynamical system*

$$\Big(\bigcup_{i=0}^{p-1} \big(\{i\} \times (0, x_{\boldsymbol{\beta}^{(i)}}]\big), \mathcal{L}_{\boldsymbol{\beta}}^{\mathrm{ext}}, \mu_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ (\phi_{\boldsymbol{\beta}}^{\mathrm{ext}})^{-1}, L_{\boldsymbol{\beta}}^{\mathrm{ext}}\Big)$$

is ergodic and has entropy $\frac{1}{p}\log(\beta)$.

5.3.3 Density functions and frequencies

Thanks to Theorem 5.3.6, we obtain formulae for the density function of the measure associated with the lazy dynamical system and the frequencies of digits in lazy $\boldsymbol{\beta}$-expansions.

Proposition 5.3.8. *The density function $\frac{d(\mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1})}{d(\lambda_p \circ \phi_{\boldsymbol{\beta}}^{-1})}$ of $\mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ with respect to the measure $\lambda_p \circ \phi_{\boldsymbol{\beta}}^{-1}$ is*

$$\frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p} \circ \phi_{\boldsymbol{\beta}}^{-1} = \left(\sum_{i=0}^{p-1} \left(\frac{d\mu_{\boldsymbol{\beta},i}}{d\lambda} \circ \pi_2 \right) \cdot \chi_{\{i\}\times[0,1)} \right) \circ \phi_{\boldsymbol{\beta}}^{-1}.$$

Proof. Consider $A \in \mathcal{L}_{\boldsymbol{\beta}}$. We have

$$\int_A \frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p} \circ \phi_{\boldsymbol{\beta}}^{-1}\, d(\lambda_p \circ \phi_{\boldsymbol{\beta}}^{-1}) = \int_{\phi_{\boldsymbol{\beta}}^{-1}(A)} \frac{d\mu_{\boldsymbol{\beta}}}{d\lambda_p}\, d\lambda_p = \mu_{\boldsymbol{\beta}}(\phi_{\boldsymbol{\beta}}^{-1}(A)).$$

The conclusion follows from Proposition 5.2.22 ∎

Proposition 5.3.9. *For λ-almost all $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$, the frequency of any digit d occurring in the lazy $\boldsymbol{\beta}$-expansion of x exists and is equal to*

$$\frac{1}{p}\sum_{i=0}^{p-1} \mu_{\boldsymbol{\beta},i}\Big(\big[\tfrac{\lceil\beta_i\rceil - 1 - d}{\beta_i}, \tfrac{\lceil\beta_i\rceil - d}{\beta_i}\big) \cap [0,1)\Big).$$

Proof. Let $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and let $\ell_{\boldsymbol{\beta}}(x) = a_0a_1a_2\cdots$ and $d_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - x) = b_0b_1b_2\cdots$. Consider a digit d occurring in $\ell_{\boldsymbol{\beta}}(x)$. By Remark 5.3.3, for all $k \in \mathbb{N}$, $a_k = d$ if and only if $b_k = \lceil\beta_k\rceil - 1 - d$. The conclusion follows from Proposition 5.2.23. ∎

5.4 Isomorphism with the $\boldsymbol{\beta}$-shifts

The aim of this section is to generalize the isomorphism from Theorem 1.4.50 between the greedy β-transformation T_β and the β-shift S_β (using the notion of cylinders from Definition 1.4.49) to the framework of alternate bases. Moreover, the analogue lazy result is given.

Definition 5.4.1. Consider the σ-algebra

$$\mathcal{G}_{\boldsymbol{\beta}} = \bigcup_{i=0}^{p-1} \big(\{i\} \times (\mathcal{C}_{A_{\boldsymbol{\beta}}} \cap S_{\boldsymbol{\beta}^{(i)}})\big)$$

on $\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$. We define

$$\sigma_p\colon \bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}),\ (i,w) \mapsto ((i+1) \bmod p, \sigma(w))$$

$$\psi_{\boldsymbol{\beta}}\colon [\![0,p-1]\!] \times [0,1) \to \bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}),\ (i,x) \mapsto (i, d_{\boldsymbol{\beta}^{(i)}}(x)).$$

Note that the transformation σ_p is well defined by Proposition 2.3.39.

Theorem 5.4.2. *The map $\psi_{\boldsymbol{\beta}}$ defines an isomorphism between the dynamical systems*

$$\big([\![0,p-1]\!] \times [0,1), \mathcal{T}_p, \mu_{\boldsymbol{\beta}}, T_{\boldsymbol{\beta}}\big)$$

and

$$\left(\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p\right).$$

Proof. It is easily seen that

$$\psi_{\boldsymbol{\beta}} \circ T_{\boldsymbol{\beta}} = \sigma_p \circ \psi_{\boldsymbol{\beta}} \tag{5.17}$$

and that $\psi_{\boldsymbol{\beta}}$ is injective. Moreover, we have

$$\psi_{\boldsymbol{\beta}}\big([\![0,p-1]\!] \times [0,1)\big) = \cup_{i=0}^{p-1}(\{i\} \times D_{\boldsymbol{\beta}^{(i)}})$$

and

$$\mu_{\boldsymbol{\beta}}(\psi_{\boldsymbol{\beta}}^{-1}(\cup_{i=0}^{p-1}(\{i\} \times D_{\boldsymbol{\beta}^{(i)}})) = 1.$$

■

Note that (5.17) is a 2-component version of Proposition 2.3.10.

However, although $\psi_{\boldsymbol{\beta}}$ is continuous, it does not define a topological isomorphism since it is not surjective.

Remark 5.4.3. In view of Theorems 5.4.2 and 1.4.50, the set $\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}})$ can be seen as the "greedy $\boldsymbol{\beta}$-shift", that is, the generalization of the greedy β-shift S_β to alternate bases. However, in Chapter 2, what we called the greedy $\boldsymbol{\beta}$-shift is the union $\Sigma_{\boldsymbol{\beta}} = \bigcup_{i=0}^{p-1} S_{\boldsymbol{\beta}^{(i)}}$. This definition was motivated by Theorem 3.4.6. In summary, we can say that there are two ways to extend the notion of β-shift to alternate bases $\boldsymbol{\beta}$, depending on the way we look at it: either as a combinatorial object or as a dynamical object.

Thanks to Theorem 5.4.2, we obtain an analogue of Theorem 5.2.15 for the transformation σ_p.

Theorem 5.4.4. *The measure $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is the unique σ_p-invariant probability measure on $\mathcal{G}_{\boldsymbol{\beta}}$ that is absolutely continuous with respect to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$. Furthermore, $\mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}$ is equivalent to $\lambda_p \circ \psi_{\boldsymbol{\beta}}^{-1}$ on $\mathcal{G}_{\boldsymbol{\beta}}$ and the dynamical system*

$$\Big(\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p\Big)$$

is ergodic and has entropy $\frac{1}{p}\log(\beta)$.

In order to have an analogue of Theorem 5.4.4 in the lazy framework, we now define an isomorphism between the dynamical system

$$\Big(\bigcup_{i=0}^{p-1}(\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p\Big)$$

and its analogue lazy one.

Definition 5.4.5. We consider the σ-algebra

$$\mathcal{G}'_{\boldsymbol{\beta}} = \bigcup_{i=0}^{p-1} \big(\{i\} \times (\mathcal{C}_{A_{\boldsymbol{\beta}}} \cap S'_{\boldsymbol{\beta}^{(i)}})\big)$$

on $\bigcup_{i=0}^{p-1}(\{i\} \times S'_{\boldsymbol{\beta}^{(i)}})$ and we define the maps

$$\sigma'_p \colon \bigcup_{i=0}^{p-1}(\{i\} \times S'_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1}(\{i\} \times S'_{\boldsymbol{\beta}^{(i)}}),\ (i,w) \mapsto ((i+1) \bmod p, \sigma(w))$$

$$\Theta_{\boldsymbol{\beta}} \colon \bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}) \to \bigcup_{i=0}^{p-1} (\{i\} \times S'_{\boldsymbol{\beta}^{(i)}}),\ (i,a) \mapsto (i, \theta_{\boldsymbol{\beta}^{(i)}}(a)).$$

Theorem 5.4.6. *The map $\Theta_{\boldsymbol{\beta}}$ defines an isomorphism between the dynamical systems*

$$\left(\bigcup_{i=0}^{p-1} (\{i\} \times S_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1}, \sigma_p\right)$$

and

$$(\bigcup_{i=0}^{p-1} (\{i\} \times S'_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}'_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1} \circ \Theta_{\boldsymbol{\beta}}^{-1}, \sigma'_p)$$

Proof. This immediately follows from Proposition 2.4.39. ∎

The following result is a consequence of Theorems 5.3.2, 5.4.2 and 5.4.6.

Corollary 5.4.7. *The map $\Theta_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}$ is an isomorphism between the dynamical systems*

$$(\bigcup_{i=0}^{p-1} (\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]), \mathcal{L}_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}, L_{\boldsymbol{\beta}})$$

and

$$(\bigcup_{i=0}^{p-1} (\{i\} \times S'_{\boldsymbol{\beta}^{(i)}}), \mathcal{G}'_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}}^{-1} \circ \Theta_{\boldsymbol{\beta}}^{-1}, \sigma'_p).$$

It is easy to check that, as expected, that for all $(i,x) \in \bigcup_{i=0}^{p-1} \big(\{i\} \times (x_{\boldsymbol{\beta}^{(i)}} - 1, x_{\boldsymbol{\beta}^{(i)}}]$, we have

$$\Theta_{\boldsymbol{\beta}} \circ \psi_{\boldsymbol{\beta}} \circ \phi_{\boldsymbol{\beta}}^{-1}(i,x) = (i, \ell_{\boldsymbol{\beta}^{(i)}}(x)).$$

5.5 $\boldsymbol{\beta}$-Expansions and some $(\beta_0 \dots \beta_{p-1})$-expansions

We can see the greedy and lazy $\boldsymbol{\beta}$-expansions of real numbers as β-representations, with $\beta = \prod_{i=0}^{p-1} \beta_i$, over the digit set $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$ from Chapter 4 (see Definition 4.2.2), where

$$\boldsymbol{D} = \left(\overline{[\![0, \lceil \beta_0 \rceil - 1]\!], \dots, [\![0, \lceil \beta_{p-1} \rceil - 1]\!]}\right).$$

In fact, by rewriting Equality (3.1) from Chapter 3, we get

$$\begin{aligned} x = {} & \frac{a_0\beta_1\cdots\beta_{p-1} + a_1\beta_2\cdots\beta_{p-1} + \cdots + a_{p-1}}{\beta} \\ & + \frac{a_p\beta_1\cdots\beta_{p-1} + a_{p+1}\beta_2\cdots\beta_{p-1} + \cdots + a_{2p-1}}{\beta^2} \\ & + \cdots . \end{aligned} \tag{5.18}$$

For the sake of simplicity, from now on, we fix $\boldsymbol{D}$ as the alternate alphabet above and we omit the dependence on $\boldsymbol{D}$ in the writing $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$, that is, we write

$$\mathrm{Dig}(\boldsymbol{\beta}) = \left\{ \sum_{i=0}^{p-1} c_i\beta_{i+1}\cdots\beta_{p-1} : \forall i \in [\![0, p-1]\!],\ c_i \in [\![0, \lceil\beta_i\rceil - 1]\!] \right\}.$$

5.5.1 Digit set built thanks to all the p-tuples

In this section, we examine some cases where by considering the greedy (resp., lazy) $\boldsymbol{\beta}$-expansion of a real number $x \in [0, 1)$ (resp., $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$) and rewriting it as (5.18), the obtained representation is the greedy (resp., lazy) $(\beta, \mathrm{Dig}(\boldsymbol{\beta}))$-expansion of x (see Section 1.4.5).

Definition 5.5.1. We define the map

$$f_{\boldsymbol{\beta}} \colon \prod_{i=0}^{p-1} [\![0, \lceil\beta_i\rceil - 1]\!] \to \mathbb{R},\ (c_0, \ldots, c_{p-1}) \mapsto \sum_{i=0}^{p-1} c_i\beta_{i+1}\cdots\beta_{p-1}.$$

The map $f_{\boldsymbol{\beta}}$ is in fact the map $f_{\boldsymbol{\beta},\boldsymbol{D}}$ from Remark 4.2.11 where $\boldsymbol{D} = \big([\![0, \lceil\beta_0\rceil - 1]\!], \ldots, [\![0, \lceil\beta_{p-1}\rceil - 1]\!]\big)$.

Note that $f_{\boldsymbol{\beta}}$ is not injective in general. The digit set $\mathrm{Dig}(\boldsymbol{\beta})$ has cardinality at most $\prod_{i=0}^{p-1} \lceil\beta_i\rceil$ and can be rewritten $\mathrm{Dig}(\boldsymbol{\beta}) = \mathrm{im}(f_{\boldsymbol{\beta}})$. Let us write

$$\mathrm{Dig}(\boldsymbol{\beta}) = \{d_0, d_1 \ldots, d_m\}$$

with $d_0 < d_1 < \cdots < d_m$. We have

$$\begin{aligned} d_0 &= f_{\boldsymbol{\beta}}(0, \ldots, 0) = 0, \\ d_1 &= f_{\boldsymbol{\beta}}(0, \ldots, 0, 1) = 1 \end{aligned}$$

and

$$d_m = f_{\boldsymbol{\beta}}(\lceil\beta_0\rceil - 1, \ldots, \lceil\beta_{p-1}\rceil - 1).$$

In what follows, as in Section 5.1.1, we suppose that $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ is equipped with the lexicographic order.

Recall that all along this section, we let β denote the product $\prod_{i=0}^{p-1} \beta_i$.

Lemma 5.5.2. *The set* $\mathrm{Dig}(\boldsymbol{\beta})$ *is an allowable digit set for* β*.*

Proof. We need to check Condition (1.4), which means in this case

$$\max_{k \in [\![0, m-1]\!]} (d_{k+1} - d_k) \leq \frac{d_m - d_0}{\beta - 1}.$$

We have $d_0 = 0$ and

$$d_m = f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1, \dots, \lceil \beta_{p-1} \rceil - 1) \geq \sum_{i=0}^{p-1} (\beta_i - 1)\beta_{i+1} \cdots \beta_{p-1} = \beta - 1,$$

Therefore, it suffices to show that for all $k \in [\![0, m-1]\!]$, $d_{k+1} - d_k \leq 1$. Thus, we only have to show that $f(c'_0, \dots, c'_{p-1}) - f(c_0, \dots, c_{p-1}) \leq 1$ where $(c_0, \dots, c_{p-1})$ and $(c'_0, \dots, c'_{p-1})$ are lexicographically consecutive elements of $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$. For such p-tuples, there exists $j \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \dots, c_{j-1} = c'_{j-1}$, $c_j = c'_j - 1$, $c_{j+1} = \lceil \beta_{j+1} \rceil - 1, \dots, c_{p-1} = \lceil \beta_{p-1} \rceil - 1$ and $c'_{j+1} = \cdots = c'_{p-1} = 0$. Then

$$\begin{aligned}
&f(c'_0, \dots, c'_{p-1}) - f(c_0, \dots, c_{p-1}) \\
&= \beta_{j+1} \cdots \beta_{p-1} - \sum_{i=j+1}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \\
&\leq \beta_{j+2} \cdots \beta_{p-1} - \sum_{i=j+2}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \\
&\ \ \vdots \\
&\leq \beta_{p-1} - (\lceil \beta_{p-1} \rceil - 1) \\
&\leq 1.
\end{aligned}$$

■

Since $x_{\boldsymbol{\beta}} = \frac{d_m}{\beta - 1}$, it follows from Lemma 5.5.2 that every point in $[0, x_{\boldsymbol{\beta}})$ admits a greedy $(\beta, \mathrm{Dig}(\boldsymbol{\beta}))$-expansion.

Proposition 5.5.3. *For all* $x \in [0, x_{\boldsymbol{\beta}})$*, we have*

$$T_{\beta, \mathrm{Dig}(\boldsymbol{\beta})}(x) \leq \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x) \tag{5.19}$$

and

$$L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) \geq \pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x). \tag{5.20}$$

Proof. Let $x \in [0, x_{\boldsymbol{\beta}})$. On the one hand, we have

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) = \beta x - d$$

where d is the greatest digit in $\mathrm{Digits}(\boldsymbol{\beta})$ such that $\frac{d}{\beta} \leq x$. On the other hand, by rephrasing Proposition 5.1.8 in terms of the map $f_{\boldsymbol{\beta}}$ when n equals p, we get

$$\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x) = \beta x - f_{\boldsymbol{\beta}}(c)$$

where c is the lexicographically greatest p-tuple in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ such that $\frac{f_{\boldsymbol{\beta}}(c)}{\beta} \leq x$. By definition of d, we get $d \geq f_{\boldsymbol{\beta}}(c)$. Therefore, we obtain (5.19). The inequality (5.20) then follows from Theorem 5.3.2. ■

In what follows, we provide some conditions under which the inequalities of Proposition 5.5.3 happen to be equalities.

Proposition 5.5.4. *The transformations*

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} \quad \textit{and} \quad \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$$

coincide if and only if the transformations

$$L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} \quad \textit{and} \quad \pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$$

do.

Proof. We only show the forward direction, the backward direction being similar. Suppose that $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ and let $x \in (0, x_{\boldsymbol{\beta}}]$. Since $x_{\boldsymbol{\beta}} = \frac{d_m}{\beta - 1}$ and $\mathrm{Dig}(\boldsymbol{\beta}) = \widetilde{\mathrm{Dig}(\boldsymbol{\beta})}$, we successively obtain that

$$\begin{aligned}
L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) &= L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} \circ \phi_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x_{\boldsymbol{\beta}} - x) \\
&= \phi_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} \circ T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x_{\boldsymbol{\beta}} - x) \\
&= \phi_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} \circ \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x_{\boldsymbol{\beta}} - x) \\
&= \pi_2 \circ \phi_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x_{\boldsymbol{\beta}} - x) \\
&= \pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \phi_{\boldsymbol{\beta}}^{\mathrm{ext}} \circ \delta_0(x_{\boldsymbol{\beta}} - x) \\
&= \pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0(x).
\end{aligned}$$

■

The next result provides us with a sufficient condition under which the transformations $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ coincide. Here, the non-decreasingness of the map $f_{\boldsymbol{\beta}}$ refers to the lexicographic order: for all $c, c' \in \prod_{i=0}^{p-1} [\![0, \lceil\beta_i\rceil - 1]\!]$, $c <_{\mathrm{lex}} c' \implies f_{\boldsymbol{\beta}}(c) \le f_{\boldsymbol{\beta}}(c')$.

Theorem 5.5.5. *If the map $f_{\boldsymbol{\beta}}$ is non-decreasing then*

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}.$$

Proof. We keep the same notation as in the proof of Proposition 5.5.3. Let $c' \in \prod_{i=0}^{p-1} [\![0, \lceil\beta_i\rceil - 1]\!]$ such that $d = f_{\boldsymbol{\beta}}(c')$. By definition of c, we get $c \ge_{\mathrm{lex}} c'$. Now, if $f_{\boldsymbol{\beta}}$ is non-decreasing then $f_{\boldsymbol{\beta}}(c) \ge f_{\boldsymbol{\beta}}(c') = d$. Hence the conclusion. ■

The following example shows that considering the length-p alternate base $\boldsymbol{\beta} = (\overline{\beta, \ldots, \beta})$ with $p \in \mathbb{N}_{\ge 3}$, it may happen that $T_{\beta^p,\mathrm{Dig}(\boldsymbol{\beta})}$ differs from $\pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. This result was already proved in [DdVKL12, Proposition 2.1].

Example 5.5.6. Consider the alternate base $\boldsymbol{\beta} = (\overline{\varphi^2, \varphi^2, \varphi^2})$. Then

$$\mathrm{Dig}(\boldsymbol{\beta}) = \{\varphi^4 c_0 + \varphi^2 c_1 + c_2 : c_0, c_1, c_2 \in \{0,1,2\}\}.$$

Dajani et al. [DdVKL12, Proposition 2.1] proved that $T_{\beta^n,\mathrm{Dig}(\boldsymbol{\beta})} = T^n_\beta$ for all $n \in \mathbb{N}$ if and only if $f_{\boldsymbol{\beta}}$ is non-decreasing. Since

$$f_{\boldsymbol{\beta}}(0,2,2) = 2\varphi^2 + 2 > \varphi^4 = f_{\boldsymbol{\beta}}(1,0,0),$$

the transformations $T_{\varphi^6,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ differ.

Whenever $f_{\boldsymbol{\beta}}$ is not non-decreasing, the transformations $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ can either coincide or not. The following two examples illustrate both cases. In particular, Example 5.5.8 shows that the sufficient condition given in Theorem 5.5.5 is not necessary.

Example 5.5.7. Consider the alternate base $\boldsymbol{\beta} = (\overline{\varphi, \varphi, \sqrt{5}})$. Then

$$\mathrm{Dig}(\boldsymbol{\beta}) = \{\sqrt{5}\varphi c_0 + \sqrt{5} c_1 + c_2 : c_0, c_1 \in \{0,1\},\ c_2 \in \{0,1,2\}\}.$$

However, $f_{\boldsymbol{\beta}}(0,1,2) = \sqrt{5} + 2 \simeq 4.23$ and $f_{\boldsymbol{\beta}}(1,0,0) = \sqrt{5}\varphi \simeq 3.61$. It can be easily checked that there exists $x \in [0, x_{\boldsymbol{\beta}})$ such that

$$T_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}(x) \ne \pi_2 \circ (T^{\mathrm{ext}}_{\boldsymbol{\beta}})^3 \circ \delta_0(x).$$

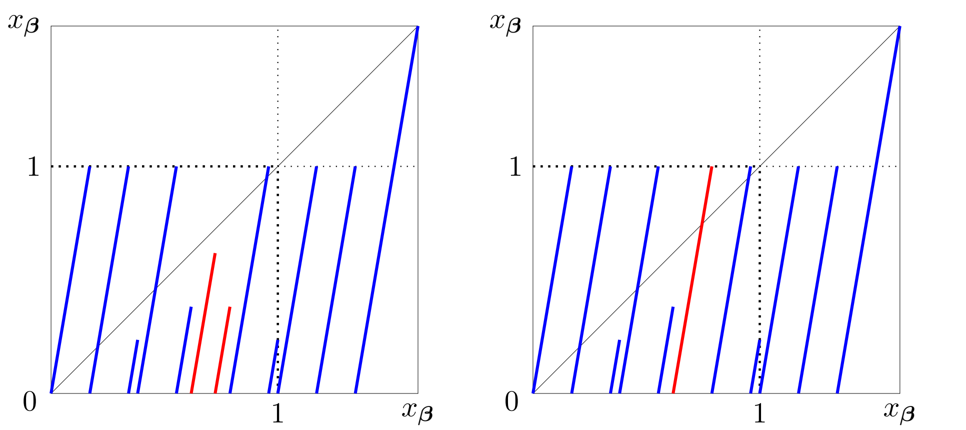

Figure 5.7: The transformations $T_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}$ (left) and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (right) with $\boldsymbol{\beta} = (\overline{\varphi, \varphi, \sqrt{5}})$.

For example, we can compute $T_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}(0.75) \simeq 0.15$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0(0.75) \simeq 0.77$. The transformations $T_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ are depicted in Figure 5.7, where the red lines show the images of the interval $\left[\frac{\sqrt{5}+2}{\sqrt{5}\varphi^2}, \frac{\sqrt{5}\varphi+1}{\sqrt{5}\varphi^2}\right) \simeq [0.72, 0.78)$, that is, where the two transformations differ. Similarly, the transformations $L_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ are depicted in Figure 5.8. As illustrated in red, the two transformations differ on the interval $\phi_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}\left(\left[\frac{\sqrt{5}+2}{\sqrt{5}\varphi^2}, \frac{\sqrt{5}\varphi+1}{\sqrt{5}\varphi^2}\right)\right) \simeq (0.82, 0.89]$.

Example 5.5.8. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{3}{2}, \frac{3}{2}, 4})$. We have $\mathrm{Dig}(\boldsymbol{\beta}) = [\![0, 13]\!]$. The map $f_{\boldsymbol{\beta}}$ is not non-decreasing since we have $f_{\boldsymbol{\beta}}(0,1,3) = 7$ and $f_{\boldsymbol{\beta}}(1,0,0) = 6$. However, $T_{9,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ and $L_{9,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. The transformation $T_{9,\mathrm{Dig}(\boldsymbol{\beta})}$ is depicted in Figure 5.9.

The next example illustrates that it may happen that the transformations $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}$ and $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^p \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ indeed coincide on $[0,1)$ but not on $[0, x_{\boldsymbol{\beta}})$.

Example 5.5.9. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{\sqrt{5}}{2}, \frac{\sqrt{6}}{2}, \frac{\sqrt{7}}{2}})$. Then $f_{\boldsymbol{\beta}}(0,1,1) > f_{\boldsymbol{\beta}}(1,0,0)$ and it can be checked that the maps $T_{\frac{\sqrt{210}}{8},\mathrm{Dig}(\boldsymbol{\beta})}$ and

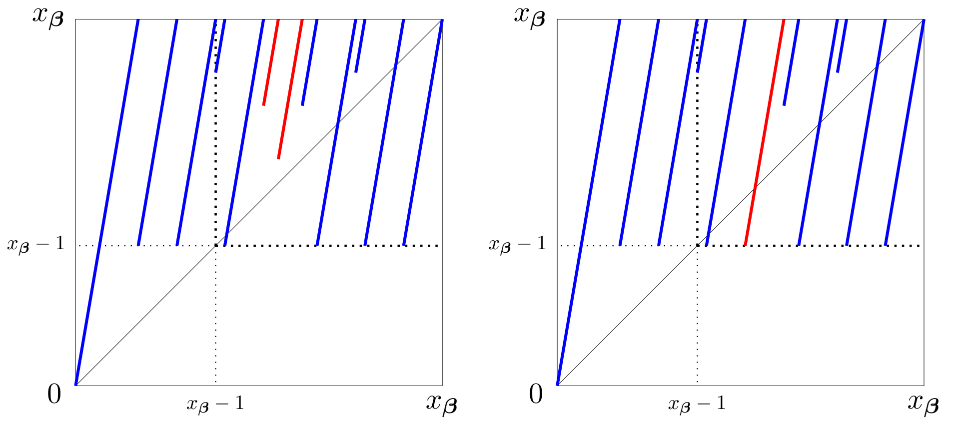

Figure 5.8: The transformations $L_{\sqrt{5}\varphi^2,\mathrm{Dig}(\boldsymbol{\beta})}$ (left) and $\pi_2 \circ (L_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (right) with $\boldsymbol{\beta} = (\overline{\varphi, \varphi, \sqrt{5}})$.

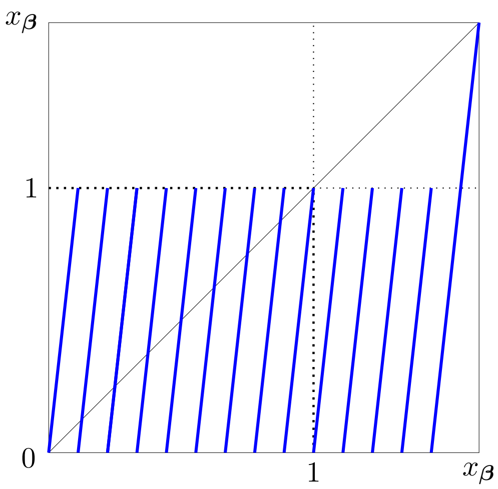

Figure 5.9: The transformations $T_{9,\mathrm{Dig}(\boldsymbol{\beta})}$ where $\boldsymbol{\beta} = (\overline{\tfrac{3}{2}, \tfrac{3}{2}, 4})$.

$\pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^3 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ differ on the interval $\left[\frac{f_{\boldsymbol{\beta}}(0,1,1)}{\beta_2\beta_1\beta_0}, \frac{f_{\boldsymbol{\beta}}(1,0,1)}{\beta_2\beta_1\beta_0}\right) \simeq [1.28, 1.44)$. However, the two maps coincide on $[0,1)$.

Finally, we provide a necessary and sufficient condition for the map $f_{\boldsymbol{\beta}}$ to be non-decreasing.

Proposition 5.5.10. *The map $f_{\boldsymbol{\beta}}$ is non-decreasing if and only if for all $j \in [\![1, p-2]\!]$,*

$$\sum_{i=j}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \leq \beta_j \cdots \beta_{p-1}. \tag{5.21}$$

Proof. If the map $f_{\boldsymbol{\beta}}$ is non-decreasing then for all $j \in [\![1, p-2]\!]$,

$$\begin{aligned}\sum_{i=j}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} &= f_{\boldsymbol{\beta}}(0, \ldots, 0, 0, \lceil \beta_j \rceil - 1, \ldots, \lceil \beta_{p-1} \rceil - 1) \\ &\leq f_{\boldsymbol{\beta}}(0, \ldots, 0, 1, 0, \ldots, 0) \\ &= \beta_j \cdots \beta_{p-1}.\end{aligned}$$

Conversely, suppose that (5.21) holds for all $j \in [\![1, p-2]\!]$ and consider two p-tuples $(c_0, \ldots, c_{p-1})$ and $(c'_0, \ldots, c'_{p-1})$ in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ such that $(c_0, \ldots, c_{p-1}) <_{\mathrm{lex}} (c'_0, \ldots, c'_{p-1})$. Then there exists $j \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \ldots, c_{j-1} = c'_{j-1}$ and $c_j \leq c'_j - 1$. We get

$$\begin{aligned}f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1}) &\leq \sum_{i=0}^{j} c'_i \beta_{i+1} \cdots \beta_{p-1} - \beta_{j+1} \cdots \beta_{p-1} \\ &\qquad + \sum_{i=j+1}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} \\ &\leq \sum_{i=0}^{j} c'_i \beta_{i+1} \cdots \beta_{p-1} \\ &\leq f_{\boldsymbol{\beta}}(c'_0, \ldots, c'_{p-1}).\end{aligned}$$

■

Corollary 5.5.11. *If $p = 2$ then $T_{\beta_0\beta_1, \mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (T_{\boldsymbol{\beta}}^{\mathrm{ext}})^2 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$. In particular, $T_{\beta_0\beta_1, \mathrm{Dig}(\boldsymbol{\beta})}\big|_{[0,1)} = T_{\beta_1} \circ T_{\beta_0}$.*

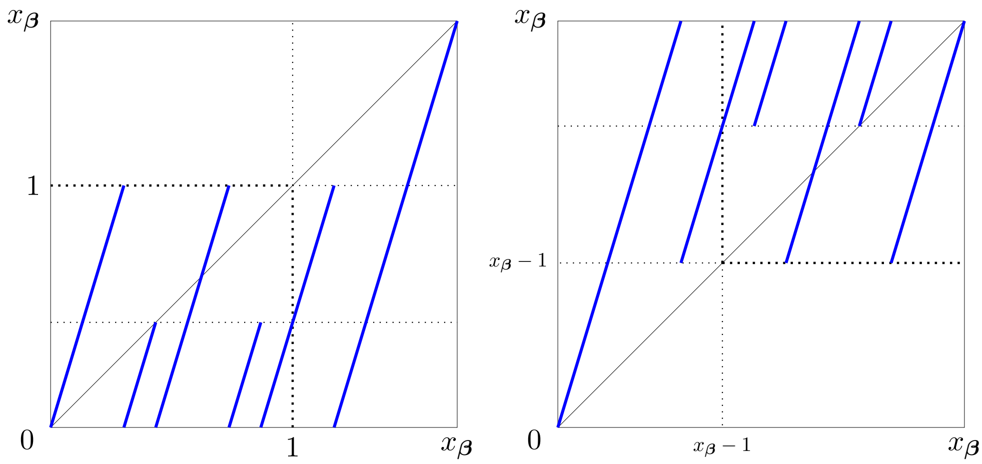

Figure 5.10: The transformations $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^2 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ (left) and $\pi_2 \circ (L_{\boldsymbol{\beta}}^{\text{ext}})^2 \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ (right) for $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$.

Proof. This follows from Theorem 5.5.5 and Proposition 5.5.10. ■

Example 5.5.12. Consider once more the alternate base $\boldsymbol{\beta} = \overline{(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})}$ from Example 5.1.1. Then $\text{Dig}(\boldsymbol{\beta}) = \{0, 1, \beta_1, \beta_1 + 1, 2\beta_1, 2\beta_1 + 1\}$ and $x_{\boldsymbol{\beta}} = \frac{2\beta_1+1}{\beta_1\beta_0-1} = \frac{5+7\sqrt{13}}{18}$. The transformations $\pi_2 \circ (T_{\boldsymbol{\beta}}^{\text{ext}})^2 \circ \delta_0\big|_{[0,x_{\boldsymbol{\beta}})}$ and $\pi_2 \circ (L_{\boldsymbol{\beta}}^{\text{ext}})^2 \circ \delta_0\big|_{(0,x_{\boldsymbol{\beta}}]}$ are depicted in Figure 5.10. By Corollary 5.5.11, they coincides with $T_{\beta_0\beta_1,\text{Dig}(\boldsymbol{\beta})}$ and $L_{\beta_0\beta_1,\text{Dig}(\boldsymbol{\beta})}$ respectively.

5.5.2 Digit sets built thanks to admissible p-tuples

As said in the previous section, by considering the greedy $\boldsymbol{\beta}$-expansion of a real number $x \in [0, 1)$ and rewriting it as (5.18), the obtained representation is a $(\beta, \text{Dig}(\boldsymbol{\beta}))$-representation of x. However, since the greedy $\boldsymbol{\beta}$-expansions of real numbers in $[0, 1)$ are greedy $\boldsymbol{\beta}$-admissible sequences characterized by the combinatorial property given in Theorem 2.3.33, not all p-tuples of letters can be the preimage of a digit. Since the set $D_{\boldsymbol{\beta}}$ is closed for the power-p of the shift map, that is $\sigma^p(D_{\boldsymbol{\beta}}) = D_{\boldsymbol{\beta}}$, a p-tuple $(c_0, c_1, \dots, c_{p-1})$ of letters can appear if $c_0c_1 \cdots c_{p-1} \in \text{Pref}(D_{\boldsymbol{\beta}})$. Hence, the obtained representation is more precisely a $(\beta, \mathrm{Adm}(\boldsymbol{\beta}))$-representation of x where

$$\mathrm{Adm}(\boldsymbol{\beta}) = \left\{ \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} : c_0 c_1 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}}) \right\}.$$

Similarly, by considering the lazy $\boldsymbol{\beta}$-expansion of a real number x in $(x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ and rewriting it as (5.18), the obtained representation is a $(\beta, \mathrm{Adm}'(\boldsymbol{\beta}))$-representation of x where

$$\mathrm{Adm}'(\boldsymbol{\beta}) = \left\{ \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} : c_0 c_1 \cdots c_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}}) \right\}.$$

Clearly, the digit sets $\mathrm{Adm}(\boldsymbol{\beta})$ and $\mathrm{Adm}'(\boldsymbol{\beta})$ are subsets of the digit set $\mathrm{Dig}(\boldsymbol{\beta})$ studied in the previous section. Moreover, we have

$$\mathrm{Adm}(\boldsymbol{\beta}) = f_{\boldsymbol{\beta}}\big(\{(c_0, \ldots, c_{p-1}) : c_0 c_1 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})\}\big)$$

and

$$\mathrm{Adm}'(\boldsymbol{\beta}) = f_{\boldsymbol{\beta}}\big(\{(c_0, \ldots, c_{p-1}) : c_0 c_1 \cdots c_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})\}\big).$$

The goal of this section is to study whether while considering a greedy (resp., lazy) $\boldsymbol{\beta}$-expansion of a real number $x \in [0,1)$ (resp., $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$) and rewriting it as (5.18), the obtained representation is the greedy $(\beta, \mathrm{Adm}(\boldsymbol{\beta}))$-expansion (resp., the lazy $(\beta, \mathrm{Adm}'(\boldsymbol{\beta}))$-expansion) of x.

We start with the study of the digit sets $\mathrm{Adm}(\boldsymbol{\beta})$ and $\mathrm{Adm}'(\boldsymbol{\beta})$.

Lemma 5.5.13. *For any real number a, we have*

$$a \in \mathrm{Adm}(\boldsymbol{\beta}) \iff \sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1) \beta_{i+1} \cdots \beta_{p-1} - a \ \in \mathrm{Adm}'(\boldsymbol{\beta}).$$

Proof. Consider a real number $a \in \mathrm{Adm}(\boldsymbol{\beta})$. There exists a p-tuple $(c_0, \ldots, c_{p-1})$ such that $c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})$ and $a = f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1})$. By Proposition 2.4.12, we get $(\lceil \beta_0 \rceil - 1 - c_0) \cdots (\lceil \beta_{p-1} \rceil - 1 - c_{p-1}) \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})$. Therefore, the real number $f_{\boldsymbol{\beta}}\big((\lceil \beta_0 \rceil - 1 - c_0) \cdots (\lceil \beta_{p-1} \rceil - 1 - c_{p-1})\big)$ belongs to the digit set $\mathrm{Adm}'(\boldsymbol{\beta})$ where

$$\begin{aligned} & f_{\boldsymbol{\beta}}\big((\lceil \beta_0 \rceil - 1 - c_0) \cdots (\lceil \beta_{p-1} \rceil - 1 - c_{p-1})\big) \\ = & \sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1 - c_i) \beta_{i+1} \cdots \beta_{p-1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} - \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} \\
&= \sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1)\beta_{i+1} \cdots \beta_{p-1} - f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1}).
\end{aligned}
$$

This ends the forward direction, the backward direction being similar. ■

As a consequence, we obtain that the digit sets $\mathrm{Adm}(\boldsymbol{\beta})$ and $\mathrm{Adm}'(\boldsymbol{\beta})$ have the same cardinality. Let us write

$$
\mathrm{Adm}(\boldsymbol{\beta}) = \{a_0, a_1, \dots, a_n\} \quad \text{and} \quad \mathrm{Adm}'(\boldsymbol{\beta}) = \{a'_0, a'_1, \dots, a'_n\}
$$

with $a_0 < a_1 < \cdots < a_n$ and $a'_0 < a'_1 < \cdots < a'_n$. Since, in general, not all p-tuples in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ are admissible, we have $n \le m$, where $m + 1$ is the cardinality of the digit set $\mathrm{Dig}(\boldsymbol{\beta})$ from the previous section.

A major difference with respect to the previous section is given by the following proposition, where the increasingness of the map $f_{\boldsymbol{\beta}}$ refers to the lexicographic order: for all $c, c' \in \prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$, $c <_{\mathrm{lex}} c' \implies f_{\boldsymbol{\beta}}(c) < f_{\boldsymbol{\beta}}(c')$.

Proposition 5.5.14. *The map $f_{\boldsymbol{\beta}}$ is increasing when restricted to the sets*

$$
\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})\}
$$

and

$$
\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})\}.
$$

Proof. First, consider two p-tuples $(c_0, \dots, c_{p-1})$ and $(c'_0, \dots, c'_{p-1})$ such that $(c_0, \dots, c_{p-1}) <_{\mathrm{lex}} (c'_0, \dots, c'_{p-1})$ and $c_0 \cdots c_{p-1},\ c'_0 \cdots c'_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})$. There exists $j \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \dots, c_{j-1} = c'_{j-1}$, $c_j \le c'_j - 1$. Moreover, since $c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})$, we have $c_{j+1} \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}^{(j+1)}})$. Hence, we have $\sum_{i=j+1}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} < \beta_{j+1} \cdots \beta_{p-1}$. We have

$$
\begin{aligned}
&f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1}) \\
&= \sum_{i=0}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} \\
&= \sum_{i=0}^{j-1} c_i \beta_{i+1} \cdots \beta_{p-1} + c_j \beta_{j+1} \cdots \beta_{p-1} + \sum_{i=j+1}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{j-1} c_i' \beta_{i+1} \cdots \beta_{p-1} + (c_j' - 1)\beta_{j+1} \cdots \beta_{p-1} + \sum_{i=j+1}^{p-1} c_i \beta_{i+1} \cdots \beta_{p-1} \\
&< \sum_{i=0}^{j-1} c_i' \beta_{i+1} \cdots \beta_{p-1} + c_j' \beta_{j+1} \cdots \beta_{p-1}.
\end{aligned}
$$

We obtain

$$
f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1}) < \sum_{i=0}^{j} c_i' \beta_{i+1} \cdots \beta_{p-1} \leq \sum_{i=0}^{p-1} c_i' \beta_{i+1} \cdots \beta_{p-1} = f_{\boldsymbol{\beta}}(c_0', \ldots, c_{p-1}').
$$

Second, consider two p-tuples $(c_0, \ldots, c_{p-1})$ and $(c_0', \ldots, c_{p-1}')$ such that $(c_0, \ldots, c_{p-1}) <_{\mathrm{lex}} (c_0', \ldots, c_{p-1}')$ and $c_0 \cdots c_{p-1},\ c_0' \cdots c_{p-1}' \in \mathrm{Pref}(D_{\boldsymbol{\beta}}')$. The p-tuples

$$
(\lceil \beta_0 \rceil - 1 - c_0, \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1})
$$

and

$$
(\lceil \beta_0 \rceil - 1 - c_0', \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}')
$$

are such that

$$
(\lceil \beta_0 \rceil - 1 - c_0, \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}) >_{\mathrm{lex}} (\lceil \beta_0 \rceil - 1 - c_0', \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}')
$$

and the length-p words $(\lceil \beta_0 \rceil - 1 - c_0) \cdots (\lceil \beta_{p-1} \rceil - 1 - c_{p-1})$ and $(\lceil \beta_0 \rceil - 1 - c_0') \cdots (\lceil \beta_{p-1} \rceil - 1 - c_{p-1}')$ belong to $\mathrm{Pref}(D_{\boldsymbol{\beta}})$ by Proposition 2.4.12. By the first part of the proof, we have

$$
f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1 - c_0, \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}) > f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1 - c_0', \ldots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}').
$$

We get

$$
\begin{aligned}
&\sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1) \beta_{i+1} \cdots \beta_{p-1} - f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1}) \\
>&\sum_{i=0}^{p-1} (\lceil \beta_i \rceil - 1) \beta_{i+1} \cdots \beta_{p-1} - f_{\boldsymbol{\beta}}(c_0', \ldots, c_{p-1}'),
\end{aligned}
$$

that is,

$$
f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1}) < f_{\boldsymbol{\beta}}(c_0', \ldots, c_{p-1}').
$$

■

From now on, for all $i \in [\![0, p-1]\!]$, we write $d^*_{\boldsymbol{\beta}^{(i)}}(1) = t_0^{(i)} t_1^{(i)} t_2^{(i)} \cdots$ and $\ell^*_{\boldsymbol{\beta}^{(i)}}(x_{\boldsymbol{\beta}^{(i)}} - 1) = \ell_0^{(i)} \ell_1^{(i)} \ell_2^{(i)} \cdots$. As a consequence of the previous proposition, we have

$$a_0 = f_{\boldsymbol{\beta}}(0, \dots, 0) = 0$$

and

$$a_n = f_{\boldsymbol{\beta}}(t_0^{(0)}, \dots, t_{p-1}^{(0)})$$

since $t_0^{(0)} \cdots t_{p-1}^{(0)}$ belongs to $\mathrm{Pref}(D_{\boldsymbol{\beta}})$ by Corollary 2.3.48 and, by Theorem 2.3.33, no lexicographically larger length-p word do. Similarly, we have

$$a'_0 = f_{\boldsymbol{\beta}}(\ell_0^{(0)}, \dots, \ell_{p-1}^{(0)})$$

since $\ell_0^{(0)} \cdots \ell_{p-1}^{(0)}$ belongs to $\mathrm{Pref}(D'_{\boldsymbol{\beta}})$ by Proposition 2.4.44 and, by Theorem 2.4.41, no lexicographically smaller length-p word do, and

$$a'_n = f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1, \dots, \lceil \beta_{p-1} \rceil - 1).$$

It is important to note that, in general, we have $\widetilde{\mathrm{Adm}(\boldsymbol{\beta})} \neq \mathrm{Adm}'(\boldsymbol{\beta})$.

Example 5.5.15. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}})$. By Example 2.4.26, we have $d^*_{\boldsymbol{\beta}}(1) = 200(10)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = (10)^\omega$, $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = 012(02)^\omega$ and $\ell_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = (02)^\omega$. Hence, by Theorems 2.3.33 and 2.4.41, we respectively get

$$\begin{aligned}
\mathrm{Adm}(\boldsymbol{\beta}) &= f_{\boldsymbol{\beta}}\big(\{(0,0),(0,1),(1,0),(1,1),(2,0)\}\big) \\
&= \{0, 1, \beta_1, \beta_1 + 1, 2\beta_1\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Adm}'(\boldsymbol{\beta}) &= f_{\boldsymbol{\beta}}\big(\{(0,1),(1,0),(1,1),(2,0),(2,1)\}\big) \\
&= \{1, \beta_1, \beta_1 + 1, 2\beta_1, 2\beta_1 + 1\}.
\end{aligned}$$

Moreover, we have

$$\widetilde{\mathrm{Adm}(\boldsymbol{\beta})} = \{0, \beta_1 - 1, \beta_1, 2\beta_1 - 1, 2\beta_1\} \neq \mathrm{Adm}'(\boldsymbol{\beta}).$$

Lemma 5.5.16. *We have*

$$\frac{a_0}{\beta - 1} = 0, \qquad \frac{a_n}{\beta - 1} \geq 1,$$
$$\frac{a'_0}{\beta - 1} \leq x_{\boldsymbol{\beta}} - 1 \quad \textit{and} \quad \frac{a'_n}{\beta - 1} = x_{\boldsymbol{\beta}}$$

Proof. The first equality is straightforward since $a_0 = 0$ and the fourth one is immediate by definition of $x_{\boldsymbol{\beta}}$. Moreover, we have

$$\begin{aligned} a_n &= f_{\boldsymbol{\beta}}(t_0^{(0)}, \dots, t_{p-1}^{(0)}) \\ &= \sum_{i=0}^{p-1} t_i^{(0)} \beta_{i+1} \cdots \beta_{p-1} \\ &= \Big(\sum_{m\in\mathbb{N}} \frac{t_m^{(0)}}{\prod_{k=0}^{m} \beta_k}\Big)\beta - \sum_{m=p}^{+\infty} \frac{t_m^{(0)}}{\prod_{k=p}^{m} \beta_k} \\ &\geq \beta - 1. \end{aligned}$$

Finally, we have

$$\begin{aligned} a'_0 &= f_{\boldsymbol{\beta}}(\ell_0^{(0)}, \dots, \ell_{p-1}^{(0)}) \\ &= f_{\boldsymbol{\beta}}(\lceil\beta_0\rceil - 1 - t_0^{(0)}, \dots, \lceil\beta_{p-1}\rceil - 1 - t_{p-1}^{(0)}) \\ &= f_{\boldsymbol{\beta}}(\lceil\beta_0\rceil - 1, \dots, \lceil\beta_{p-1}\rceil - 1) - f_{\boldsymbol{\beta}}(t_0^{(0)}, \dots, t_{p-1}^{(0)}) \\ &= a'_n - a_n. \end{aligned}$$

Therefore, we get

$$\frac{a'_0}{\beta - 1} = \frac{a'_n - a_n}{\beta - 1} = x_{\boldsymbol{\beta}} - \frac{a_n}{\beta - 1} \leq x_{\boldsymbol{\beta}} - 1.$$

This ends the proof. ∎

Lemma 5.5.17. *The sets* $\mathrm{Adm}(\boldsymbol{\beta})$ *and* $\mathrm{Adm}'(\boldsymbol{\beta})$ *are allowable digit sets for* β*.*

Proof. First, we consider the set $\mathrm{Adm}(\boldsymbol{\beta})$. We need to check Condition (1.4), which means in our case

$$\max_{k\in[\![0,n-1]\!]} (a_{k+1} - a_k) \leq \frac{a_n}{\beta - 1}.$$

By Proposition 5.5.14 and Lemma 5.5.16, it suffices to show that for all $k \in [\![0, m-1]\!]$, $f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1}) - f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1}) \leq 1$ where $(c_0, \dots, c_{p-1})$ and $(c'_0, \dots, c'_{p-1})$ are lexicographically consecutive elements in $\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})\}$. There exists $j \in [\![0, p-1]\!]$ such that $c_0 = c'_0, \dots, c_{j-1} = c'_{j-1}$, $c_j = c'_j - 1$, $c_{j+1} = t_0^{(j+1)}, \dots, c_{p-1} = t_{p-j-2}^{(j+1)}$ and $c'_{j+1} = \cdots = c'_{p-1} = 0$. We have

$$f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1}) - f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1})$$

$$
\begin{aligned}
&= c'_j \beta_{j+1} \cdots \beta_{p-1} - (c'_j - 1)\beta_{j+1} \cdots \beta_{p-1} - \sum_{i=j+1}^{p-1} t^{(j+1)}_{i-j-1} \beta_{i+1} \cdots \beta_{p-1} \\
&= \beta_{j+1} \cdots \beta_{p-1} - \Big(\big(\sum_{m \in \mathbb{N}} \frac{t^{(j+1)}_m}{\prod_{k=0}^{m} \beta_{j+1+k}} \big) \beta_{j+1} \cdots \beta_{p-1} \\
&\qquad\qquad - \sum_{m=p-j-1}^{+\infty} \frac{t^{(j+1)}_m}{\prod_{k=p-j-1}^{m} \beta_{j+1+k}} \Big) \\
&= \sum_{m=p-j-1}^{+\infty} \frac{t^{(j+1)}_m}{\prod_{k=p-j-1}^{m} \beta_{j+1+k}} \\
&\leq 1.
\end{aligned}
$$

Second, we consider the set $\mathrm{Adm}'(\boldsymbol{\beta})$. In this case, Condition (1.4) means

$$\max_{k \in [\![0, n-1]\!]} (a'_{k+1} - a'_k) \leq \frac{a'_n - a'_0}{\beta - 1},$$

where, by Lemma 5.5.16, we have

$$\frac{a'_n - a'_0}{\beta - 1} \geq 1.$$

Again, it suffices to show that for all $k \in [\![0, m-1]\!]$, $f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1}) - f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1}) \leq 1$ where $(c_0, \dots, c_{p-1})$ and $(c'_0, \dots, c'_{p-1})$ are lexicographically consecutive elements in $\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})\}$. The p-tuples $(\lceil \beta_0 \rceil - 1 - c'_0, \dots, \lceil \beta_{p-1} \rceil - 1 - c'_{p-1})$ and $(\lceil \beta_0 \rceil - 1 - c_0, \dots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1})$ are lexicographically consecutive elements in $\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})\}$. The conclusion follows by the first part of the proof since we have

$$f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1 - c_0, \dots, \lceil \beta_{p-1} \rceil - 1 - c_{p-1}) = a'_n - f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1})$$

and

$$f_{\boldsymbol{\beta}}(\lceil \beta_0 \rceil - 1 - c'_0, \dots, \lceil \beta_{p-1} \rceil - 1 - c'_{p-1}) = a'_n - f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1}).$$

■

By the previous two lemmas, we get that any real number $x \in [0, 1)$ has a greedy $(\beta, \mathrm{Adm}(\boldsymbol{\beta}))$-expansion and any real number $x \in (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$ has a lazy $(\beta, \mathrm{Adm}'(\boldsymbol{\beta}))$-expansion. We are now ready to prove the main result of this section.

Theorem 5.5.18. *We have*

$$T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})} = \pi_2 \circ (T_{\boldsymbol{\beta}})^p \circ \delta_0 \quad \textit{on } [0,1)$$

and

$$L_{\beta,\mathrm{Adm}'(\boldsymbol{\beta})} = \pi_2 \circ (L_{\boldsymbol{\beta}})^p \circ \delta_0 \quad \textit{on } (x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}].$$

Proof. First, consider the greedy maps and let $x \in [0,1)$. On the one hand, we have

$$T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})}(x) = \beta x - d$$

where d is the greatest digit in $\mathrm{Adm}(\boldsymbol{\beta})$ such that $\frac{d}{\beta} \le x$. On the other hand, by rephrasing Proposition 5.1.8 in terms of the map $f_{\boldsymbol{\beta}}$ when the parameter n from the statement equals p, we get

$$\pi_2 \circ (T_{\boldsymbol{\beta}})^p \circ \delta_0(x) = \beta x - f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1})$$

where $(c_0, \ldots, c_{p-1})$ is the lexicographically greatest p-tuple in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ such that $\frac{f_{\boldsymbol{\beta}}(c_0,\ldots,c_{p-1})}{\beta} \le x$. By Proposition 2.3.15, the word $c_0 \cdots c_{p-1}$ is the length-p prefix of $d_{\boldsymbol{\beta}}(x)$. Hence, the p-tuple $(c_0, \ldots, c_{p-1})$ belongs to the set $\{(c_0, \ldots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})\}$. By definition of d, we get $d \ge f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1})$. Let $(c'_0, \ldots, c'_{p-1})$ such that $c'_0 \cdots c'_{p-1} \in \mathrm{Pref}(D_{\boldsymbol{\beta}})$ and $d = f_{\boldsymbol{\beta}}(c'_0, \ldots, c'_{p-1})$. By definition of $(c_0, \ldots, c_{p-1})$, we get $(c_0, \ldots, c_{p-1}) \ge_{\mathrm{lex}} (c'_0, \ldots, c'_{p-1})$. By Proposition 5.5.14, we get

$$f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1}) \ge f_{\boldsymbol{\beta}}(c'_0, \ldots, c'_{p-1}) = d.$$

Second, consider $x \in (x_{\boldsymbol{\beta}}-1, x_{\boldsymbol{\beta}}]$. We have

$$L_{\beta,\mathrm{Adm}'(\boldsymbol{\beta})}(x) = \beta x - d$$

where d is the least digit in $\mathrm{Adm}'(\boldsymbol{\beta})$ such that

$$\frac{d}{\beta} + \sum_{k=1}^{+\infty} \frac{a'_n}{\beta^{k+1}} \ge x.$$

By Lemma 5.5.16, this inequality can be rephrased as

$$\frac{d + x_{\boldsymbol{\beta}}}{\beta} \ge x.$$

On the other hand, by rephrasing Proposition 5.1.13 by using (5.7) and in terms of the map $f_{\boldsymbol{\beta}}$ when the parameter n from the statement equals p, we get

$$\pi_2 \circ (L_{\boldsymbol{\beta}})^p \circ \delta_0(x) = \beta x - f_{\boldsymbol{\beta}}(c_0, \ldots, c_{p-1})$$

where $(c_0, \dots, c_{p-1})$ is the lexicographically least p-tuple in $\prod_{i=0}^{p-1} [\![0, \lceil \beta_i \rceil - 1]\!]$ such that $\frac{f_{\boldsymbol{\beta}}(c_0,\dots,c_{p-1})+x_{\boldsymbol{\beta}}}{\beta} \geq x$. By Proposition 2.4.18, the word $c_0 \cdots c_{p-1}$ is the length-p prefix of $\ell_{\boldsymbol{\beta}}(x)$. Hence, the p-tuple $(c_0, \dots, c_{p-1})$ belongs to the set $\{(c_0, \dots, c_{p-1}) : c_0 \cdots c_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})\}$. By definition of d, we get $d \leq f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1})$. Let $(c'_0, \dots, c'_{p-1})$ such that $c'_0 \cdots c'_{p-1} \in \mathrm{Pref}(D'_{\boldsymbol{\beta}})$ and $d = f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1})$. By definition of $(c_0, \dots, c_{p-1})$, we get $(c_0, \dots, c_{p-1}) \leq_{\mathrm{lex}} (c'_0, \dots, c'_{p-1})$. By Proposition 5.5.14, we get

$$f_{\boldsymbol{\beta}}(c_0, \dots, c_{p-1}) \leq f_{\boldsymbol{\beta}}(c'_0, \dots, c'_{p-1}) = d.$$

Hence the conclusion. ■

Note that, compared to the proofs in Section 5.5.1, in Theorem 5.5.18, the lazy equality cannot be immediately deduced from the greedy one since $\widetilde{\mathrm{Adm}(\boldsymbol{\beta})} \neq \mathrm{Adm}'(\boldsymbol{\beta})$.

As a consequence of the previous theorem, we get and improvement of Theorem 5.5.5.

Lemma 5.5.19. *We have*

$$\mathrm{Adm}(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta}) \cap [0, \beta)$$

and

$$\mathrm{Adm}'(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta}) \cap (a'_n - \beta, a'_n].$$

Moreover, we have

$$\mathrm{Adm}(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap [0, \beta) \quad \iff \quad \mathrm{Adm}'(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap (a'_n - \beta, a'_n].$$

Proof. We have $\mathrm{Adm}(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta})$ and $\mathrm{Adm}'(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta})$. Moreover, we have $\mathrm{Adm}(\boldsymbol{\beta}) = \{a_0, \dots, a_n\}$ with $0 = a_0 < \cdots < a_n$ and

$$a_n = f_{\boldsymbol{\beta}}(t_0^{(0)}, \dots, t_{p-1}^{(0)}) < \beta.$$

Hence, we get $\mathrm{Adm}(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta}) \cap [0, \beta)$. By Lemma 5.5.13, for any real number a, we have $a \in \mathrm{Adm}(\boldsymbol{\beta})$ if and only if $a'_n - a \in \mathrm{Adm}'(\boldsymbol{\beta})$. We obtain $\mathrm{Adm}'(\boldsymbol{\beta}) \subseteq \mathrm{Dig}(\boldsymbol{\beta}) \cap (a'_n - \beta, a'_n]$.

Now, we turn to the second part of the statement. Since $\widetilde{\mathrm{Dig}(\boldsymbol{\beta})} = \mathrm{Dig}(\boldsymbol{\beta})$, for any real number d, we have $d \in \mathrm{Dig}(\boldsymbol{\beta})$ if and only if $d_m - d \in \mathrm{Dig}(\boldsymbol{\beta})$, where $d_m = a'_n$. We get

$$\mathrm{Adm}(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap [0, \beta)$$

$$\begin{aligned}&\iff \{a'_n - a \colon a \in \mathrm{Adm}(\boldsymbol{\beta})\} = \{a'_n - d \colon a \in \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta)\}\\ &\iff \mathrm{Adm}'(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap (a'_n - \beta, a'_n].\end{aligned}$$

∎

Proposition 5.5.20. *The transformations*

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}\big|_{[0,1)} \quad \textit{and} \quad \pi_2 \circ (T_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{[0,1)}$$

coincide if and only if the transformations

$$L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}\big|_{(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]} \quad \textit{and} \quad \pi_2 \circ (L_{\boldsymbol{\beta}})^p \circ \delta_0\big|_{(x_{\boldsymbol{\beta}}-1,x_{\boldsymbol{\beta}}]}$$

do.

Proof. The same proof as that of Proposition 5.5.4 can be applied. ∎

Theorem 5.5.21. *We have*

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (T_{\boldsymbol{\beta}})^p \circ \delta_0 \quad \textit{on } [0,1)$$

if and only if

$$\mathrm{Adm}(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta).$$

Similarly, we have

$$L_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} = \pi_2 \circ (L_{\boldsymbol{\beta}})^p \circ \delta_0 \quad \textit{on } (x_{\boldsymbol{\beta}} - 1, x_{\boldsymbol{\beta}}]$$

if and only if

$$\mathrm{Adm}'(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap (a'_n - \beta, a'_n].$$

Proof. First, we consider the greedy part of the statement. Suppose that

$$\mathrm{Adm}(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta).$$

By Theorem 5.5.18, it is sufficient to prove that we have $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})} = T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})}$ on $[0,1)$. For all $x \in [0,1)$, we have $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) = \beta x - d$ where d is the greatest digit in $\mathrm{Dig}(\boldsymbol{\beta})$ such that $\frac{d}{\beta} \le x$. By assumption we have $x < 1$, hence we obtain $d < \beta$. Therefore, since $\mathrm{Adm}(\boldsymbol{\beta}) = \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta)$, the digit d is the greatest digit in $\mathrm{Adm}(\boldsymbol{\beta})$ such that $\frac{d}{\beta} \le x$. We obtain

$$T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) = T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})}(x).$$

Conversely, suppose that

$$\mathrm{Adm}(\boldsymbol{\beta}) \neq \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta).$$

By Lemma 5.5.19, there exists a digit

$$d \in \big(\mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta)\big) \setminus \mathrm{Adm}(\boldsymbol{\beta}).$$

If there exists $k \in [\![0,n-1]\!]$ such that $a_k < d < a_{k+1}$. Without loss of generality, we suppose that there is no other digit $d' \in \big(\mathrm{Dig}(\boldsymbol{\beta}) \cap [0,\beta)\big) \setminus \mathrm{Adm}(\boldsymbol{\beta})$ such that $d < d' < a_{k+1}$ (otherwise we consider d' instead of d). Then, for all $x \in [\frac{d}{\beta}, \frac{a_{k+1}}{\beta})$, we have $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) = \beta x - d$ whereas $T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})}(x) = \beta x - a_k$. If $d > a_n$, then for all $x \in [\frac{d}{\beta}, 1)$, we have $T_{\beta,\mathrm{Dig}(\boldsymbol{\beta})}(x) = \beta x - d$ whereas $T_{\beta,\mathrm{Adm}(\boldsymbol{\beta})}(x) = \beta x - a_n$. Hence, the two maps differ and we conclude by Theorem 5.5.18.

The lazy part of the statement follows by the first part of the proof by using Lemma 5.5.19 and Proposition 5.5.20. ■

Example 5.5.22. Consider the alternate base $\boldsymbol{\beta} = (\overline{\frac{3}{2}, \frac{3}{2}, 4})$ from Example 5.5.8. We have $\mathrm{Dig}(\boldsymbol{\beta}) = [\![0,13]\!]$. Since we have $d^*_{\boldsymbol{\beta}}(1) = (102)^\omega$, $d^*_{\boldsymbol{\beta}^{(1)}}(1) = 11(102)^\omega$ and $d^*_{\boldsymbol{\beta}^{(2)}}(1) = 3(102)^\omega$, we get

$$\begin{aligned}
\mathrm{Adm}(\boldsymbol{\beta}) &= f_{\boldsymbol{\beta}}\big(\{(0,0,0),(0,0,1),(0,0,2),(0,0,3),(0,1,0),(0,1,1),\\
&\qquad\quad (1,0,0),(1,0,1),(1,0,2)\}\big)\\
&= [\![0,8]\!]\\
&= \mathrm{Dig}(\boldsymbol{\beta}) \cap [0,9).
\end{aligned}$$

Moreover, we have $\ell^*_{\boldsymbol{\beta}}(x_{\boldsymbol{\beta}} - 1) = (011)^\omega$, $\ell^*_{\boldsymbol{\beta}^{(1)}}(x_{\boldsymbol{\beta}^{(1)}} - 1) = 02(011)^\omega$ and $\ell^*_{\boldsymbol{\beta}^{(2)}}(x_{\boldsymbol{\beta}^{(2)}} - 1) = 0(011)^\omega$. Therefore, we get

$$\begin{aligned}
\mathrm{Adm}'(\boldsymbol{\beta}) &= f_{\boldsymbol{\beta}}\big(\{(0,1,1),(0,1,2),(0,1,3),(1,0,2),(1,0,3),(1,1,0),\\
&\qquad\quad (1,1,1),(1,1,2),(1,1,3)\}\big)\\
&= [\![5,13]\!]\\
&= \mathrm{Dig}(\boldsymbol{\beta}) \cap (4,13].
\end{aligned}$$

By Theorem 5.5.21, the maps $T_{9,\mathrm{Dig}(\boldsymbol{\beta})}$ on $[0,1)$ and $L_{9,\mathrm{Dig}(\boldsymbol{\beta})}$ on $[0,x_{\boldsymbol{\beta}})$ respectively coincide with $\pi_2 \circ (T_{\boldsymbol{\beta}})^3 \circ \delta_0$ on $[0,1)$ and $\pi_2 \circ (L_{\boldsymbol{\beta}})^3 \circ \delta_0$ on $[0,x_{\boldsymbol{\beta}})$. This agrees with what was observed in Example 5.5.8.

Remark 5.5.23. I found the results from this section after the publication of the article [CCD21]. A direct consequence of Theorem 5.5.18 is that, for all $i \in [\![0,p-1]\!]$, the measure $\mu_{\boldsymbol{\beta},i}$ from Definition 5.2.5 is the unique $T_{\beta,\mathrm{Adm}(\boldsymbol{\beta}^{(i)})}$-invariant absolutely continuous probability measure given by [DK10, Theorem 2.10]. However, note that the digit set $\mathrm{Dig}(\boldsymbol{\beta})$ has its own advantage since, compared to $\mathrm{Adm}(\boldsymbol{\beta})$ and $\mathrm{Adm}'(\boldsymbol{\beta})$, it can be constructed without any prior combinatorial check. In fact, in order to construct the set $\mathrm{Adm}(\boldsymbol{\beta})$ (resp., $\mathrm{Adm}'(\boldsymbol{\beta})$), one needs to first compute the quasi-greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 (resp., quasi-lazy $\boldsymbol{\beta}^{(i)}$-expansions of $x_{\boldsymbol{\beta}^{(i)}} - 1$) for all $i \in [\![0, p-1]\!]$.

PERSPECTIVES

During this doctoral research, we studied questions related to a generalization of β-representations which have been studied a lot since 1960. Real base representations have many related research groundwork to be generalized to Cantor and alternate base frameworks. Still, a great deal of work remains to be achieved in this new theory, which is good news for future research. We end this dissertation by a brief summary of some potential future research questions.

1. Real base expansions were generalized to the context of negative bases in 2009 by Ito and Sadahiro [IS09]. In [CD20], in order to generalize real bases and Cantor (integer) bases, Caalima and Demegillo work with sequences of real numbers composed of positive and negative bases. However, not all properties of this book had been studied by Caalima and Demegillo who concentrated on a generalization of Parry's theorem characterizing greedy admissible sequences. In particular, they did not work on lazy expansions, greedy and lazy $\boldsymbol{\beta}$-shifts, normalization or dynamics. Hence, an open question is to generalize results from this dissertation to sequences $\boldsymbol{\beta} = (\beta_n)_{n\in\mathbb{N}}$ allowing positive and negative bases.

2. In Chapter 3, we proved that an alternate base $\boldsymbol{\beta}$ is a Parry alternate base if and only if the associated greedy (resp., lazy) $\boldsymbol{\beta}$-shift is sofic. Moreover, we illustrated that there exist non-finite type greedy

$\boldsymbol{\beta}$-shifts that are based on alternate bases $\boldsymbol{\beta}$ such that all greedy $\boldsymbol{\beta}^{(i)}$-expansions of 1 are finite. An open question is to elucidate whether all $\boldsymbol{\beta}^{(i)}$-expansions of 1 have to be finite in order to have a greedy $\boldsymbol{\beta}$-shift of finite type.

3. In the real base case, for every sequence of non-negative digits $a = a_0a_1a_2\cdots$ satisfying the lexicographic condition $a_na_{n+1}a_{n+2}\cdots \leq_{\text{lex}} a$ for all $n \in \mathbb{N}$, there exists a unique $\beta > 1$ such that $d_\beta(1) = a$ [Par60]. It is not clear yet whether for p integer digit sequences $a^{(0)}, \ldots, a^{(p-1)}$ satisfying analogous lexicographic conditions, there exists a unique alternate base $\overline{(\beta_0, \ldots, \beta_{p-1})}$ such that $d_{\boldsymbol{\beta}^{(i)}}(1) = a^{(i)}$ for $i = 0, 1, \ldots, p-1$. Corollary 4.3.10 represents a first step towards this direction.

4. For $\beta > 1$, let $\mathrm{Per}(\beta)$ denote the set of real numbers in $[0, 1)$ having an ultimately periodic β-expansion. Schmidt [Sch80] proved that if $\mathrm{Per}(\beta) \supset \mathbb{Q} \cap [0, 1)$ then β is either a Pisot number or a Salem number (that is a real algebraic integer greater than 1 such that all of its Galois conjugates have modulus less than or equal 1 and at most one of them has modulus exactly 1); and that if β is a Pisot number, then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0, 1)$. The question here is to generalize such results, that is to understand which are the ultimately periodic $\boldsymbol{\beta}$-expansions for an alternate base $\boldsymbol{\beta}$, and in particular, for which alternate bases $\boldsymbol{\beta}$ do all rational numbers have ultimately periodic $\boldsymbol{\beta}$-expansions. We currently work on this question with Émilie Charlier and Savinien Kreczman.

5. In Chapter 4, we proved that if $\beta = \prod_{i=0}^{p-1} \beta_i$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1}$ belong to $\mathbb{Q}(\beta)$, then $\boldsymbol{\beta}$ is a Parry alternate base (see Theorem 4.3.14). We have illustrated that β being Pisot is not a necessary condition. For $p = 1$, Solomyak obtained algebraic properties of Parry numbers [Sol94]. It would be interesting to study the analogy in the context of alternate bases. In particular, to find bounds on the algebraic conjugates of β.

6. In Chapter 4, we proved that if $\beta = \prod_{i=0}^{p-1} \beta_i$ is a Pisot number and $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$, then the greedy and lazy normalization functions are computable by finite Büchi automata. An open question is to investigate if whether or not this sufficient condition is also necessary.

7. For a real base $\beta > 1$, Theorem 1.4.30 tells us that if $d > \beta - 1$, then the spectrum $X^d(\beta)$ has no accumulation point in $\mathbb{R}$ if and only if β is Pisot. In view of Theorems 4.2.10 and 4.4.2 from Chapter 4, we leave

the following two questions open. Let $\boldsymbol{\beta} = (\overline{\beta_0, \dots, \beta_p})$ be an alternate base, let $\beta = \prod_{i=0}^{p-1} \beta_i$, let $\boldsymbol{D} = (\overline{D_0, \dots, D_{p-1}})$ be an alternate alphabet and let $\mathrm{Dig}(\boldsymbol{\beta}, \boldsymbol{D})$ be the associated real digit set.

(a) Suppose that the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$ and that $D_i \supseteq [\![- \lfloor \beta_i \rfloor, \lfloor \beta_i \rfloor]\!]$ for all $i \in [\![0, p-1]\!]$. Can we deduce that the product β is a Pisot number and that $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$?

(b) Suppose that the spectrum $X^{\mathrm{Dig}(\boldsymbol{\beta},\boldsymbol{D})}(\beta)$ has no accumulation point in $\mathbb{R}$ and that $D_i \supseteq [\![- \lceil \beta_i \rceil + 1, \lceil \beta_i \rceil - 1]\!]$ for all $i \in [\![0, p-1]\!]$, (with or without the hypothesis that there exists $j \in [\![0, p-1]\!]$ such that $d_j \geq \lceil \beta \rceil - 1$, depending on item (a)). Can we deduce that β is a Pisot number and that $\beta_i \in \mathbb{Q}(\beta)$ for all $i \in [\![0, p-1]\!]$?

8. In Chapter 5, we concentrated on measure theoretical aspects of alternate base expansions. A natural question would be to consider the topological point of view. For example, it would be of interest to prove that the topological entropies of the topological dynamical systems under consideration coincide with the measure theoretical entropy $\frac{1}{p}\log(\beta)$ found, where $\beta = \prod_{i=0}^{p-1} \beta_i$. In particular, this would prove that the measure theoretical dynamical systems studied in Chapter 5 are all of maximal entropy.

9. In Chapters 3, 4 and 5, we studied the properties of alternate base expansions. An extension of these works can be investigated while considering Cantor bases $\boldsymbol{\beta}$ that take only finitely many values and such that for all $n \in \mathbb{N}$, the value of β_n can be "interpreted using a computable method". Since automata are in some way the simplest model of computation, a first step can be to investigate the behaviors of Cantor bases such that $(\beta_n)_{n\in\mathbb{N}}$ is an automatic sequence [AS03]. A famous example of such a Cantor base is a Thue-Morse Cantor base (2.2). Related open questions are the following ones.

   - Do such systems define sofic $\boldsymbol{\beta}$-shifts?
   - What are the algebraic properties of the natural extension of the spectra associated with such Cantor real bases?
   - What kind of algebraic properties do we get?
   - Given such a Cantor real base, can we find associated greedy and lazy transformation, iterations of which generate the greedy and lazy expansion respectively? Moreover, can we prove the existence of associated unique absolutely continuous invariant measures?

10. Consider the set

$$\mathbb{Z}_\beta = \{\pm \sum_{n=0}^{\ell-1} a_n \beta^{\ell-1-n} : \ell \in \mathbb{N},\ a_0 a_1 \dots a_{\ell-1} 0^\omega \in D_\beta\}.$$

Clearly, the set $\mathbb{Z}_\beta$ is a subset of the spectrum $X^{\lceil\beta\rceil-1}(\beta)$ of β over the alphabet $[\![-\lceil\beta\rceil+1, \lceil\beta\rceil-1]\!]$. This set $\mathbb{Z}_\beta$ was introduced by Gazeau [Gaz97] and is called the set of *β-integers*. The set $\mathbb{Z}_\beta$ has a lot of properties.

- $\mathbb{Z}_\beta = \mathbb{Z}$ when β is an integer greater than or equal to 2.
- $\mathbb{Z}_\beta$ has no accumulation point.
- $\mathbb{Z}_\beta$ is self-similar thus $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$.
- $\mathbb{Z}_\beta$ is not invariant under translation if $\beta \notin \mathbb{N}$.
- $\mathbb{Z}_\beta$ forms a Meyer set if β is a Pisot number [BFGK98], that is, $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$ for a finite set $F \subset \mathbb{R}$.
- $\mathbb{Z}_\beta = X^{\lceil\beta\rceil-1}(\beta)$ if and only if β is a confluent Parry number (sometimes called generalized multinacci numbers), that is, zeros greater than 1 of polynomials

  $$x^d - mx^{d-1} - mx^{d-2} - \cdots - mx - n$$

  where $d \geq 1$ and $m \geq n \geq 1$.

From the first item, we know that when β is an integer greater than or equal to 2, the distances between neighboring elements in $\mathbb{Z}_\beta$ is always 1. If β is not an integer, the situation changes significantly, but still the distances between neighboring elements can be characterized. Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence such that

$$\mathbb{Z}_\beta = \pm\{x_n : n \in \mathbb{N}\}.$$

The set $\mathbb{Z}_\beta$ has finitely many distances $x_{n+1} - x_n$ if and only if β is a Parry number [Thu89]. If β is a Parry number, there exist a positive integer k and real numbers $\Delta_0, \dots, \Delta_k$ such that $\{\Delta_i : i \in [\![0,k]\!]\}$ is the set of distances of $\mathbb{Z}_\beta$. This set of distances is intimately linked with the values of prefixes of $d_\beta(1)$. We define an infinite word $u_\beta = (u_n)_{n\in\mathbb{N}}$ defined by $u_n = i$, with $i \in [\![0,k]\!]$, if $x_{n+1} - x_n = \Delta_i$, for all $n \in \mathbb{N}$. The word u_β is the fixed point of a morphism [Fab95]. Moreover, the infinite word u_β is Sturmian, that is aperiodic of minimal factor complexity, if and only if β is a quadratic Pisot number. An interesting research project I started working on with Émilie Charlier, Zuzana Masáková and Edita Pelantová is to study properties of analogue $\boldsymbol{\beta}$-integers when $\boldsymbol{\beta}$ is an alternate base.

11. Let U be the set of *univoque bases* consisting of real numbers $\beta > 1$ such that 1 has a unique β-expansion. The set U has been widely studied for nearly 25 years. To cite just a few:

    - Erdös, Joó and Komornik [EJK90] showed that U is uncountable and of zero Lebesgue measure.
    - Daróczy and Kátai [DK95] showed that U has full Hausdorff dimension.
    - Komornik and Loreti [KL98, KL02, KL07] found its smallest element q_{KL}, which is now called the Komornik-Loreti constant and is related to the Thue-Morse sequence, and proved that the topological closure of U is a Cantor set, that is a non-empty compact set having neither interior nor isolated points.
    - Dajani, Komornik, Kong and Li [DKKL18] proved that the algebraic difference $U - U$ contains an interval.

    A vast potential research project is to define and study an analogue set of univoque Cantor (or alternate) bases.

BIBLIOGRAPHY

[Abr59] Leonid Mikhaĭlovich Abramov. The entropy of a derived automorphism. *Dokl. Akad. Nauk SSSR*, 128:647–650, 1959.

[AK13] Shigeki Akiyama and Vilmos Komornik. Discrete spectra and Pisot numbers. *J. Number Theory*, 133(2):375–390, 2013.

[AS03] Jean-Paul Allouche and Jeffrey Shallit. *Automatic sequences. Theory, applications, generalizations.* Cambridge University Press, Cambridge, 2003.

[Bas02] Frédérique Bassino. Beta-expansions for cubic Pisot numbers. In *LATIN 2002: Theoretical informatics (Cancun)*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 141–152. Springer, Berlin, 2002.

[Ber77] Anne Bertrand. Développements en base de Pisot et répartition modulo 1. *C. R. Acad. Sci. Paris Sér. A-B*, 285(6):A419–A421, 1977.

[BFGK98] Čestmir Burdík, Christiane Frougny, Jean-Pierre Gazeau, and Rudolph Krejcar. Beta-integers as natural counting systems for quasicrystals. *J. Phys. A*, 31(30):6449–6472, 1998.

[BG97] Abraham Boyarsky and Pawel Góra. *Laws of chaos.* Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA,

1997. Invariant measures and dynamical systems in one dimension.

[BM86] Anne Bertrand-Mathis. Développement en base θ ; répartition modulo un de la suite $(x\theta^n)_{n\geq 0}$; langages codés et θ-shift. *Bull. Soc. Math. France*, 114(3):271–323, 1986.

[BR10] Valérie Berthé and Michel Rigo. *Combinatorics, words and symbolic dynamics*, volume 159 of *Encyclopedia Math. Appl.* Cambridge Univ. Press, Cambridge, 2010.

[Büc60] Julius Richard Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.

[Can69] Georg Cantor. Über die einfachen Zahlensysteme. *Z. Math. Phys.*, 14:121–128, 1869.

[CC21] Émilie Charlier and Célia Cisternino. Expansions in Cantor real bases. *Monatsh. Math.*, 195:585–610, 2021.

[CCD21] Émilie Charlier, Célia Cisternino, and Karma Dajani. Dynamical behavios of expansions in alternate bases. *Ergodic Theory and Dynamical Systems*, pages 1–34, 2021.

[CCMP22] Émilie Charlier, Célia Cisternino, Zuzana Masáková, and Edita Pelantová. Spectrum, algebraicity and normalization in alternate bases. arXiv:2202.03718, 2022.

[CD20] Jonathan Caalima and Shiela Demegillo. Beta Cantor series expansion and admissible sequences. *Acta Polytechnica*, 60(3):214–224, 2020.

[Cis21] Célia Cisternino. Combinatorial properties of lazy expansions in Cantor real bases. arXiv:2202.04938, 2021.

[DdVKL12] Karma Dajani, Martijn de Vries, Vilmos Komornik, and Paola Loreti. Optimal expansions in non-integer bases. *Proc. Amer. Math. Soc.*, 140(2):437–447, 2012.

[DK95] Zoltán Daróczy and Imre Kátai. On the structure of univoque numbers. *Publ. Math. Debrecen*, 46(3-4):385–408, 1995.

[DK02a] Karma Dajani and Cor Kraaikamp. *Ergodic theory of numbers*, volume 29 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 2002.

[DK02b] Karma Dajani and Cor Kraaikamp. From greedy to lazy expansions and their driving dynamics. *Expo. Math.*, 20(4):315–327, 2002.

[DK07] Karma Dajani and Charlene Kalle. Random β-expansions with deleted digits. *Discrete Contin. Dyn. Syst.*, 18(1):199–217, 2007.

[DK10] Karma Dajani and Charlene Kalle. A note on the greedy β-transformation with arbitrary digits. In *École de Théorie Ergodique*, volume 20 of *Sémin. Congr.*, pages 83–104. Soc. Math. France, Paris, 2010.

[DK21] Karma Dajani and Charlene Kalle. *A first course in ergodic theory.* Chapman and Hall/CRC, 2021.

[DKKL18] Karma Dajani, Vilmos Komornik, Derong Kong, and Wenxia Li. Algebraic sums and products of univoque bases. *Indag. Math. (N.S.)*, 29(4):1087–1104, 2018.

[EJK90] Pál Erdös, István Joó, and Vilmos Komornik. Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems. *Bull. Soc. Math. France*, 118(3):377–390, 1990.

[ER59] Pál Erdős and Alfréd Rényi. Some further statistical properties of the digits in Cantor's series. *Acta Math. Acad. Sci. Hungar.*, 10:21–29, 1959.

[Fab95] Stéphane Fabre. Substitutions et β-systèmes de numération. *Theoret. Comput. Sci.*, 137(2):219–236, 1995.

[Fen16] De-Jun Feng. On the topology of polynomials with bounded integer coefficients. *J. Eur. Math. Soc. (JEMS)*, 18(1):181–193, 2016.

[FP18] Christiane Frougny and Edita Pelantová. Two applications of the spectrum of numbers. *Acta Math. Hungar.*, 156(2):391–407, 2018.

[Fro92] Christiane Frougny. Representations of numbers and finite automata. *Math. Systems Theory*, 25(1):37–60, 1992.

[FS10] Christiane Frougny and Jacques Sakarovitch. Number representation and finite automata. In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 34–107. Cambridge Univ. Press, Cambridge, 2010.

[Fur81] Hillel Furstenberg. *Recurrence in ergodic theory and combinatorial number theory.* Princeton University Press, Princeton, N.J., 1981. M. B. Porter Lectures.

[Gal76] János Galambos. *Representations of real numbers by infinite series.* Lecture Notes in Mathematics, Vol. 502. Springer-Verlag, Berlin-New York, 1976.

[Gaz97] Jean-Pierre Gazeau. Pisot-cyclotomic integers for quasilattices. In *The mathematics of long-range aperiodic order (Waterloo, ON, 1995)*, volume 489 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 175–198. Kluwer Acad. Publ., Dordrecht, 1997.

[Gel59] Alexandre Ossipovitch Gel'fond. A common property of number systems. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 23:809–814, 1959.

[Gór09] Pawel Góra. Invariant densities for piecewise linear maps of the unit interval. *Ergodic Theory Dynam. Systems*, 29(5):1549–1583, 2009.

[HMV18] Kevin Hare, Zuzana Masáková, and Tomáš Vávra. On the spectra of Pisot-cyclotomic numbers. *Lett. Math. Phys.*, 108(7):1729–1756, 2018.

[IS09] Shunji Ito and Taizo Sadahiro. Beta-expansions with negative bases. *Integers*, 9:A22, 239–259, 2009.

[IT74] Shunji Ito and Yōichirō Takahashi. Markov subshifts and realization of β-expansions. *J. Math. Soc. Japan*, 26:33–55, 1974.

[KL98] Vilmos Komornik and Paola Loreti. Unique developments in non-integer bases. *Amer. Math. Monthly*, 105(7):636–639, 1998.

[KL02] Vilmos Komornik and Paola Loreti. Subexpansions, superexpansions and uniqueness properties in non-integer bases. *Period. Math. Hungar.*, 44(2):197–218, 2002.

[KL07] Vilmos Komornik and Paola Loreti. On the topological structure of univoque sets. *J. Number Theory*, 122(1):157–183, 2007.

[KLZ21] Vilmos Komornik, Jian Lu, and Yuru Zou. Expansions in multiple bases over general alphabets. arXiv:2102.10051, 2021.

[KT84] Peter Kirschenhofer and Robert Franz Tichy. On the distribution of digits in Cantor representations of integers. *J. Number Theory*, 18(1):121–134, 1984.

[Li21] Yao-Qiang Li. Expansions in multiple bases. *Acta Math. Hungar.*, 163(2):576–600, 2021.

[Lot97] M. Lothaire. *Combinatorics on words*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1997.

[Lot02] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.

[LS12] Lingmin Liao and Wolfgang Steiner. Dynamical properties of the negative beta-transformation. *Ergodic Theory Dynam. Systems*, 32(5):1673–1690, 2012.

[LY82] A. Lasota and James A. Yorke. Exact dynamical systems and the Frobenius-Perron operator. *Trans. Amer. Math. Soc.*, 273(1):375–384, 1982.

[Mor21] Harold Calvin Marston Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.*, 22:84–100, 1921.

[Mul63] David E. Muller. Infinite sequences and finite machines. In *Proceedings of the Fourth Annual Symposium on Switching Circuit Theory and Logical Design (swct 1963)*, pages 3–16, 1963.

[Neu21] Jörg Neunhäuserer. Non-uniform expansions of real numbers. *Mediterr. J. Math.*, 18(2):Paper No. 70, 8, 2021.

[Par60] William Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.*, 11:401–416, 1960.

[Ped05] Marco Pedicini. Greedy expansions and sets with deleted digits. *Theoret. Comput. Sci.*, 332(1-3):313–336, 2005.

[PP04] Dominique Perrin and Jean-Éric Pin. *Infinite Words. Automata, Semigroups, Logic and Games*, volume 141. Elsevier Academic Press, 2004.

[Rén56] Alfréd Rényi. On the distribution of the digits in Cantor's series. *Mat. Lapok*, 7:77–100, 1956.

[Rén57] Alfréd Rényi. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.*, 8:477–493, 1957.

[Rig14] Michel Rigo. *Formal languages, automata and numeration systems. 1.* Networks and Telecommunications Series. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014. Introduction to combinatorics on words.

[Roh61] Vladimir Abramovich Rohlin. Exact endomorphisms of a Lebesgue space. *Izv. Akad. Nauk SSSR Ser. Mat.*, 25:499–530, 1961.

[Sch80] Klaus Schmidt. On periodic expansions of Pisot numbers and Salem numbers. *Bull. London Math. Soc.*, 12(4):269–278, 1980.

[Sha48] Claude Elwood Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423, 623–656, 1948.

[Sid03] Nikita Sidorov. Almost every number has a continuum of β-expansions. *Amer. Math. Monthly*, 110(9):838–842, 2003.

[Sol94] Boris Solomyak. Conjugates of beta-numbers and the zero-free domain for a class of analytic functions. *Proc. London Math. Soc. (3)*, 68(3):477–498, 1994.

[Thu12] Axel Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Norske. vid. Selk. Skr. Mat. Nat. Kl.*, 1:1–67, 1912.

[Thu89] William Paul Thurston. Groups, tilings and finite state automata. *Geometry supercomputer project research report GCG1. University of Minnesota*, 1989.

[Váv21] Tomáš Vávra. Periodic representations in Salem bases. *Israel J. Math.*, 242(1):83–95, 2021.

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Notation

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