Iterative solvers and stabilisation for mixed electrostatic and magnetostatic formulations *

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Abstract

Mixed electrostatic and magnetostatic finite element formulations are considered. Solution methods for the resulting indefinite algebraic systems are investigated. Methods developed for the mixed formulations of the Stokes equations are modified in order to apply to the Maxwell equations: an efficient block preconditioner is proposed and a stabilised formulation is described. The different methods are applied to 2D and 3D examples.

Keywords: mixed finite elements, Maxwell equations, MINRES, stabilisation
AMS classification:

1 Introduction

This paper deals with finite element methods for the numerical solution of the Maxwell equations in electromagnetism. Potential formulations are traditionally used in the finite element analysis of the Maxwell equations. Thanks to the definition of a potential, such as the electric scalar potential V or the magnetic vector potential $\underline{a}$, one of the equations can be eliminated from this system of partial differential equations. In electrostatics and magnetostatics, the system then reduces to one single equation for the potential. Mixed formulations are based on a different approach. One equation of the system is considered as a constraint, which is imposed by means of Lagrange multipliers and the constrained problem is rewritten as a saddle point problem.
The main advantage of potential formulations is that only one equation must be solved. However, the unknown field, i.e. a scalar or vector potential, is of little practical interest and must be differentiated in order to yield a physically relevant quantity like the electric field $\underline{e}$ or the magnetic induction $\underline{b}$. These quantities are computed directly with mixed formulations. The higher computational cost due to the additional equation is then offset by a higher accuracy [5]. Another advantage of the mixed formulation is the simplified treatment of the curl-free condition for the magnetic field $\underline{h}$ in multiply connected non-conducting regions. This condition can easily be expressed with Lagrange multipliers [6, 3]; it is much more cumbersome to impose it in a direct way. The main challenge of mixed formulations is the solution of an indefinite linear saddle point problem. This article focuses on the solution of such indefinite systems arising from the mixed $\underline{d}-V$ formulation, where the electric displacement $\underline{d}$ is discretised with Whitney face elements, and from the mixed $\underline{h}-\underline{a}$ formulation where $\underline{h}$ and $\underline{a}$ are both discretised with Whitney

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edge elements. We also address the stabilisation, which, in mixed finite element approaches, causes unphysical oscillations to be present in the solution.
The structure of the paper is as follows. In section 2, we present an overview of electrostatic and magnetostatic formulations. In section 3, we introduce the MINRES solver and design a block preconditioner suited for the electrostatic and magnetostatic problems. In section 4, we present the stabilised mixed formulations and propose original expressions for the stabilisation parameters. In section 5, the results of 2D and 3D simulations are presented. Section 6 concludes the paper.

2 An overview of electrostatic and magnetostatic formulations

2.1 Maxwell equations

The steady-state Maxwell equations can be uncoupled into the electrostatic equations,

$$\nabla \times \underline{e} = 0\,, \quad \nabla \cdot \underline{d} = \rho\,, \quad \underline{d} = \varepsilon \underline{e}\,, \tag{2.1}$$

for the electric field $\underline{e}$ and the electric displacement $\underline{d}$, and the magnetostatic equations,

$$\nabla \times \underline{h} = \underline{j}\,, \quad \nabla \cdot \underline{b} = 0\,, \quad \underline{b} = \mu \underline{h}\,, \tag{2.2}$$

for the magnetic field $\underline{h}$ and the magnetic induction $\underline{b}$. Each system contains two equations and a constitutive law. The latter relates the two corresponding fields, respectively $\underline{e}$, $\underline{d}$ and $\underline{h}$, $\underline{b}$ to one another. Four different formulations can be derived from each system: one can solve for $\underline{h}$ (resp. $\underline{e}$) or $\underline{b}$ (resp. $\underline{d}$) and in each case, one may either define a potential and reduce the system to one equation or consider one equation as a constraint which is imposed by means of Lagrange multipliers. The former approach leads to potential formulations, which are the most frequently used in practice, whereas the latter leads to mixed formulations. All four possibilities are represented in Table 1 and Table 2. The functions ϕ and V are magnetic and electric scalar potentials. The vector fields $\underline{a}$ and $\underline{w}$ are magnetic and electric vector potentials. These potentials are related to the original variables by the following formulas:

$$\underline{b} = \nabla \times \underline{a}\,, \quad \underline{h} = \underline{h}_s - \nabla\phi\,, \quad \underline{d} = \underline{d}_s + \nabla \times \underline{w}\,, \quad \underline{e} = -\nabla V\,, \tag{2.3}$$

where $\underline{h}_s$ is a magnetic source field such that $\nabla \times \underline{h}_s = \underline{j}$ and $\underline{d}_s$ is an electric source field such that $\nabla \cdot \underline{d}_s = \rho$.

	E	**D**
Potential	$\nabla \cdot (-\varepsilon \nabla V) = \rho$	$\nabla \times (\varepsilon^{-1} (\underline{d}_s + \nabla \times \underline{w})) = 0$
Mixed	$\varepsilon \underline{e} - \underline{d}_s - \nabla \times \underline{w} = 0$ $\nabla \times \underline{e} = 0$	$\varepsilon^{-1} \underline{d} + \nabla V = 0$ $\nabla \cdot \underline{d} = \rho$

Table 1: Four equivalent electrostatic formulations.

2.2 Mixed finite element formulations

We will focus in this paper on the mixed $\underline{d} - V$ formulation and the mixed $\underline{h} - \underline{a}$ formulation. These formulations have the advantage of not requiring the construction of a source field.

	H	**B**
Potential	$\nabla\cdot(\mu(\underline{h}_s-\nabla\phi))=0$	$\nabla\times(\mu^{-1}\nabla\times\underline{a})=\underline{j}$
Mixed	$\mu\underline{h}-\nabla\times\underline{a}=0$ $\nabla\times\underline{h}=\underline{j}$	$\mu^{-1}\underline{b}-\underline{h}_s+\nabla\phi=0$ $\nabla\cdot\underline{b}=0$

Table 2: Four equivalent magnetostatic formulations.

In case of the $\underline{d}-V$ formulation, the unknown fields $\underline{d}$, V and the test fields $\underline{d}'$, V' are defined in the function spaces (2.4) and (2.5) respectively,

$$\mathrm{H}_d(\Omega)=\{\underline{d}\in\mathrm{H}(\mathrm{div};\Omega):\underline{n}\cdot\underline{d}|_{\Gamma_d}=d_{n0}\}\quad,\quad \mathrm{H}_V(\Omega)=\{V\in\mathrm{H}(\mathrm{grad};\Omega):V|_{\Gamma_e}=V_0\}\,,\qquad(2.4)$$
$$\mathrm{H}_{d0}(\Omega)=\{\underline{d}\in\mathrm{H}(\mathrm{div};\Omega):\underline{n}\cdot\underline{d}|_{\Gamma_d}=0\}\quad,\quad \mathrm{H}_{V0}(\Omega)=\{V\in\mathrm{H}(\mathrm{grad};\Omega):V|_{\Gamma_e}=0\}\,.\qquad(2.5)$$

Here, the functions d_{n0} and V_0 are known and specified along a boundary segment Γ_d and Γ_e respectively. The weak form, which is at the basis of the mixed finite element formulation, reads as follows:

Find $\underline{d}\in\mathrm{H}_d(\Omega)$ *and* $V\in\mathrm{H}_V(\Omega)$ *such that* $\forall\underline{d}'\in\mathrm{H}_{d0}(\Omega)$ *and* $\forall V'\in\mathrm{H}_{V0}(\Omega)$,

$$\int_\Omega \varepsilon^{-1}\underline{d}\cdot\underline{d}'\,\mathrm{d}\Omega-\int_\Omega V\,\nabla\cdot\underline{d}'\,\mathrm{d}\Omega = -\int_{\Gamma_V} V\,\underline{d}'\,\mathrm{d}\Gamma\,,\qquad(2.6)$$
$$\int_\Omega \nabla\cdot\underline{d}\,V'\,\mathrm{d}\Omega = \int_\Omega \rho V'\,\mathrm{d}\Omega\,.\qquad(2.7)$$

The fields $\underline{d}$ and V will typically be discretised with face elements (Fig. 1) and nodal elements respectively.
In case of the mixed $\underline{h}-\underline{a}$ formulation, the unknown fields $\underline{h}$ and $\underline{a}$ are defined in the function spaces (2.8) and (2.9) respectively,

$$\mathrm{H}_h(\Omega)=\{\underline{h}\in\mathrm{H}(\mathrm{curl};\Omega):\underline{n}\times\underline{h}|_{\Gamma_h}=\underline{h}_{t0}\}\quad,\quad \mathrm{H}_a(\Omega)=\{\underline{a}\in\mathrm{H}(\mathrm{curl};\Omega):\underline{n}\times\underline{a}|_{\Gamma_b}=\underline{a}_{t0}\}\,,\qquad(2.8)$$
$$\mathrm{H}_{h0}(\Omega)=\{\underline{h}\in\mathrm{H}(\mathrm{curl};\Omega):\underline{n}\times\underline{h}|_{\Gamma_h}=0\}\quad,\quad \mathrm{H}_{a0}(\Omega)=\{\underline{a}\in\mathrm{H}(\mathrm{curl};\Omega):\underline{n}\times\underline{a}|_{\Gamma_b}=0\}\,.\qquad(2.9)$$

The functions $\underline{h}_{t0}$ and $\underline{a}_{t0}$ are given along Γ_h and Γ_b respectively. The weak form reads:

Find $\underline{h}\in\mathrm{H}_h(\Omega)$ *and* $\underline{a}\in\mathrm{H}_a(\Omega)$ *such that* $\forall\underline{h}'\in\mathrm{H}_{h0}(\Omega)$ *and* $\forall\underline{a}'\in\mathrm{H}_{a0}(\Omega)$,

$$\int_\Omega \mu\,\underline{h}\cdot\underline{h}'\,\mathrm{d}\Omega-\int_\Omega \underline{a}\cdot\nabla\times\underline{h}'\,\mathrm{d}\Omega = \int_{\Gamma_b}\underline{a}\times\underline{h}'\,\mathrm{d}\Gamma\,,\qquad(2.10)$$
$$\int_\Omega \nabla\times\underline{h}\cdot\underline{a}'\,\mathrm{d}\Omega = \int_\Omega \underline{j}\cdot\underline{a}'\,\mathrm{d}\Omega\,.\qquad(2.11)$$

Both $\underline{h}$ and $\underline{a}$ are discretised with edge elements (Fig. 1).

3 A preconditioned MINRES solver for the indefinite system

The discretisation of the formulations (2.6-2.7) and (2.10-2.11) leads to symmetric indefinite algebraic systems of the form

$$\begin{bmatrix} K & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix}\,,\qquad(3.12)$$

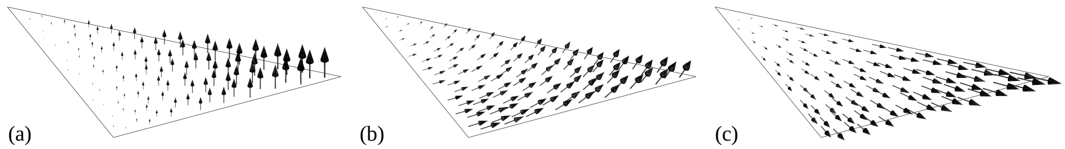

Figure 1: Example of 2D shape functions for (a) the vector potential $\underline{a}$, (b) the magnetic field $\underline{h}$ and (c) the electric displacement $\underline{d}$.

which is characteristic of mixed formulations. A system of the form (3.12) is called a saddle point problem. One of the iterative solvers that can efficiently solve such systems is MINRES, a Krylov subspace method based on the symmetric Lanczos procedure [2]. It requires a preconditioner preserving the symmetry of the original matrix. When (3.12) is preconditioned with the block diagonal matrix

$$\begin{bmatrix} K & 0 \\ 0 & S \end{bmatrix}, \tag{3.13}$$

where $S = BK^{-1}B^T$ is the Schur complement, MINRES converges in three iterations whatever the size of the problem [1]. In practice, however, constructing and inverting S is much too expensive and one must find a computable approximation $\hat{S}$ of S. The choice of $\hat{S}$, which is critical to the performance of the preconditioner (3.13), is strongly problem-dependent.

The preconditioner suggested in this paper is constructed by analogy to a preconditioner that is used succesfully for a mixed formulation of the Stokes problem in fluid dynamics. For the velocity-pressure formulation of the Stokes equations, the pressure mass matrix M_p is an 'optimal' approximation of $\hat{S}$ if the interpolation functions of the velocity and pressure satisfy the Ladyzhenskaya-Babuška-Brezzi (LBB) condition [2]. With use of that preconditioner, the number of MINRES iterations will be higher than 3, but it will be bounded by a finite (small) number independent of the size of the problem. This important property results from the spectral equivalence between S and M_p [1]. The following definition of spectral equivalence is taken from [4].

Definition 3.1 *Let $H \subset (0,\infty)$ be an index set with $0 \in \overline{H}$ (e.g., H: set of all grid sizes). If $\{A_h\}_{h\in H}$ and $\{B_h\}_{h\in H}$ are two families of regular matrices, then $\{A_h\}_{h\in H}$ and $\{B_h\}_{h\in H}$ are called spectrally equivalent if there is a constant C independent of $h \in H$ such that* $\mathrm{cond}_2(B_h^{-1}A_h) \leq C, \forall h \in H$.

For the electrostatic and magnetostatic cases, appropriate preconditioners $\hat{S}_{\mathrm{elec}}$ and $\hat{S}_{\mathrm{magn}}$ to the corresponding Schur complement are necessary. The expressions proposed in this article are

$$\hat{S}_{\mathrm{elec}} = \int_\Omega \varepsilon \nabla V \cdot \nabla V' \,\mathrm{d}\Omega\,, \tag{3.14}$$

$$\hat{S}_{\mathrm{magn}} = \int_\Omega \mu^{-1} \nabla \times \underline{a} \cdot \nabla \times \underline{a}' \,\mathrm{d}\Omega\,, \tag{3.15}$$

which correspond respectively to the weak form of the scalar potential V and the vector potential $\underline{a}$ formulations. In order to conserve the mesh-independent rate of convergence that was obtained for the Stokes problem, the Schur complement and its approximations $\hat{S}_{\mathrm{elec}}$ and $\hat{S}_{\mathrm{magn}}$ must be spectrally equivalent. Proving that such a relation holds for (3.14-3.15) is not an easy task. A simplified numerical test has been adopted here: the expressions $\mathrm{cond}_2(\hat{S}_{\mathrm{elec}}^{-1}S)$ and $\mathrm{cond}_2(\hat{S}_{\mathrm{mag}}^{-1}S)$ have been computed for a set of meshes of the 2D models shown in Fig. 2 with an increasing number of elements (Table 3). The results show that $\mathrm{cond}_2(\hat{S}^{-1}S)$ is not proportional to the size of the mesh. It seems therefore reasonable to assume that spectral equivalence between S and its approximations (3.14-3.15) has been achieved.

Size(S)	475	546	747	933	1238	1670
$\mathrm{cond}_2(\hat{S}_{\mathrm{elec}}^{-1} S)$	53.6	64.2	80.8	75.6	73.8	96.7
Size(S)	253	303	373	508	705	963
$\mathrm{cond}_2(\hat{S}_{\mathrm{mag}}^{-1} S)$	86.8	106.2	143.2	120.1	86.4	130.5

Table 3: Computation of $\mathrm{cond}_2(\hat{S}^{-1} S)$ for a set of 2D meshes.

4 Stabilised mixed formulations

4.1 *Pressure-Stabilised Petrov-Galerkin*

Mixed finite element formulations are subject to the LBB stability condition, which limits the choice of the shape functions for the unknown fields. The best example is the velocity-pressure formulation of the Stokes equations: if the velocity $\underline{v}$ and the pressure p are both discretised with first order shape functions, the pressure solution is contaminated by oscillations. To avoid this problem, one should discretise $\underline{v}$ with second order basis functions. Alternatively, one can use the Pressure-Stabilised Petrov-Galerkin (PSPG) formulation: due to an additional term in the test functions for the momentum equation,

$$\underline{v}' \to \underline{v}' + \tau_e \nabla p' , \tag{4.16}$$

equal order shape functions can be used for $\underline{v}$ and p. The choice of the free parameter τ_e determines the accuracy of the formulation. Several expressions of τ_e exist for the Stokes problem: they depend on parameters such as the local mesh length and the viscosity of the fluid. These empirical functions are not suited for the mixed formulations of Maxwell's equations which have a different structure and different properties. Recently, expressions for τ_e based on the element matrices of the finite element formulation have been proposed in [7]. It has been shown that such formulae perform well for the Navier-Stokes equations. In this paper, the same approach is applied to the Maxwell equations.
The test functions $\underline{h}'$ and $\underline{d}'$ are modified in a similar way as (4.16),

$$\underline{h}' \quad \to \quad \underline{h}' + \tau_{e,\mathrm{mag}} \nabla \times \underline{a}' , \tag{4.17}$$
$$\underline{d}' \quad \to \quad \underline{d}' + \tau_{e,\mathrm{elec}} \nabla V' . \tag{4.18}$$

Applying (4.17) to the $\underline{h} - \underline{a}$ formulation, one finds the stabilised magnetostatic formulation:

Find $\underline{h} \in \mathrm{H}_h(\Omega)$ *and* $\underline{a} \in \mathrm{H}_a(\Omega)$ *such that* $\forall \underline{h}' \in \mathrm{H}_{h0}(\Omega)$ *and* $\forall \underline{a}' \in \mathrm{H}_{a0}(\Omega)$,

$$\int_\Omega \left(\underline{h}' \cdot (\mu \underline{h}) - \nabla \times \underline{h}' \cdot \underline{a} + \tau_{e,\mathrm{mag}} \nabla \times \underline{a}' \cdot (\mu \underline{h} - \nabla \times \underline{a}) \right) \mathrm{d}\Omega = \int_{\Gamma_b} \underline{a} \times \underline{h}' \,\mathrm{d}\Gamma , \tag{4.19}$$
$$- \int_\Omega \underline{a}' \cdot \nabla \times \underline{h} \,\mathrm{d}\Omega = - \int_\Omega \underline{a}' \cdot \underline{j} \,\mathrm{d}\Omega . \tag{4.20}$$

The stabilised electrostatic formulation is obtained by applying (4.18) to the $\underline{d} - V$ formulation:

Find $\underline{d} \in \mathrm{H}_d(\Omega)$ *and* $V \in \mathrm{H}_V(\Omega)$ *such that* $\forall \underline{d}' \in \mathrm{H}_{d0}(\Omega)$ *and* $\forall V' \in \mathrm{H}_{v0}(\Omega)$,

$$\int_\Omega \left(\varepsilon^{-1} \underline{d}' \cdot \underline{d} - \nabla \cdot \underline{d}' V + \tau_{e,\mathrm{elec}} \nabla V' \cdot (\varepsilon^{-1} \underline{d} + \nabla V) \right) \mathrm{d}\Omega = - \int_{\Gamma_V} V \underline{d}' \,\mathrm{d}\Gamma , \tag{4.21}$$
$$- \int_\Omega V' \nabla \cdot \underline{d} \,\mathrm{d}\Omega = - \int_\Omega V' \rho_c \,\mathrm{d}\Omega . \tag{4.22}$$

The additional term in the test functions modifies the structure of the system (3.12) which becomes

$$\begin{bmatrix} K & B^T \\ B+\tau_e C & \tau_e D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g+\tau_e h \end{bmatrix}. \tag{4.23}$$

The matrix is not symmetric anymore but it is now definite thanks to the term $\tau_e D$ on the diagonal.

4.2 Stabilisation parameter

The stabilisation parameter proposed in [7] is defined in such a way that the entries in the additional blocks C and D in (4.23) are of the same order of magnitude as those in B. To achieve this, the expression of τ_e is defined by

$$\tau_e = \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)^{-1}, \quad \text{with} \quad \tau_1 = \frac{|B|_e}{|C|_e}, \quad \tau_2 = \frac{|B|_e}{|D|_e}, \tag{4.24}$$

where $|.|_e$ is the order of magnitude of the entries of the block evaluated in element e. Applied to the magnetostatic formulation, this yields

$$\tau_{1,\text{mag}} = \frac{|\int_{\Omega_e} \nabla \times \underline{h}' \cdot \underline{a}\, d\Omega|}{|\int_{\Omega_e} \nabla \times \underline{a}' \cdot (\mu\, \underline{h})\, d\Omega|}, \quad \tau_{2,\text{mag}} = \frac{|\int_{\Omega_e} \nabla \times \underline{h}' \cdot \underline{a}\, d\Omega|}{|\int_{\Omega_e} \nabla \times \underline{a}' \cdot (\nabla \times \underline{a})\, d\Omega|}, \tag{4.25}$$

and for the electrostatic formulation, one has

$$\tau_{1,\text{elec}} = \frac{|\int_{\Omega_e} \nabla \cdot \underline{d}' V\, d\Omega|}{|\int_{\Omega_e} \varepsilon^{-1} \underline{d} \cdot \nabla V'\, d\Omega|}, \quad \tau_{2,\text{elec}} = \frac{|\int_{\Omega_e} \nabla \cdot \underline{d}' V\, d\Omega|}{|\int_{\Omega_e} \nabla V' \cdot \nabla V\, d\Omega|}. \tag{4.26}$$

The magnitude of the integrals on a mesh element Ω_e is evaluated as the product of a characteristic magnitude of the integrand times the volume of the element. The characteristic magnitude of the nodal shape functions N_i, the edge elements $\underline{E}_i$, the face elements $\underline{F}_i$ and their derivatives are $|N_i| = 1$, $|\underline{E}_i| = l_e^{-1}$, $|\underline{F}_i| = l_e^{-2}$, $|\nabla N_i| = l_e^{-1}$, $|\nabla \times \underline{E}_i| = l_e^{-2}$, $|\nabla \cdot \underline{F}_i| = l_e^{-3}$, where l_e is a characteristic length of the element Ω_e, such as the diameter of the circumscribed circle. The resulting expressions of τ_e read

$$\tau_{e,\text{elec}} = \left(\frac{1}{\varepsilon} + l_e\right)^{-1}, \tag{4.27}$$

$$\tau_{e,\text{mag}} = \left(\mu + \frac{1}{l_e}\right)^{-1}. \tag{4.28}$$

5 Numerical results

5.1 Two-dimensional results

The different methods have been tested on two problems with a simple geometry, so as to be able to progressively refine the mesh to study the convergence properties. The electrostatic model represents a 3-phase cable where each of the three internal wires is surrounded by two layers of insulating material. The magnetostatic model is a classical iron C-core with a coil wound around the central part and an iron block in the airgap (Fig. 2). The material parameters ε and μ are discontinuous; their minimum and maximum values are in a ratio of 1 to 100 for ε and 1 to 10^4 for μ. Both problems have been solved with MINRES, first without preconditioner and then with the block preconditioner (3.13) making use of the approximations (3.14) or (3.15). The number of MINRES iterations is given in Table 4. It is larger than

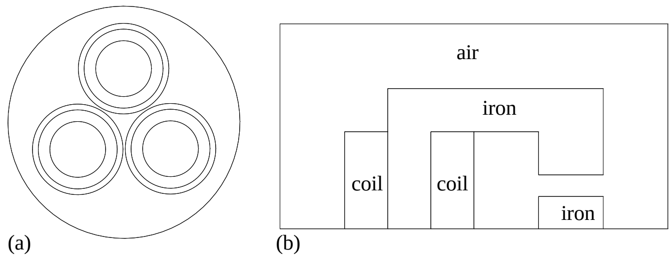

Figure 2: Geometry of (a) the 2D electrostatic problem and (b) the 2D magnetostatic problem.

3 due to the approximated Schur complement but it remains acceptable and more importantly, it does not increase with the size of the problem.

An attempt has been made to use a diagonal preconditioner for the K-block, which is a mass matrix of edge and face elements respectively in the magnetic and electric cases. The results show that the convergence of MINRES is strongly influenced by the approximation of K, at least for the electrostatic problem.

Electrostatic			**Magnetostatic**		
K (14 930 unk.)	**S** (5 120 unk.)	**# iter**	**K** (13 600 unk.)	**S** (4 630 unk.)	**# iter**
no precond.	*no precond.*	2144	*no precond.*	*no precond.*	1768
K	$\hat{S}_{\text{elec}}$	106	K	$\hat{S}_{\text{magn}}$	71
$\text{diag}(K)$	$\hat{S}_{\text{elec}}$	1228	$\text{diag}(K)$	$\hat{S}_{\text{magn}}$	122

Table 4: Number of iterations of MINRES with different block preconditioners, applied to the 2D electrostatic and magnetostatic problems (formulations (2.6-2.7) and (2.10-2.11) respectively).

5.2 *Three-dimensional results*

5.2.1 Electrostatic

The 3D models are constructed by extruding the 2D models in Fig. 2 in the direction perpendicular to the plane. In 3D, the number of unknowns of $\underline{d}$ and V is equal respectively to the number of faces and the number of nodes. In our examples, the former is typically 3 to 10 times larger than the latter. System (3.12) has been solved with MINRES and the block preconditioner (3.13). The approximation (3.14) of the Schur complement performs in 3D equally well as in 2D: Table 5 shows that the spectral equivalence can also be assumed in 3D.

Size(S)	535	586	674	741	871	1005	1155	1365
$\text{cond}_2(\hat{S}_{\text{elec}}^{-1} S)$	370	363	162	221	221	193	239	212

Table 5: Value of $\text{cond}_2(\hat{S}_{\text{elec}}^{-1} S)$ for a set of 3D meshes.

CG tolerance (K)	CG tolerance (S)	# MINRES iterations
10^{-10}	10^{-10}	123
10^{-10}	10^{-4}	123
10^{-6}	10^{-4}	250

Table 6: Number of MINRES iterations with different preconditioners for the 3D electrostatic problem.

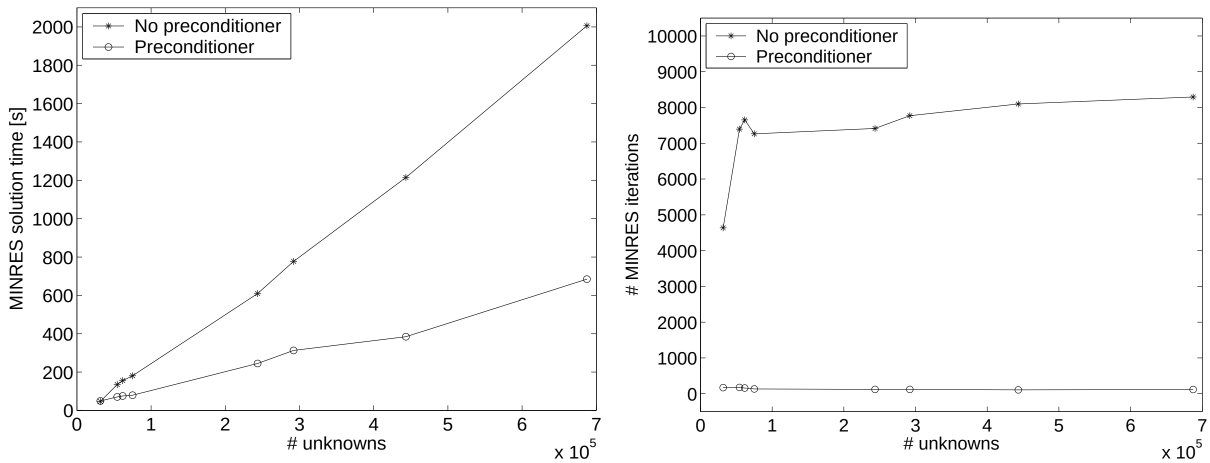

Figure 3: Solution time (left) and number of iterations (right) of the 3D electrostatic problem solved with MINRES, with and without preconditioner, for different meshes.

The preconditioner (3.13) uses CG steps for both K and S blocks. The number of CG steps must be larger for K than for S because the number of MINRES iterations is much more dependent on the former, as outlined in Table (6). For the examples studied in this paper, the most efficient preconditioner is obtained by taking the CG tolerance equal to 10^{-10} for K and 10^{-2} for S. The solution time and the number of iterations of MINRES with and without preconditioner are plotted in Fig. 3 for different meshes.
No stability problems have been encountered with the mixed electrostatic formulation. The PSPG formulation (4.21-4.22) has nevertheless been solved in order to compare its solution time with that of the Galerkin formulation. For each system, the best available combination of preconditioner and solver has been used: MINRES and the block preconditioner described above for the non-stabilised system (Galerkin formulation), BiCGstab and ILU for the non-symmetric stabilised system (PSPG formulation). Fig. 4 shows that the non-stabilised system is solved much faster than the stabilised one. A well-designed block preconditioner for the stabilised system would probably be more efficient than ILU. However, in the absence of stability problems, it is simpler and probably faster anyway to solve the indefinite system resulting from the Galerkin formulation.

*5.2.*2 Magnetostatic

In 3D magnetostatic problems, both $\underline{h}$ and $\underline{a}$ are discretised with edge elements and have therefore the same number of degrees of freedom. This is different from the 2D case, where the vector potential $\underline{a}$ is perpendicular to the plane and is discretised with nodal shape functions. This explains the oscillations of $\underline{a}$ that are observed in 3D but not in 2D. Similar oscillations of the pressure occur in Stokes problems when equal-order shape functions are used for $\underline{v}$ and p. Stabilisation is therefore necessary for the $\underline{h}-\underline{a}$ formulation since both $\underline{h}$ and $\underline{a}$ are discretised with first order edge elements. The system resulting

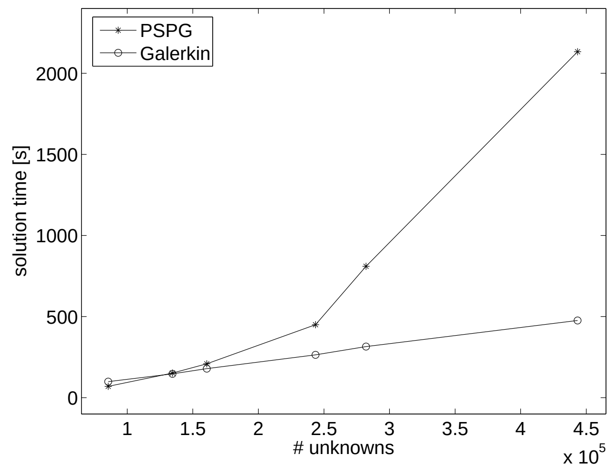

Figure 4: Comparison of the solution time of the 3D electrostatic problem with the stabilised formulation (PSPG) solved with ILU-BiCGstab and the non-stabilised formulation (Galerkin) solved with MINRES and a block preconditioner.

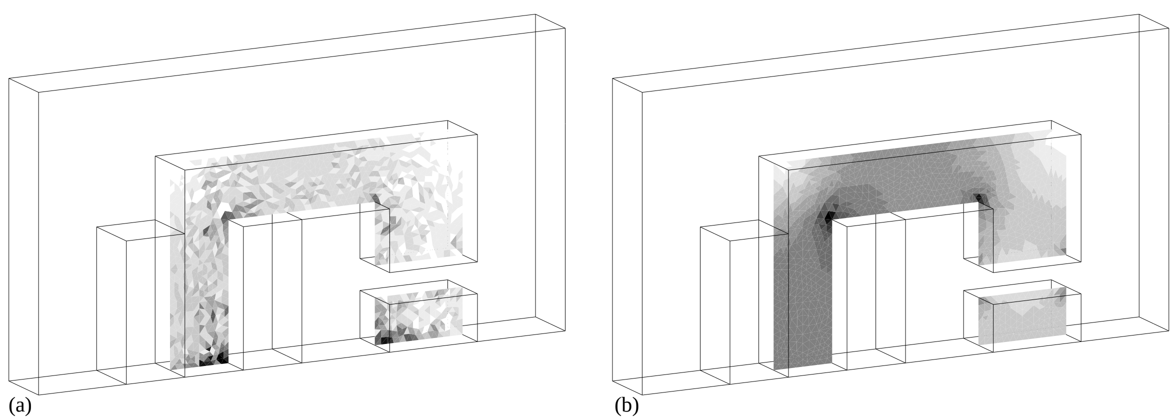

Figure 5: 2D view of $\|\nabla \times \underline{a}\|$ in the iron domain for the 3D magnetostatic problem, (a) without stabilisation (Galerkin) and (b) with stabilisation (PSPG).

from the PSPG magnetostatic formulation (4.19-4.20) has been solved. The expression (4.28) of the τ_e parameter results in an efficient stabilisation of the problem as illustrated in Fig. 5.
As before, the iterative solution of the stabilised system is problematic. Beyond a certain problem size the ILU-BiCGstab solver fails to converge in an acceptable number of iterations. A more efficient (block-)preconditioner is called for, using, e.g., a number of CG iterations for the upper diagonal block K (a mass matrix of edge elements) and a multigrid cycle for the lower diagonal block.

6 Conclusion

This paper focuses on the solution of electrostatic and magnetostatic mixed formulations. Solution methods originally developed for the velocity-pressure formulation of the Navier-Stokes equations are adapted to the Maxwell equations.
A block preconditioner used in combination with MINRES to solve symmetric indefinite systems arising from mixed formulations is described. One component of this preconditioner is the Schur complement

of the system, which must be approximated. The choice of a sensible approximation is critical to the efficiency of the method and it is strongly problem-dependent. Approximations are proposed for the electrostatic and magnetostatic formulations. It is shown that they are spectrally equivalent to the exact Schur complement, which ensures that the rate of convergence of MINRES is mesh-independent.
Pressure-stabilised Petrov-Galerkin formulations of the electrostatic and magnetostatic equations are proposed. Appropriate expressions of the stabilisation parameter τ_e are found by application of the method developed in fluid mechanics.
The efficiency of the block preconditioner is first tested with 2D simulations. According to our expectations, the number of MINRES iterations is small and nearly constant whatever the size of the problem. In 3D, the preconditioner is equally efficient for the electrostatic problem. The simulations also show that it is faster to solve the indefinite system with MINRES and the appropriate preconditioner than the stabilised (definite) system with ILU and BiCGstab. In the 3D magnetostatic case, the presence of oscillations indicate that the stabilised formulation must be used. The expression of the parameter τ_e proposed in this paper gives satisfactory results.

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