Full waveform inversion in the frequency domain

Truncated Newton methods for simultaneous permittivity and conductivity reconstruction

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Problem statement

Consider

- a model m
- a wave propagation operator $\mathcal{F}$
- a wavefield u
- a mesurement operator $\mathcal{R}$
- a dataset d

Full wave inversion consists in finding m^* such that

$$\mathcal{R}(u) = d \text{ with } \mathcal{F}(m^*)u = f$$

through the **optimization problem**

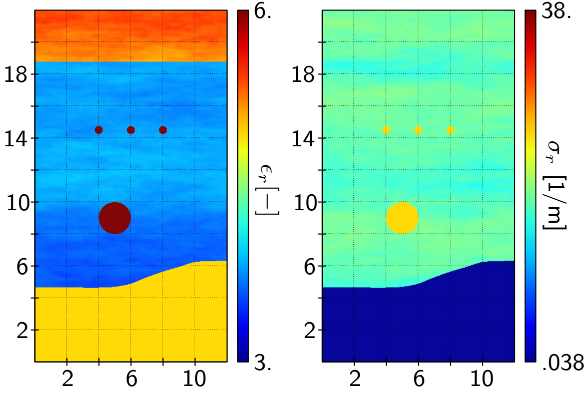
$$m^* = \arg\min_m \mathcal{J}(m) \triangleq \arg\min_m \text{dist}(\mathcal{R}(u(m)), d)$$

Crosshole radar example

Model space

The unknown model is here composed of both **permittivity** and **conductivity** distributions, *i.e.*

$$m \triangleq (\epsilon_r(\boldsymbol{x}), \sigma_r(\boldsymbol{x})).$$



Typical distributions[1] features highly contrasted targets ($\epsilon_{r,\text{pipe}} = 80$) embedded in a layered background with stochastic fluctuations.

[1]Ernst et al., "Full-Waveform Inversion of Crosshole Radar Data Based on 2-D Finite-Difference Time-Domain Solutions of Maxwell's Equations".

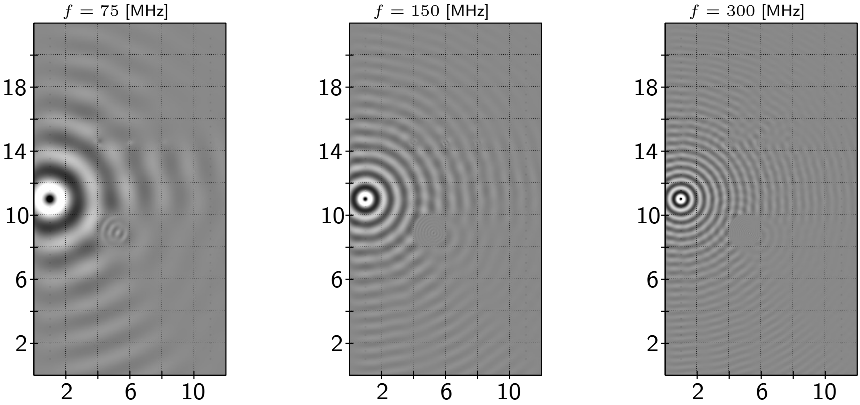
Wavefield space

In electromagnetics (2D TE), the wavefield is an **electric field** whose propagation can be modelled by the **Helmholtz equation**, *i.e.*

$$u \triangleq e(\boldsymbol{x})$$

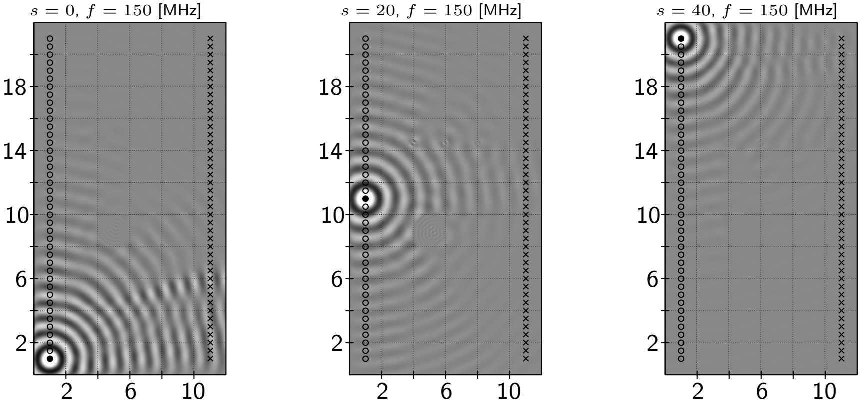
and

$$\mathcal{F}(u, m) \triangleq \Delta e + k^2 \left(\epsilon_r - j \frac{\sigma_r}{k}\right) e$$



Data space I

Several **emitters** (○) are buried in a first hole. Each emitter (•) is successively excited and the response is recorded at all the **receivers** (×), buried in another hole.



The response is recorded for several frequencies.

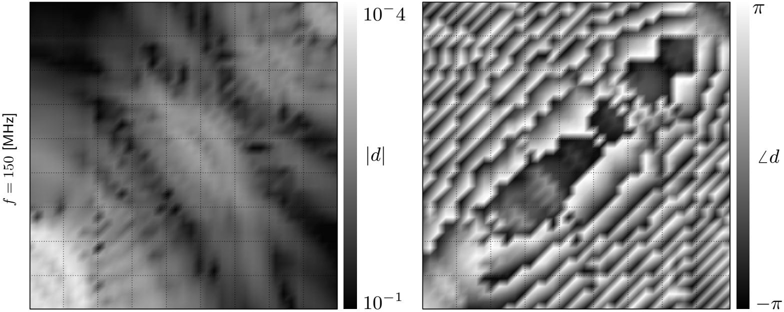
Data space II

A dataset d is thus a $n_s \times n_r \times n_f$ **complex-valued matrix**, *i.e*

$$d \in \mathbb{C}^{n_s \times n_r \times n_f}$$

which is obtained by **pointwise** measurements at the receivers, *i.e.*

$$[\mathcal{R}(u)]_{s,r,f} \triangleq e_{s,f}(\boldsymbol{x}_r).$$



Minimization problem

Descent direction I

Local optimization techniques must be used in practice because the search space is large (*e.g.* $\epsilon_r(\boldsymbol{x})$ and $\sigma_r(\boldsymbol{x})$).

Most algorithms are built on an approximation of the **Newton direction**

$$p \approx -\mathcal{H}^{-1}(j')$$

with j' the gradient and $\mathcal{H}$ the Hessian operator.

Both kernels result from the Taylor expansion of the performance functional

$$\begin{aligned}\mathcal{J}(m+\delta m) &\approx \mathcal{J}(m) + \{D_m\mathcal{J}\}(\delta m) + \frac{1}{2}\{D^2_{mm}\mathcal{J}\}(\delta m, \delta m)\\ &\approx \mathcal{J}(m) + \left\langle j', \delta m\right\rangle_M \quad + \frac{1}{2}\left\langle \mathcal{H}(\delta m), \delta m\right\rangle_M\end{aligned}$$

and thus **strongly depend on the chosen inner product** $\langle , \rangle_M$.

Descent direction II

Gradients and **Hessian operators** are defined by

$$\{D_m \mathcal{J}\}(\delta m) \triangleq \langle j', \delta m \rangle_M , \forall \delta m$$

and

$$\{D^2_{mm} \mathcal{J}\}(\delta m_1, \delta m_2) \triangleq \langle \mathcal{H}(\delta m_1), \delta m_2 \rangle_M , \forall \delta m_1 \forall \delta m_2$$

Changing the inner product is equivalent to **applying a preconditionner** to both kernels[2]. Preconditioning does not change the exact Newton direction but does change approximate solutions.

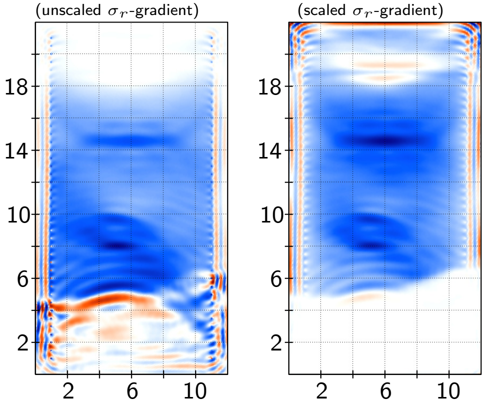
Incorporating **prior knowledge** in this inner product to encourage desired features can thus yield better approximate directions.

[2]Zuberi and Pratt, "Mitigating nonlinearity in full waveform inversion using scaled-Sobolev pre-conditioning".

Inner product examples

Example 1: spatial scaling

$$\langle \cdot, \cdot \rangle_M = \left\langle \cdot \sqrt{h_a(\boldsymbol{x})}, \sqrt{h_a(\boldsymbol{x})} \cdot \right\rangle_{L_2}$$



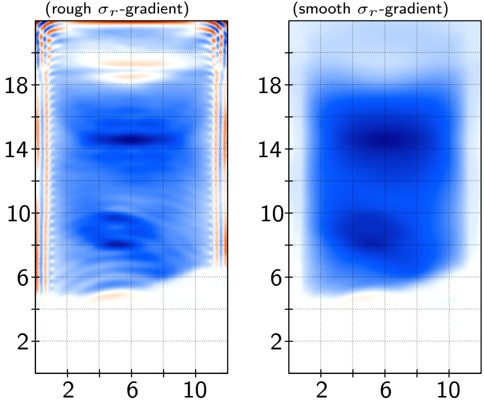
Consider an appropriate **spatial weight**, *e.g.* the diagonal of the Gauss-Newton Hessian[3] $h_a(\boldsymbol{x})$.

This inner product **restores balance** between gradient contributions everywhere.

[3] Pan, Innanen, and Liao, "Accelerating Hessian-free Gauss-Newton full-waveform inversion via l-BFGS preconditioned conjugate-gradient algorithm".

Example 2: spatial smoothing

$$\langle \cdot, \cdot \rangle_M = \left\langle \cdot \sqrt{h_a(\boldsymbol{x})}, \sqrt{h_a(\boldsymbol{x})} \cdot \right\rangle_{L_2} + \mu \left\langle \boldsymbol{\nabla} \cdot, \boldsymbol{\nabla} \cdot \right\rangle_{L_2}$$



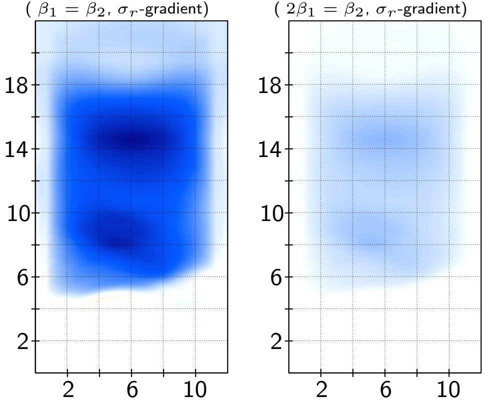
Adding such a **regularization term** penalizes rough models. Kernels w.r.t this inner product are therefore smoother.

Encouraging **smooth updates** early in inversion processes is a strategy to avoid local minima trapping[4].

[4] Zuberi and Pratt, "Mitigating nonlinearity in full waveform inversion using scaled-Sobolev pre-conditioning".

Example 3: parameter weighting

$$\langle m_a, m_b \rangle_M = \beta_1^2 \langle \epsilon_a, \epsilon_b \rangle + \beta_2^2 \langle \sigma_a, \sigma_b \rangle$$



The weight β governs the **relative importance** given to permittivities and conductivities.

By changing its value, it is possible to **reconstruct one parameter first** and again to avoid local minima trapping[5].

[5] Lavoué et al., "Two-dimensional permittivity and conductivity imaging by full waveform inversion of multioffset GPR data: a frequency-domain quasi-Newton approach".

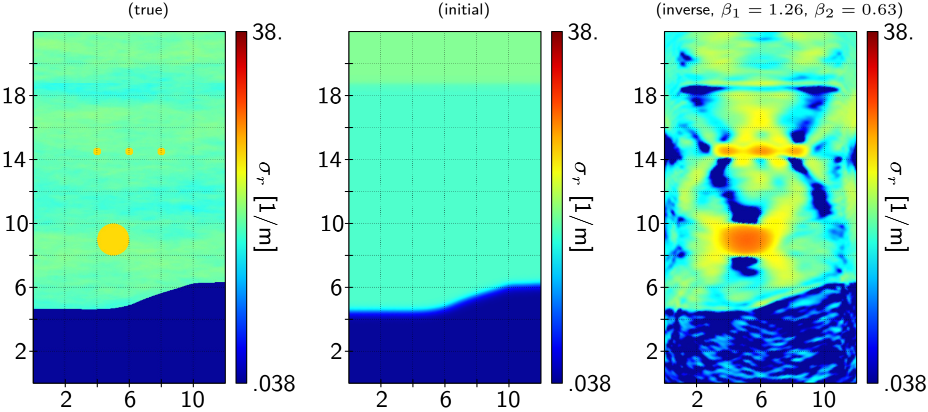
Crosshole radar application (synthetic)

Application choices

- 41 emitters and 41 receivers (spacing $\lessapprox$ background wavelength)
- 7 frequencies ($75, 90, 120, 150, 180, 225, 300$ [MHz])
- Individual/Sequential inversion from low to high frequency
- Minimization by a trust-region Newton method[6]
- Scaled, smoothed and weighted inner products
- Distance between data measured using least square norm
- Forward problems solved using finite elements
- Measured data generated synthetically (*i.e* with a numerical model)

[6] Adriaens, Métivier, and Geuzaine, "A trust-region Newton method for frequency-domain full waveform inversion Trust-Region Newton Method".

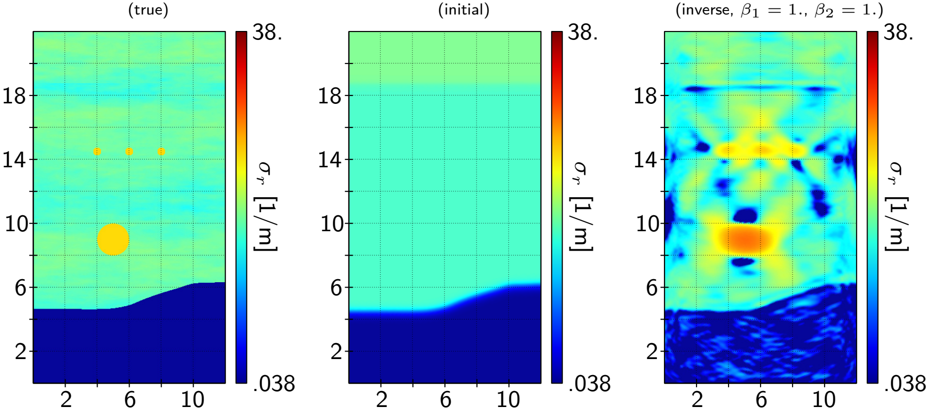
Results: conductivity



- Locations are correct
- Amplitudes are wrong
- Background is not resolved
- Background features artifacts

Reconstruction quality depends on β

Results: conductivity



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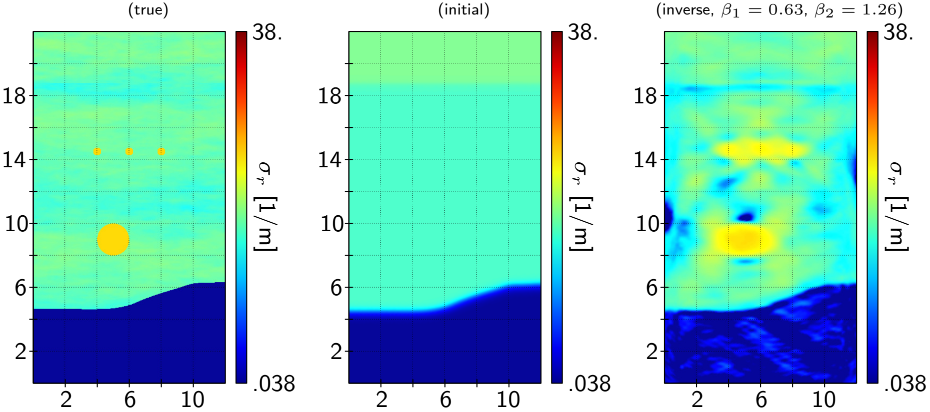

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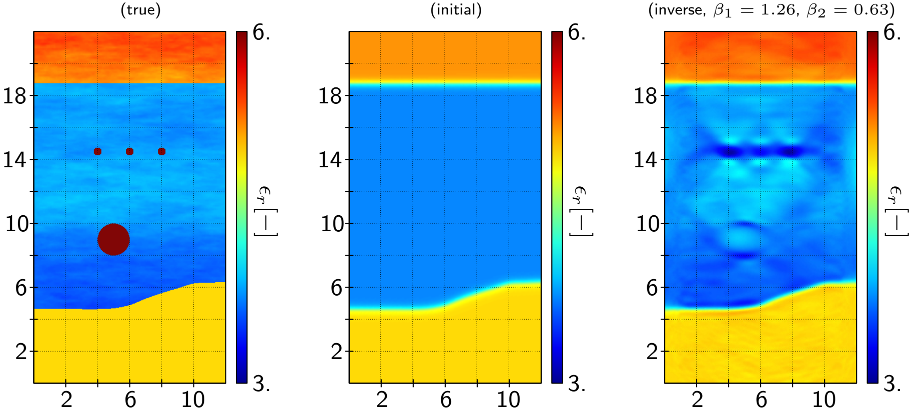
Results: permittivity



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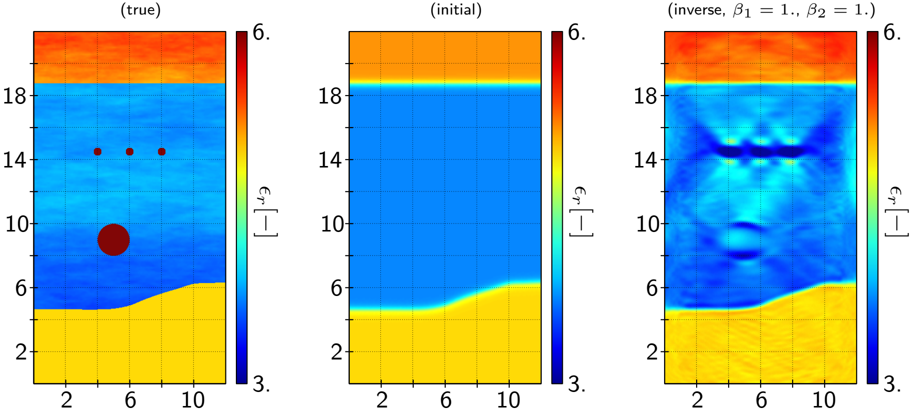

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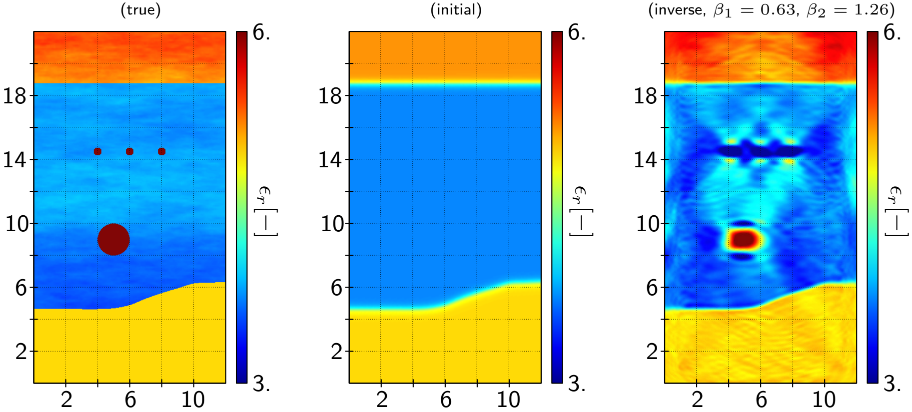

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Conclusion

1. Pure Newton methods are attractive but prohibitively expensive
2. Truncated Newton methods must be used instead
3. Truncated Newton methods however require a preconditioning strategy
4. Modifying the inner product is an elegant preconditioning strategy
5. Modifying the inner product should be investigated further, especially to retrieve highly contrasted media

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Thank you for your attention