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A Riesz basis of wavelets and its dual with quintic deficient splines

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Abstract. In this note, the dual of the Riesz basis of quintic splines wavelets obtained in [1] is explicitely constructed.

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Introduction

It is well known that for any natural number m, the cardinal B-spline $N_{m+1} = \chi_{[0,1]} * \dots * \chi_{[0,1]}$ ($m+1$factors) can be used as a scaling function to construct orthogonal and biorthogonal bases of wavelets in $L^2(\mathbb{R})$, with different properties (see for example [3], [8]).

But, in approximation theory for instance, other splines are also very popular: the deficient splines (see some recent results in [5], [9]). In the paper [1], one can find a direct approach of the problem of the explicit construction of scaling functions, multiresolution analysis and wavelets with symmetry properties and compact support, involving deficient splines of degree 5 and regularity 3. Other results can also be found in [6], [7].

The present paper is a continuation of [1]. It gives an explicit construction of the dual basis of the deficient splines wavelets basis obtained in [1]. The dual is also generated by two wavelets, which are deficient splines with symmetry properties and exponential decay.

1 Definitions, notations, deficient spline wavelets

For $m \in \mathbb{N}$, the set of deficient splines of degree $2m+1$ is the set

$$V_0 = \{f \in L_2(\mathbb{R}): \ f|_{[k,k+1]} = P_k^{(2m+1)}, k \in \mathbb{Z} \text{and } f \in C_{m+1}(\mathbb{R})\}.$$

For $m = 1$, it is the set of classical cardinal cubic splines. For $\boxed{m = 2}$ we denote it as the set of

$$\boxed{\textit{deficient quintic splines}}$$

and, in this note, we only consider this case.

In this section, we recall the explicit and direct construction of a Riesz basis of wavelets consisting of deficient splines wavelets with compact support and symmetry property of [1].

1 Proposition. *The following functions φ_a andφ_s*

$$\varphi_a(x) = \begin{cases} x^4 - \frac{11}{15}x^5 & \text{if } x \in [0,1] \\ -\frac{9}{8}(x-\frac{3}{2}) + 3(x-\frac{3}{2})^3 - \frac{38}{15}(x-\frac{3}{2})^5 & \text{if } x \in [1,2] \\ -(3-x)^4 + \frac{11}{15}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

$$\varphi_s(x) = \begin{cases} x^4 - \frac{3}{5}x^5 & \text{if } x \in [0,1] \\ \frac{57}{80} - \frac{3}{2}(x-\frac{3}{2})^2 + (x-\frac{3}{2})^4 & \text{if } x \in [1,2] \\ (3-x)^4 - \frac{3}{5}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

are respectively antisymmetric and symmetric with respect to$\frac{3}{2}$ and the family

$$\{\varphi_a(.-k),\ k \in \mathbb{Z}\} \cup \{\varphi_s(.-k),\ k \in \mathbb{Z}\}$$

constitutes a Riesz basis of V_0.$\square$

For every $j \in \mathbb{Z}$ we define

$$V_j = \{f \in L^2(\mathbb{R}) \ :\ f(2^{-j}.) \in V_0\}.$$

2 Proposition. *The sequence V_j $(j \in \mathbb{Z})$ is an increasingsequence of closed sets of $L^2(\mathbb{R})$ and*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

Moreover, the functions φ_a, φ_s satisfy the followingscaling relation

$$\begin{pmatrix} \widehat{\varphi_s}(2\xi) \\ \\ \widehat{\varphi_a}(2\xi) \end{pmatrix} = M_0(\xi) \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \\ \widehat{\varphi_a}(\xi) \end{pmatrix}$$

where $M_0(\xi)$ is the matrix (called filter matrix)

$$M_0(\xi) = \frac{e^{-3i\xi/2}}{64} \begin{pmatrix} 51\cos(\frac{\xi}{2}) + 13\cos(\frac{3\xi}{2}) & -9i(\sin(\frac{\xi}{2}) + \sin(\frac{3\xi}{2})) \\ \\ i(11\sin(\frac{3\xi}{2}) + 21\sin(\frac{\xi}{2})) & -7\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) \end{pmatrix}.$$

$\square$

For every $j \in \mathbb{Z}$, we denote by W_j the orthogonal complement of V_j in V_{j+1}. Using standard techniques of Fourier analysis in the context of wavelets, one obtains the following result.

3 Proposition. *A function f belongs to W_0 if and only ifthere existsp,* $q \in L^2_{loc}$*, $2\pi-$ periodic such that*

$$\widehat{f}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi)$$

and

$$\overline{M_0(\xi)}\ \overline{W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}\ \overline{W(\xi+\pi)} \begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \ \ a.e.$$

where M_0 is the filter matrix obtained in Proposition 2 and $W(\xi)$ is the matrix

$$W(\xi) = \begin{pmatrix} \omega_s(\xi) & \omega_m(\xi) \\ \overline{\omega_m(\xi)} & \omega_a(\xi) \end{pmatrix}$$

with

$$\begin{aligned}
\omega_a(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_a}(\xi+2l\pi)|^2 = \frac{23247 - 21362\cos\xi - 385\cos(2\xi)}{311850} \\
\omega_s(\xi) &= \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_s}(\xi+2l\pi)|^2 = \frac{14445 + 7678\cos\xi + 53\cos(2\xi)}{34650} \\
\omega_m(\xi) &= \sum_{l=-\infty}^{+\infty} \widehat{\varphi_s}(\xi+2l\pi)\overline{\widehat{\varphi_a}(\xi+2l\pi)} = -\frac{i}{51975}\sin\xi\ (6910 + 193\cos\xi).
\end{aligned}$$

□

4 Theorem. *There exists deficient splines wavelets with support in*$[0,5]$ *and symmetry properties (with respect to $\frac{5}{2}$).*

More precisely, there exists real numbers

$$p_j^{(s)}, q_j^{(s)}, p_j^{(a)}, q_j^{(a)}, \quad j = 0, \ldots, 7$$

verifying

$$p_j^{(s)} = p_{7-j}^{(s)},\ q_j^{(s)} = -q_{7-j}^{(s)}, \quad p_j^{(a)} = -p_{7-j}^{(a)},\ q_j^{(a)} = q_{7-j}^{(a)}, \qquad j = 0,1,2,3$$

such that the family $\{\psi_s(.-k) :\ k \in \mathbb{Z}\} \cup \{\psi_a(.-k) : k \in \mathbb{Z}\}$ constitute a Riesz basis of W_0, where

$$\widehat{\psi_s}(2\xi) = \sum_{j=0}^{7} p_j^{(s)} e^{-ij\xi}\widehat{\varphi_s}(\xi)\ +\ \sum_{j=0}^{7} q_j^{(s)} e^{-ij\xi}\widehat{\varphi_a}(\xi)$$

$$\widehat{\psi_a}(2\xi) = \sum_{j=0}^{7} p_j^{(a)} e^{-ij\xi}\widehat{\varphi_s}(\xi) \ + \ \sum_{j=0}^{7} q_j^{(a)} e^{-ij\xi}\widehat{\varphi_a}(\xi).$$

$\square$

Explicit values of the coefficients can be found in [1]. It follows that the family

$$\boxed{\{2^{j/2}\psi_s(2^j.-k) : j,k \in \mathbb{Z}\} \cup \{2^{j/2}\psi_a(2^j.-k) : j,k \in \mathbb{Z}\}} \qquad (*)$$

constitute a Riesz basis of $L^2(\mathbb{R})$ of deficient splines wavelets with compact support and symmetry properties. The symmetry properties can be written as follows

$$\widehat{\widehat{\psi_s}}(\xi) = e^{-5i\xi}\widehat{\widehat{\psi_s}}(-\xi), \quad \widehat{\widehat{\psi_a}}(\xi) = -e^{-5i\xi}\widehat{\widehat{\psi_a}}(-\xi).$$

Here are pictures of φ_s, φ_a

and of ψ_s, ψ_a (up to a multiplicative constant)

2 The dual basis

The following result is classical in the context of frames and Riesz basis (see for example [2], [4]).

5 Proposition. *If f_m $(m \in \mathbb{N})$ is a Riesz basis of an Hilbert space H, there exists a unique sequence g_m $(m \in \mathbb{N})$ of elements of H such that $< f_m, g_k >= \delta_{km}$ for every $m, k \in \mathbb{N}$. More precisely one has*

$$g_m = S^{-1} f_m, \qquad m \in \mathbb{N}$$

where S is the frame operator

$$S \ : \ H \to H \quad f \mapsto \sum_{m=1}^{+\infty} < f, f_m > f_m.$$

The sequence g_m $(m \in \mathbb{N})$ is also a Riesz basis and is called the dual Riesz basis of f_m $(m \in \mathbb{N})$. It also satisfies

$$f = \sum_{m=1}^{+\infty} < f, f_m > g_m = \sum_{m=1}^{+\infty} < f, g_m > f_m$$

for every $f \in H$.$\square$

Now, we want to give an explicit construction of the dual basis of the Riesz basis (*).

But before doing so, let us install some notations and let us also briefly recall some additional properties concerning the wavelet basis (*). We denote by W_ψ the matrix similar to W (see Proposition 3) but defined using the functions ψ_a, ψ_s instead of φ_a, φ_s, i.e.

$$W_\psi(\xi) = \begin{pmatrix} \omega_{\psi_s}(\xi) & \omega_{\psi_s,\psi_a}(\xi) \\ \overline{\omega_{\psi_s,\psi_a}(\xi)} & \omega_{\psi_a}(\xi) \end{pmatrix}$$

where

$$\omega_{\psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\psi_a}(\xi + 2l\pi)|^2, \quad \omega_{\psi_s}(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\psi_s}(\xi + 2l\pi)|^2$$

$$\omega_{\psi_s,\psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} \widehat{\psi_s}(\xi + 2l\pi)\overline{\widehat{\psi_a}(\xi + 2l\pi)}.$$

These functions have the following properties.

6 Property. *The functions $\omega_{\psi_a}, \omega_{\psi_s}, \omega_{\psi_s,\psi_a}$ are 2π- periodic trigonometric polynomials such that*

$$\omega_{\psi_a}(\xi) \geq c > 0, \quad \omega_{\psi_s}(\xi) \geq c > 0, \quad \omega_{\psi_a}(-\xi) = \omega_{\psi_a}(\xi), \quad \omega_{\psi_s}(-\xi) = \omega_{\psi_s}(\xi)$$

and

$$\overline{\omega_{\psi_s,\psi_a}(\xi)} = -\omega_{\psi_s,\psi_a}(\xi) = \omega_{\psi_s,\psi_a}(-\xi)$$

for every $\xi \in \mathbb{R}$. There are also $A, B > 0$ such that

$$A \leq \det(W_\psi(\xi)) \leq B, \quad \forall \xi \in \mathbb{R}.$$

Proof. The proof is direct, using the support and the symmetry properties of the functions ψ_a, ψ_s and the Riesz condition satisfied by the basis (*). □

Since the wavelets of different levels are orthogonal to each other (that is to say, the spaces W_j and $W_{j'}$ are orthogonal if $j \neq j'$), it suffices to consider one scale (say, $j = 0$) to construct the dual. That's the reason why we present the construction of the dual as follows.

7 Theorem. *The functions $\widetilde{\psi}_1, \widetilde{\psi}_2$ defined as*

$$\widehat{\widetilde{\psi}_1}(\xi) = \alpha_1(\xi)\widehat{\psi}_a(\xi) + \beta_1(\xi)\widehat{\psi}_s(\xi), \quad \widehat{\widetilde{\psi}_2}(\xi) = \alpha_2(\xi)\widehat{\psi}_a(\xi) + \beta_2(\xi)\widehat{\psi}_s(\xi)$$

where

$$\alpha_1(\xi) = \frac{\omega_{\psi_s}(\xi)}{\det(W_\psi(\xi))}, \quad \beta_1(\xi) = \frac{\omega_{\psi_s,\psi_a}(\xi)}{\det(W_\psi(\xi))},$$

$$\alpha_2(\xi) = \overline{\beta_1(\xi)} = -\beta_1(\xi), \quad \beta_2(\xi) = \frac{\omega_{\psi_a}(\xi)}{\det(W_\psi)(\xi)}$$

are such that the family of functions

$$\left\{ 2^{j/2}\widetilde{\psi}_i(2^j. - k) \ : \ i = 1, 2; j, k \in \mathbb{Z} \right\}$$

is the dual basis of the basis of wavelets (*).

Proof. First, we look for a function $\widetilde{\psi}_1$ in W_0 such that

$$< \psi_a(. - k), \widetilde{\psi}_1 >_{L^2(\mathbb{R})} = \delta_{0k} \quad \text{and} \quad < \psi_s(. - k), \widetilde{\psi}_1 >_{L^2(\mathbb{R})} = 0$$

for every $k \in \mathbb{Z}$. Since $\{\psi_s(. - k) : \ k \in \mathbb{Z}\} \cup \{\psi_a(. - k) : k \in \mathbb{Z}\}$ constitute a Riesz basis of W_0, we look in fact for 2π-periodic and L^2_{loc} functions α_1, β_1 such that

$$\widehat{\widetilde{\psi}_1}(\xi) = \alpha_1(\xi)\widehat{\psi_a}(\xi) + \beta_1(\xi)\widehat{\psi_s}(\xi)$$

and such that

$$< e^{-ik\cdot}\widehat{\psi_a}, \alpha_1\widehat{\psi_a} + \beta_1\widehat{\psi_s} >_{L^2(\mathbb{R})} = 2\pi\delta_{0k} \quad \text{and} \quad < e^{-ik\cdot}\widehat{\psi_s}, \alpha_1\widehat{\psi_a} + \beta_1\widehat{\psi_s} >_{L^2(\mathbb{R})} = 0$$

for every $k \in \mathbb{Z}$. The last equalities are equivalent to

$$\left\{ \begin{array}{ll} \int_0^{2\pi} e^{-ik\xi} \left(\overline{\alpha_1(\xi)}\omega_{\psi_a}(\xi) + \overline{\beta_1(\xi)\omega_{\psi_s,\psi_a}(\xi)} \right) d\xi & = 2\pi\delta_{0k} \\ \int_0^{2\pi} e^{-ik\xi} \left(\overline{\alpha_1(\xi)}\omega_{\psi_s,\psi_a}(\xi) + \overline{\beta_1(\xi)}\omega_{\psi_s}(\xi) \right) d\xi & = 0 \end{array} \right. , \quad \forall k \in \mathbb{Z}$$

hence also to

$$\left\{ \begin{array}{l} \alpha_1(\xi)\omega_{\psi_a}(\xi) + \beta_1(\xi)\omega_{\psi_s,\psi_a}(\xi) = 1 \\ \alpha_1(\xi)\overline{\omega_{\psi_s,\psi_a}(\xi)} + \beta_1(\xi)\omega_{\psi_s}(\xi) = 0. \end{array} \right.$$

Using matrices, this can be rewritten as

$$\begin{pmatrix} \omega_{\psi_s}(\xi) & \overline{\omega_{\psi_s,\psi_a}(\xi)} \\ \omega_{\psi_s,\psi_a}(\xi) & \omega_{\psi_a}(\xi) \end{pmatrix} \begin{pmatrix} \beta_1(\xi) \\ \alpha_1(\xi) \end{pmatrix} = \overline{W_\psi(\xi)} \begin{pmatrix} \beta_1(\xi) \\ \alpha_1(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solutions of this system is

$$\begin{pmatrix} \beta_1(\xi) \\ \alpha_1(\xi) \end{pmatrix} = \frac{1}{\det(W_\psi(\xi))} \begin{pmatrix} -\overline{\omega_{\psi_s,\psi_a}(\xi)} \\ \omega_{\psi_s}(\xi) \end{pmatrix} = \frac{1}{\det(W_\psi(\xi))} \begin{pmatrix} \omega_{\psi_s,\psi_a}(\xi) \\ \omega_{\psi_s}(\xi) \end{pmatrix}.$$

We proceed exactly in the same way to find a function $\widetilde{\psi_2}$ in W_0 such that

$$< \psi_a(. - k), \widetilde{\psi_2} >= 0 \quad \text{and} \quad < \psi_s(. - k), \widetilde{\psi_2} >= \delta_{0k}$$

for every $k \in \mathbb{Z}$. In this case, the final system is

$$\overline{W_\psi(\xi)} \begin{pmatrix} \beta_2(\xi) \\ \alpha_2(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which gives the solutions.

Since the spaces W_j and $W_{j'}$ are orthogonal if $j \neq j'$, we obtain, for $j, j', k, k' \in \mathbb{Z}$:

$$< 2^{j/2}\psi_a(2^j.{-k}), 2^{j'/2}\widetilde{\psi_1}(2^{j'}.{-k'}) >= \delta_{jj'}\delta_{kk'}, \quad < 2^{j/2}\psi_s(2^j.{-k}), 2^{j'/2}\widetilde{\psi_1}(2^{j'}.{-k'}) >= 0$$

and

$$< 2^{j/2}\psi_a(2^j.{-k}), 2^{j'/2}\widetilde{\psi_2}(2^{j'}.{-k'}) >= 0, \quad < 2^{j/2}\psi_s(2^j.{-k}), 2^{j'/2}\widetilde{\psi_1}(2^{j'}.{-k'}) >= \delta_{jj'}\delta_{kk'}$$

hence the conclusion.□

8 Proposition. *The functions $\widetilde{\psi_1}, \widetilde{\psi_2}$ are deficient splines with exponential decay and symmetry properties ($\widetilde{\psi_1}, \widetilde{\psi_2}$ are respectively antisymmetric and symmetric relatively to* $5/2$*).*

Proof. By construction, these functions are deficient splines. Their explicit expressions in terms of the Fourier transforms of the wavelets ψ_a, ψ_s and the form of the coefficients α_i, β_i give the exponential decay and the symmetry properties.□

Here are pictures of an approximation of the dual functions $\widetilde{\psi_1}, \widetilde{\psi_2}$ (up to a constant factor).

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