An interior-point method based optimal power flow *

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Abstract

This paper deals with the solution of an *optimal power flow* (OPF) problem by the *interior point method* (IPM). The latter is the most fashionable approach of this nonlinear programming problem due to its speed of convergence and ease handling of inequality constraints. Two interior point algorithms are presented and compared, namely the pure primal-dual and the predictor-corrector respectively. Several implementation aspects of these IPM algorithms are equally discussed. The OPF is formulated in rectangular coordinates which confers some significant advantages because generally its objective and constraints are quadratic functions. Among the large variety of OPF objectives, emphasis is put on two classical ones: the minimization of generation cost and the minimization of active power losses. The solution obtained by both algorithms prove to be robust for the above OPF sub-problems as well as for a full OPF applied to the former objective, which is unanimously recognized as the hardest problem to solve. Finally, numerical results on three test systems ranging from 60 to 300 buses are provided.

Keywords: nonlinear programming, interior point method, optimal power flow
AMS classification: ???

1 Introduction

The Optimal Power Flow (OPF) problem was first introduced by Carpentier in the early 60's [1]. Since then it has become progressively an indispensable tool in power systems planning, operational planning and real-time operation, and that whatever the electricity market environment: liberalized or not [2].
The OPF is stated in a general form as a nonlinear, non-convex, large-scale, static optimization problem with both continuous and discrete variables. It acts on available control variables in order to optimize some objective while satisfying network power flow equations, physical and operational constraints.
First approaches of the complex OPF problem can be classified in: gradient methods [3], sequential quadratic programming [4, 5] and sequential linear programming [6]. The inconvenients of these techniques involve the slow convergence especially in the neighborhood of the optimum for the first two, and a rather limited field of application such as the optimization of the decoupled active control variables for the third one.
The paper by Sun [7] constitutes a breakthrough at that time in two respects. Firstly it proposes to solve directly the Karush-Kuhn Tucker (KKT) optimality conditions by the very efficient

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Newton method. Secondly it recommends sparsity techniques to considerably speed up the computations. The weakness of this approach lies on the difficulty to identify inequality constraints that are active at the optimum.
Although emerged in the middle 50's [8] but largely developed by Fiacco and McCormick [9] the Interior Point Method (IPM) was not very popular up to the outstanding paper by Karmarkar [10]. The latter proved that IPM outperforms to a very large extent the classical simplex method especially for large linear-programming problems. The application of IPM also to nonlinear programming was the next logical step. IPM has thus various application in power systems, e.g. minimization of generation cost [20, 43], minimization of active power losses [21, 20], minimization of load shedding [22], maximization of loadability limit [23, 28], minimization of congestion cost [41], maximization of social benefit [35, 34].
Meanwhile the IPM reached maturity. The primal-dual predictor-corrector interior point algorithm proposed by Merhotra [11] became the benchmark of IPM. However, more recently, a variant of this algorithm namely the multiple centrality correction technique, initially proposed for linear programming [14], proved to be as good as the reference also for nonlinear programming applications [26].
The salient features of the IPM concern its fast convergence and its adequate handling of inequality constraints by logarithm barrier functions. As any method there are also drawbacks: the heuristic of decreasing the barrier parameter and the fact that slack variables and their corresponding dual variables must remain positive at every iteration which may drastically shorten the Newton step length. To overcome these IPM flaws, non-interior point approaches of OPF emerged such as: the unlimited point algorithm [29], the complementarity method [27], the Jacobain Smoothing [28], etc. Although not perfectly rigurous mathematically some of them prove convergence properties comparables with best interior-point algorithms.
The solution of an OPF encompasses three steps. The OPF is first solved by treating all variables as continuous. Then some heuristic are used to set up the values of discrete variables [18, 17, 19]. If the obvious technique of rounding-off of the latter to their nearest discrete value has generally little effect on objective and constraints for on-load tap changer transformer (OLTC) ratio, it requires some caution for shunt susceptance and phase shifter angle due to their larger range of variation. Finally after setting the discrete variables a OPF is run again to re-optimize continuous variables and ensure that all constraints are enforced following to the set-up of discrete variables. Alternatively, to deal with all kind of variables simultaneously, stochastic optimization techniques may be used such as evolutionary programming (e.g. genetic algorithms [37, 36]), simulating annealing [38, 39], tabu search [40], etc.
As far as the IPM is of concern, two approaches were proposed to handle discrete variables: either they are appended to the objective through penalty functions [30] or by using ordinal theory [31].
Note also that the difficulty of solving an OPF increases as one wishes to take into account constraints stemming from credible contingencies analysis, as in *security constrained* OPF [15, 16, 33, 34].
In this paper we solve various OPF problems while comparing two interior point (IP) based algorithms, namely the infeasible [1] pure primal-dual algorithm and the infeasible primal-dual predictor-corrector algorithm, called for the sake of brevity in the remaining of the paper PDIPA and PCIPA respectively.
The paper is organized as follows. Section 2 introduces the OPF problem. Section 3 offers a detailed overview of both PDIPA and PCIPA. Section 4 provides some numerical results while some conclusions are drawn in Section 5.

[1] Infeasible reffers to the fact that a strictly feasible initial point is not mandatory

2 Statement of the Optimal Power Flow problem

2.1 Generalities

In this Section we formulate the OPF problem with voltages expressed in rectangular coordinates, choice which will be explained later on.
Let us recall that the OPF is a nonlinear programming problem which aims at optimizing some objective by acting on given control variables while satisfying some equality and inequality constraints. All these concepts are successively described in the sequel.
Let $\mathcal{N}$ be the set of buses, $\mathcal{G}$ the set of generators, $\mathcal{C}$ the set of consumers (loads), $\mathcal{B}$ the set of branches, $\mathcal{L}$ the set of lines, $\mathcal{T}$ the set of transformers, $\mathcal{O}$ the set of transformers with OLTC, $\mathcal{P}$ the set of phase shifters and $\mathcal{S}$ the set of shunt elements. Let also nn, ng, nc, nb, nl, nt, no, np and ns be the size of these sets.
In rectangular coordinates the complex voltage $\underline{V}_i$ is expressed as:

$$\underline{V}_i = e_i + jf_i \quad \forall i \in \mathcal{N} \tag{2.1}$$

where e_i and f_i are its real and imaginary part respectively, while the active and reactive bus power injections take on the form:

$$P_i = -V_i^2 \sum_{j \in \mathcal{N}_i} (G_{si} + G_{ij}) + \sum_{j \in \mathcal{N}_i} [(e_i e_j + f_i f_j) G_{ij} + (f_i e_j - e_i f_j) B_{ij}] \tag{2.2}$$

$$Q_i = +V_i^2 \sum_{j \in \mathcal{N}_i} (B_{si} + B_{ij}) - \sum_{j \in \mathcal{N}_i} [(e_i e_j + f_i f_j) B_{ij} + (e_i f_j - f_i e_j) G_{ij}] \tag{2.3}$$

where G_{si} and B_{si} are the shunt conductance and susceptance at bus i, G_{ij} and B_{ij} are the conductance and susceptance of the branch linking buses i and j, $V_i^2 = e_i^2 + f_i^2$ is the module of voltage at bus i and $\mathcal{N}_i$ is the set of buses connected by branches to the bus i.

2.2 Objective function

Many OPF objectives can be thought of, let us quote among the most usual ones: minimization of generation cost, minimization of active power losses, minimization of reactive power losses, minimization of cost to remove a congestion, minimization of load shedding amount, maximization of loadability limit, maximization of social benefit, etc.
In this paper we deal with two classical objectives, namely minimum generation cost (2.4) and minimum active power losses (2.5).

$$\min \sum_{i \in \mathcal{G}} c_{0i} + c_{1i} P_{gi} + c_{2i} P_{gi}^2 + \sum_{i \in \mathcal{C}} c_i \phi_i P_{ci} \tag{2.4}$$

$$\min \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} G_{ij}^2 \left[(e_i - e_j)^2 + (f_i - f_j)^2 \right] \tag{2.5}$$

where c_{0i}, c_{1i}, c_{2i}, are cost coefficients describing the cost curve (usually assumed to do not include terms higher than quadratic) of the i-th generator and P_{gi} is its active output, while for the load i, c_i represents a compensation price (requested in case of curtailment), ϕ_i its maximum allowable percentage of load shedding and P_{ci} its active consumption. Note that the last term in (2.4) has the role to help the OPF to converge in a situation when without curtailing load some constraints can not be enforced and hence there is no optimal solution. These compensation prices are set to very large values such that, unless necessary, no load is curtailed $\phi_i \to 0$, $\forall i \in \mathcal{C}$ and therefore the optimal generation dispatch is not disturbed. Incidentally the presence of this term makes sense in some liberalised electricity markets.

2.3 Control variables

At this point of the presentation we only enumerate most common power system controls: generators active output, generators reactive output or voltage, OLTCs transformer ratio, phase shifters angle, shunts element susceptance, loads active and reactive power, TCSC reactance, SVC susceptance, etc., while a deeper insight will be provided in Section *2.6*.

2.4 Equality constraints

Equality constraints mainly involve nodal active and reactive power balance equations:

$$P_{gi} - (1 - \phi_i)P_{ci} - P_i = 0 \quad i \in \mathcal{N} \tag{2.6}$$

$$Q_{gi} - (1 - \phi_i)Q_{ci} - Q_i = 0 \quad i \in \mathcal{N} \tag{2.7}$$

where P_{gi} and Q_{gi} are the active and reactive power of the i-th generator P_{ci} and Q_{ci} are the active and reactive consumption of the i-th load, and P_i, Q_i are the power injections at the i-th bus. Additional equality constraints concern: maintaining a generator voltage at an imposed value (if requested):

$$e_i^2 + f_i^2 - (V_i^{imp})^2 = 0 \quad i \in \mathcal{G} \tag{2.8}$$

maintaining the ratio of a pure phase shifter to an imposed value:

$$a_{1i}^2 + a_{2i}^2 - (a_i^{imp})^2 = 0 \quad i \in \mathcal{P} \tag{2.9}$$

where $\underline{a_i} = a_{1i} + ja_{2i}$ $i \in \mathcal{P}$ is the phase shifter ratio in rectangular coordinates, and finally keeping a power transfer through an interface $\mathcal{F}$ (e.g. between neighbor countries) to a scheduled value:

$$\sum_{i \in \mathcal{F}_j} T_i - T_j^{imp} = 0 \quad j \in \mathcal{F}_j \tag{2.10}$$

where $\mathcal{F}_j$ is the set of such interfaces.

2.5 Inequality constraints

An OPF encompasses two types of inequality constraints: operational (aimed to ensure a secure operation of the system) and physical limits of equipments. The former involve limits on branches current and voltages magnitude:

$$\left(G_{ij}^2 + B_{ij}^2\right)\left[(e_i - e_j)^2 + (f_i - f_j)^2\right] \leq (I_{ij}^{max})^2 \quad i,j \in \mathcal{N} \tag{2.11}$$

$$(V_i^{min})^2 \leq e_i^2 + f_i^2 \leq (V_i^{max})^2 \quad i \in \mathcal{N} \tag{2.12}$$

We have chosen to express constraints on (longitudinal) branch current rather than on active power flowing through the branch because overcurrent protections and conductor heating have to do with Ampères and not MegaWatts. Note, however, that active power flow constraints can be easily incorporated if needed.
Finally, physical bounds of some power system devices can be expressed as:

$$P_{gi}^{min} \leq P_{gi} \leq P_{gi}^{max} \quad i \in \mathcal{G} \tag{2.13}$$

$$Q_{gi}^{min} \leq Q_{gi} \leq Q_{gi}^{max} \quad i \in \mathcal{G} \tag{2.14}$$

$$r_i^{min} \leq r_i \leq r_i^{max} \quad i \in \mathcal{O} \tag{2.15}$$

$$\phi_i^{min} \leq \phi_i \leq \phi_i^{max} \quad i \in \mathcal{C} \tag{2.16}$$

$$b_i^{min} \leq b_i \leq b_i^{max} \quad i \in \mathcal{S} \tag{2.17}$$

$$\tan(\varphi_i^{min}) \leq \frac{a_{2i}}{a_{1i}} \leq \tan(\varphi_i^{max}) \quad i \in \mathcal{P} \tag{2.18}$$

where for the i-th generator P_{gi}^{min}, P_{gi}^{max} (resp. Q_{gi}^{min}, Q_{gi}^{max}) are its active (resp. reactive) output limits, for the i-th OLTC transformer r_i^{min} and r_i^{max} are bounds on its ratio, for the i-th load ϕ_i^{min} and ϕ_i^{max} are limits on its allowed curtailment percentage, for the i-th shunt b_i^{min} and b_i^{max} are bounds on its susceptance, and finally for the i-th phase shifter φ_i^{min} and φ_i^{max} are the minimal and maximal value of its angle. Functional inequality (2.18) is valid as long as the $\varphi \in (-\pi/2, \pi/2)$, which is hopefully the case of real-life phase shifters.
For the sake of clarity we have restricted ourselves to the most widespread control variables and their relative constraints only. Thus FACTS devices such as: SVC, TCSC, UPFC, etc. are ignored albeit our OPF can take into account most of them. For instance SVC and TCSC susceptance can be handled by a constraint of type (2.17).
We implemented the OPF expressing voltages in rectangular coordinates due to the fact that in many OPF applications (e.g. objectives (2.4) and (2.5), minimization of reactive power losses, etc.) the objective and constraints are quadratic functions which exhibit three advantages:

1. Hessian matrix of a quadratic function is constant (e.g. second derivatives of load flow equations, branch current and voltage bounds constraints, etc.)

2. the Taylor series development of a quadratic function terminates at second order terms without truncation error

3. higher order terms are straightforwardly evaluated.

On the other hand, a (small) disadvantage is the handling bus voltage magnitudes constraints (2.8,2.12) and phase shifter angle constraints (2.9,2.18) as functional constraints instead of simple bounds as when using polar coordinates.
Very good results have been reported with a rectangular OPF for several optimization problems [25, 32].

2.6 Full OPF and decoupled sub-problems

The benchmark approach when solving an OPF problem is to make use of all means of control available and take into account all the constraints of the problem, what is known as "full" OPF, e.g. the optimization of (2.4) or (2.5), subject to (2.6-2.18). An inconvenient of this approach is its dimensionality while not all controls can considerably improve the objective.
Alternatively, based on the physically (generally) weak coupling in the transmission grid between active powers and voltage magnitudes and between reactive powers and voltage angles, in literature a distinction is often made between two types of OPF sub-problems: optimization of active power flows and that of reactive power flows [18, 16]. According to the problem they primarily affect, control variables can in turn be divided into: active power variables (e.g. generator active output, load curtailment, phase shifter angle, etc.) and reactive power variables (e.g. generator reactive power or voltage, OLTC transfomer ratio, shunt susceptance, etc.). Optimizing (2.4), subject to (2.6,2.11,2.13,2.16,2.18) constitutes an example of active power optimization sub-problem, while optimizing (2.5), subject to (2.6,2.12,2.14,2.15,2.17) is an example of reactive power optimization sub-problem.

3 Primal-Dual Interior Point Method

3.1 Obtaining the optimality conditions

The OPF formulation (2.4 or 2.5 and 2.6-2.18) can be compactly written as a general nonlinear programming problem:

$$\begin{aligned} & \min f(\mathbf{x}) && (3.19)\\ \text{subject to :} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} && (3.20)\\ & \mathbf{h}_\ell \le \mathbf{h}(\mathbf{x}) \le \mathbf{h}_u && (3.21)\\ & \mathbf{x}_\ell \le \mathbf{x} \le \mathbf{x}_u && (3.22) \end{aligned}$$

where dimension of unknowns vector $\mathbf{x}$, and functions $\mathbf{g}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are n, m and p respectively. Vectors and matrix are in bold through the whole paper.
In order to simplify the presentation we encompass the simple bound constraints (3.22) into the functional inequality constraints (3.21).
IPM encompasses four steps. First, one transforms the inequality constraints into equality constraints by adding slack variables to inequality constraints. Second, non-negativity conditions are implicitly handled by appending them to the objective function as logarithmic barrier terms. Third, one transforms the equality constrained optimization problem into an unconstrained optimization one. Fourth, one solves the *perturbed* KKT first order optimality conditions by the Newton method. It is noteworthy to remark that IPM combines three concepts: logarithmic barrier function to handle inequality constraints [8], Langrage theory of optimization subject to equality constraints [9] and Newton method.
Following what we said above one transforms the inequality constraints into equality constraints by adding slack variables to inequality constraints.

$$\begin{aligned} & \min f(\mathbf{x}) && (3.23)\\ \text{subject to :} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} && (3.24)\\ & \mathbf{h}(\mathbf{x}) - \mathbf{h}_\ell - \mathbf{s}_\ell = \mathbf{0} && (3.25)\\ & -\mathbf{h}(\mathbf{x}) + \mathbf{h}_u - \mathbf{s}_u = \mathbf{0} && (3.26)\\ & \mathbf{s}_\ell, \mathbf{s}_u \ge \mathbf{0} && (3.27) \end{aligned}$$

Now non-negativity conditions (3.27) are added to the objective function as logarithmic barrier terms, resulting the following equality constrained optimization problem:

$$\begin{aligned} & \min f(\mathbf{x}) - \mu(\ln \mathbf{s}_\ell + \ln \mathbf{s}_u) && (3.28)\\ \text{subject to :} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} && (3.29)\\ & \mathbf{h}(\mathbf{x}) - \mathbf{h}_\ell - \mathbf{s}_\ell = \mathbf{0} && (3.30)\\ & -\mathbf{h}(\mathbf{x}) + \mathbf{h}_u - \mathbf{s}_u = \mathbf{0} && (3.31) \end{aligned}$$

where μ is a positive scalar called *barrier parameter* which is gradually decreased to zero as iteration progresses. At the heart of IPM is the Fiacco & McCormick theorem [9], which proves that as μ tends to zero, the solution $\mathbf{x}(\mu)$ approaches $\mathbf{x}^\star$, the solution of the problem.
The Lagrangian of the above equality constrained optimization problem is:

$$\mathcal{L}_\mu = f(\mathbf{x}) - \mu(\ln \mathbf{s}_\ell + \ln \mathbf{s}_u) - \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}) - \boldsymbol{\pi}_\ell^T (\mathbf{h}(\mathbf{x}) - \mathbf{h}_\ell - \mathbf{s}_\ell) - \boldsymbol{\pi}_u^T (-\mathbf{h}(\mathbf{x}) + \mathbf{h}_u - \mathbf{s}_u) \quad (3.32)$$

where the vectors of Lagrange multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\pi}_\ell$ and $\boldsymbol{\pi}_u$ are called *dual variables*.

The perturbed Karush-Kuhn-Tucker (KKT) first order necessary optimality conditions of the problem are:

$$\nabla_{\mathbf{s}_\ell}\mathcal{L}_\mu = -\mu\mathbf{S}_\ell^{-1}\mathbf{e} + \boldsymbol{\pi}_\ell = \mathbf{0} \tag{3.33}$$
$$\nabla_{\mathbf{s}_u}\mathcal{L}_\mu = -\mu\mathbf{S}_u^{-1}\mathbf{e} + \boldsymbol{\pi}_u = \mathbf{0} \tag{3.34}$$
$$\nabla_{\boldsymbol{\pi}_\ell}\mathcal{L}_\mu = -\mathbf{h}(\mathbf{x}) + \mathbf{h}_\ell + \mathbf{s}_\ell = \mathbf{0} \tag{3.35}$$
$$\nabla_{\boldsymbol{\pi}_u}\mathcal{L}_\mu = \mathbf{h}(\mathbf{x}) - \mathbf{h}_u + \mathbf{s}_u = \mathbf{0} \tag{3.36}$$
$$\nabla_{\boldsymbol{\lambda}}\mathcal{L}_\mu = -\mathbf{g}(\mathbf{x}) = \mathbf{0} \tag{3.37}$$
$$\nabla_{\mathbf{x}}\mathcal{L}_\mu = \nabla f(\mathbf{x}) - \nabla\mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T - \nabla\mathbf{h}(\mathbf{x})(\boldsymbol{\pi}_\ell{}^T - \boldsymbol{\pi}_u{}^T) = \mathbf{0} \tag{3.38}$$

where $\mathbf{e} = [1, .., 1]^T$, $\mathbf{S}_\ell = diag(s_{\ell 1}, ..., s_{\ell p})$ and $\mathbf{S}_u = diag(s_{u1}, ..., s_{up})$.

3.2 Solving for Newton direction

The perturbed KKT optimality conditions ae solved by Newton method. As the goal is not to solve completely this nonlinear system for a given value of μ, one makes a single iteration solving it approximately and then diminishing the value of μ. The linear symmetric system to solve is:

$$\begin{bmatrix} \mu\mathbf{S}_\ell^{-2} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu\mathbf{S}_u^{-2} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nabla\mathbf{h}(\mathbf{x}) \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \nabla\mathbf{h}(\mathbf{x}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nabla\mathbf{g}(\mathbf{x}) \\ \mathbf{0} & \mathbf{0} & -\nabla\mathbf{h}(\mathbf{x})^T & \nabla\mathbf{h}(\mathbf{x})^T & -\nabla\mathbf{g}(\mathbf{x})^T & \nabla^2_{\mathbf{x}}\mathcal{L}_\mu \end{bmatrix} \begin{bmatrix} \Delta\mathbf{s}_\ell \\ \Delta\mathbf{s}_u \\ \Delta\boldsymbol{\pi}_\ell \\ \Delta\boldsymbol{\pi}_u \\ \Delta\boldsymbol{\lambda} \\ \Delta\mathbf{x} \end{bmatrix} = - \begin{bmatrix} \nabla_{\mathbf{s}_\ell}\mathcal{L}_\mu \\ \nabla_{\mathbf{s}_u}\mathcal{L}_\mu \\ \nabla_{\boldsymbol{\pi}_\ell}\mathcal{L}_\mu \\ \nabla_{\boldsymbol{\pi}_u}\mathcal{L}_\mu \\ \nabla_{\boldsymbol{\lambda}}\mathcal{L}_\mu \\ \nabla_{\mathbf{x}}\mathcal{L}_\mu \end{bmatrix} \tag{3.39}$$

where

$$\nabla^2_{\mathbf{x}}\mathcal{L}_\mu = \nabla^2_{\mathbf{x}} f(\mathbf{x}) - \nabla^2_{\mathbf{x}}\mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T - \nabla^2_{\mathbf{x}}\mathbf{h}(\mathbf{x})(\boldsymbol{\pi}_\ell{}^T - \boldsymbol{\pi}_u{}^T) \tag{3.40}$$

In order to reduce the dimensionality of the problem and consequently to speed up computations, alternatively one can solve first the reduced system (3.41) for $\Delta\mathbf{x}$ and $\Delta\boldsymbol{\lambda}$:

$$\begin{bmatrix} \mathbf{0} & -\nabla\mathbf{g}(\mathbf{x}) \\ -\nabla\mathbf{g}(\mathbf{x})^T & \mathbf{H_d} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{\lambda} \\ \Delta\mathbf{x} \end{bmatrix} = - \begin{bmatrix} \nabla_{\boldsymbol{\lambda}}\mathcal{L}_\mu \\ \zeta \end{bmatrix} \tag{3.41}$$

where

$$\mathbf{H_d} = \nabla^2_{\mathbf{x}}\mathcal{L}_\mu + \mu\nabla_{\mathbf{x}}\mathbf{h}(\mathbf{x})^T(\mathbf{S}_\ell^{-2} + \mathbf{S}_u^{-2})\nabla_{\mathbf{x}}\mathbf{h}(\mathbf{x}) \tag{3.42}$$

and

$$\zeta = \nabla_{\mathbf{x}}\mathcal{L}_\mu + \nabla_{\mathbf{x}}\mathbf{h}(\mathbf{x})^T[\mu(\mathbf{S}_u^{-2}\nabla_{\boldsymbol{\pi}_u}\mathcal{L}_\mu - \mathbf{S}_\ell^{-2}\nabla_{\boldsymbol{\pi}_\ell}\mathcal{L}_\mu) + \nabla_{\mathbf{s}_\ell}\mathcal{L}_\mu - \nabla_{\mathbf{s}_u}\mathcal{L}_\mu] \tag{3.43}$$

and then compute:

$$\Delta\mathbf{s}_\ell = +\nabla\mathbf{h}(\mathbf{x})\Delta\mathbf{x} - \nabla_{\boldsymbol{\pi}_\ell}\mathcal{L}_\mu \quad \Delta\boldsymbol{\pi}_\ell = -\mu\mathbf{S}_\ell^{-2}\Delta\mathbf{s}_\ell - \nabla_{\mathbf{s}_\ell}\mathcal{L}_\mu \tag{3.44}$$
$$\Delta\mathbf{s}_u = -\nabla\mathbf{h}(\mathbf{x})\Delta\mathbf{x} - \nabla_{\boldsymbol{\pi}_u}\mathcal{L}_\mu \quad \Delta\boldsymbol{\pi}_u = -\mu\mathbf{S}_u^{-2}\Delta\mathbf{s}_u - \nabla_{\mathbf{s}_u}\mathcal{L}_\mu \tag{3.45}$$

3.3 Updating of variables

At the k-th iteration primal and dual variables are updated as follows:

$$\mathbf{s}_\ell^{k+1} = \mathbf{s}_\ell^k + \alpha_p^k\Delta\mathbf{s}_\ell^k \qquad \boldsymbol{\pi}_\ell^{k+1} = \boldsymbol{\pi}_\ell^k + \alpha_d^k\Delta\boldsymbol{\pi}_\ell^k \tag{3.46}$$
$$\mathbf{s}_u^{k+1} = \mathbf{s}_u^k + \alpha_p^k\Delta\mathbf{s}_u^k \qquad \boldsymbol{\pi}_u^{k+1} = \boldsymbol{\pi}_u^k + \alpha_d^k\Delta\boldsymbol{\pi}_u^k \tag{3.47}$$
$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_p^k\Delta\mathbf{x}^k \qquad \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \alpha_d^k\Delta\boldsymbol{\lambda}^k \tag{3.48}$$

where $\alpha_p \in (0,1]$ and $\alpha_d \in (0,1]$ are the primal and dual *step length*. The maximum step length that can be taken in the Newton direction is established such that the positiveness of slack variables and their corresponding dual variables is preserved, that is:

$$\alpha_p^k = \min\left\{1, \gamma \min_{\Delta s_i^k < 0} \frac{-s_i^k}{\Delta s_i^k}\right\} \qquad \alpha_d^k = \min\left\{1, \gamma \min_{\Delta \pi_i^k < 0} \frac{-\pi_i^k}{\Delta \pi_i^k}\right\} \tag{3.49}$$

where $\gamma \in (0,1)$ is a *safety factor* aiming to ensure strict positiveness of slack variables and their corresponding dual variables and s_i^k (resp. π_i^k) stands for either $s_{\ell i}^k$ or s_{ui}^k. (resp. $\pi_{\ell i}^k$ or π_{ui}^k). Since the aim is to take the highest step possible in the Newton direction the trend is to chose a high value for this parameter, a usual value being $\gamma = 0.99995$.
The coupling of primal and dual variables into the dual feasibility conditions in the system (3.33) requires the use of a common step length:

$$\alpha_p^k = \alpha_d^k = \min\left\{\alpha_p^k, \alpha_d^k\right\} \tag{3.50}$$

Note, however, that situations of very poor centered iteration may occur (when some complementarity products are too small while other are too large) and consequently either primal or dual step length is (very) close to zero while the other may have a significant value. In such case and it would be a waste of time to do not update primal and dual variables separately. Moreover this could help the algorithm to recover faster from such spiny situation. In our implementation we take a common step length as long as $\alpha_p^k, \alpha_d^k \geq \alpha_{min}$ (tipically $\alpha_{min} = 0.1$), and separate step lengths otherwise.
Note finally that the literature reports very good results with both common [20, 25] and separate step length [21, 23] for nonlinear programming, while for the linear programming the use of separate step lenghts has proven slightly more efficient than a common step.
As a general remark, in IP algorithms all parameter choice require some tuning for a given network, objective and constraints.

3.4 Reducing the barrier parameter

The *complementarity gap* is defined as the residual of complementarity constraints:

$$\rho^k = \mathbf{s}_\ell^T \boldsymbol{\pi}_\ell + \mathbf{s}_u^T \boldsymbol{\pi}_u \tag{3.51}$$

An heuristic inspired from LP and convex QP is to reduce the barrier parameter proportionally to the complementarity gap [12]:

$$\mu^{k+1} = \sigma^k \frac{\rho^k}{2(m+p)} \tag{3.52}$$

where $\sigma \in [0,1]$ is a *centering parameter* expressing the expected, but not necessarily realized, reduction of complementarity gap. This parameter is a compromise between feasibility and optimality. The extreme value $\sigma = 0$ (resp. $\sigma = 1$) corresponds to putting emphasis on satisfying optimality (resp. feasibility) only. A typical initial value is $\sigma = 0.2$, while during the iterative process best IPM performances are reported when $\sigma \in [0.1, 0.2]$ [12, 24]. Since generally feasibility may be reached in a few iterations it makes sense to focus in the subsequent iterations more and more on attaining optimality of the feasible point, that is to gradually reduce σ towards 0.1 [24].
The choice of μ^0 is critical for the algorithm performance. There is no universal panacea for this choice, it depends on the specific problem one deals with. Its choice should avoid that constraints become active at too early during the iterative process which can worse the convergence or even

jam. Admittedly the PDIPA is more sensitive to this choice than the PCIPA one. Broadly speaking μ^0 may be chosen in the interval $[0.01, 1000]$, more comments about this initialization will be provided in Section 4.

3.5 Convergence criteria

The convergence is reached and the iterative process terminates as soon as primal feasibility (3.53), scaled dual feasibility (3.54), scaled complementarity gap (3.55) and a scaled objective function variation from an iteration to the next (3.55) fall below some tolerances:

$$\max\left\{\max\left\{\mathbf{h}_\ell - \mathbf{h}(\mathbf{x})\right\}, \left\{\mathbf{h}(\mathbf{x}) - \mathbf{h}_u\right\}, ||\mathbf{g}(\mathbf{x})||_\infty\right\} \leq \epsilon_1 \tag{3.53}$$

$$\frac{||\nabla f(\mathbf{x}) - \nabla \mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T - \nabla \mathbf{h}(\mathbf{x})(\boldsymbol{\pi}_\ell{}^T - \boldsymbol{\pi}_u{}^T)||_\infty}{1 + ||\mathbf{x}||_2 + ||\boldsymbol{\lambda}||_2 + ||\boldsymbol{\pi}_\ell||_2 + ||\boldsymbol{\pi}_u||_2} \leq \epsilon_1 \tag{3.54}$$

$$\frac{\rho}{1 + ||\mathbf{x}||_2} \leq \epsilon_1 \tag{3.55}$$

$$\frac{|f(\mathbf{x}^k) - f(\mathbf{x}^{k-1})|}{1 + |f(\mathbf{x}^k)|} \leq \epsilon_2 \tag{3.56}$$

In addition one may also require the barrier parameter to be smaller than a specified tolerance $\mu \leq \epsilon_\mu$. Typical tolerances are $\epsilon_1 = 10^{-4}$, $\epsilon_2 = 10^{-6}$ and $\epsilon_\mu = 10^{-6}$. A point which satisfies inequalities (3.53 - 3.55) is called a KKT point of accuracy ϵ_1.

3.6 Choice of an initial point

A great advantage of the IP based algorithms is that a strictly feasible initial point is not required, only the non-negativity conditions $\mathbf{s}_\ell, \mathbf{s}_u, \boldsymbol{\pi}_\ell, \boldsymbol{\pi}_u > \mathbf{0}$ must be satisfied at each iteration. Besides convergence performance of IP algorithms can be improved if some euristics are used as explained in the sequel.
We recommend to initialize $\mathbf{x}^0$ as the solution of a load flow computation. If this solution is not available beforehand, we first set bounded control variables (see eq. 3.22) at the middle point of their interval of variation, while the voltages vector is next obtained by running a standard load flow program. Conversely, chosing a flat start for voltages vector may deterioarate algorithms convergence performance.
Slack variables are initialized as suggested in [24, 25]:

$$\mathbf{s}_\ell^0 = \min\left\{\max\left\{\delta\mathbf{h}^\Delta, \mathbf{h}(\mathbf{x}^0) - \mathbf{h}_\ell\right\}, (1-\delta)\mathbf{h}^\Delta\right\} \tag{3.57}$$

$$\mathbf{s}_u^0 = \min\left\{\max\left\{\delta\mathbf{h}^\Delta, \mathbf{h}_u - \mathbf{h}(\mathbf{x}^0)\right\}, (1-\delta)\mathbf{h}^\Delta\right\} \tag{3.58}$$

where $\mathbf{h}^\Delta = \mathbf{h}_u - \mathbf{h}_\ell$ and chosing $\delta \in [0.1, 0.3]$ offers almost equally good results.
The corresponding dual variables are computed as:

$$\boldsymbol{\pi}_\ell^0 = \mu^0(\mathbf{S}_\ell^0)^{-1}\mathbf{e} \qquad \boldsymbol{\pi}_u^0 = \mu^0(\mathbf{S}_u^0)^{-1}\mathbf{e} \tag{3.59}$$

Finally, Lagrange multipliers relative to equality constraints (2.6-2.9) are set $\boldsymbol{\lambda} = \mathbf{0}$.

3.7 PDIPA summary

PDIPA can be outlined as follows:

1. Initialize primal and dual variables of the problem (see Section *3.6*), paying attention as the non-negativity conditions (3.27) be satisfied. Choose the value of safety, centering and barrier parameters.
2. Compute Newton direction by solving the system of equations (3.39).
3. Determine the step size length (3.50) and update the solution (3.46).
4. Compute the barrier parameter μ by (3.52).
5. If convergence criteria (3.53-3.56) are met, then optimal solution is found, otherwise go back to 2.

3.8 Predictor-corrector algorithm

We now describe the PCIPA which belongs to the family of higher-order IP methods [11, 13, 14]. The latter are motivated by the observation that the computational burden to factorize the extended Hessian matrix is generally much more expensive than the solution of the already factorized system. The aim of these methods is to yield an improved search direction by incorporating higher-order information into (3.33), and that with little additional computational effort.
Now instead of updating iteratively the unknown vector, say $\mathbf{y}$ as in the Newton method, we merely introduce the new point $\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta\mathbf{y}$ directly into the Newton system (3.39), obtaining:

$$\mathbf{H}\begin{bmatrix} \Delta\mathbf{s}_\ell \\ \Delta\mathbf{s}_u \\ \Delta\boldsymbol{\pi}_\ell \\ \Delta\boldsymbol{\pi}_u \\ \Delta\boldsymbol{\lambda} \\ \Delta\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mu\mathbf{S}_\ell^{-1}\mathbf{e} - \boldsymbol{\pi}_\ell - \mathbf{S}_\ell^{-1}\Delta\mathbf{S}_\ell\Delta\boldsymbol{\pi}_\ell \\ \mu\mathbf{S}_u^{-1}\mathbf{e} - \boldsymbol{\pi}_u - \mathbf{S}_u^{-1}\Delta\mathbf{S}_u\Delta\boldsymbol{\pi}_u \\ \mathbf{h}(\mathbf{x}) + \mathbf{h}(\Delta\mathbf{x}) - \mathbf{h}_\ell - \mathbf{s}_\ell \\ -\mathbf{h}(\mathbf{x}) - \mathbf{h}(\Delta\mathbf{x}) + \mathbf{h}_u - \mathbf{s}_u \\ \mathbf{g}(\mathbf{x}) + \mathbf{g}(\Delta\mathbf{x}) \\ -\nabla f(\mathbf{x}) + \nabla\mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T + \nabla\mathbf{h}(\mathbf{x})({\boldsymbol{\pi}_\ell}^T - {\boldsymbol{\pi}_u}^T) \end{bmatrix} \tag{3.60}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{S}_\ell^{-1}\boldsymbol{\Pi}_\ell & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_u^{-1}\boldsymbol{\Pi}_u & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nabla\mathbf{h}(\mathbf{x}) \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \nabla\mathbf{h}(\mathbf{x}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\nabla\mathbf{g}(\mathbf{x}) \\ \mathbf{0} & \mathbf{0} & -\nabla\mathbf{h}(\mathbf{x})^T & \nabla\mathbf{h}(\mathbf{x})^T & -\nabla\mathbf{g}(\mathbf{x})^T & \nabla^2_{\mathbf{x}}\mathcal{L}_\mu \end{bmatrix} \tag{3.61}$$

is the Hessian matrix and $\boldsymbol{\Pi}_\ell = diag(\pi_{\ell 1}, ..., \pi_{\ell p})$, $\boldsymbol{\Pi}_u = diag(\pi_{u1}, ..., \pi_{up})$.
What differs with respect to the Newton system (3.39) are the Δ terms from the right-hand side and the diagonal terms of the first two lines relative to the complementarity constraints. The latter modification results from the subsequent transformations (allowed because $s_0, s_0 + \Delta s > 0$ by definition):

$$-\mu(s_0 + \Delta s)^{-1} + \pi_0 + \Delta\pi = 0 \Leftrightarrow (s_0 + \Delta s)(\pi_0 + \Delta\pi) = \mu \Leftrightarrow$$
$$\pi_0\Delta s + s_0\Delta\pi = \mu - s_0\pi_0 - \Delta s\Delta\pi \Leftrightarrow s_0^{-1}\pi_0\Delta s + \Delta\pi = \mu s_0^{-1} - \pi_0 - s_0^{-1}\Delta s\Delta\pi$$

where s (resp. π) stands for any of $s_{\ell i}^k$ or s_{ui}^k. (resp. $\pi_{\ell i}^k$ or π_{ui}^k).
Observe that this system cannot be solved directly because the higher-order terms in (3.60) are not known in advance.
Merhotra proposes a two steps procedure involving a *predictor* and a *corrector* steps, which we describe in the sequel [11].

3.9 The predictor step

The predictor step objective is two-fold: to approximate higher-order terms in (3.60) and to dynamically estimate the barrier parameter μ.
To this purpose one solves the system (3.60) for the *affine-scaling* direction, obtained by neglecting in its right-hand side the higher-order terms and μ, that is:

$$\mathbf{H}\begin{bmatrix}\Delta\tilde{\mathbf{s}}_\ell\\ \Delta\tilde{\mathbf{s}}_u\\ \Delta\tilde{\boldsymbol{\pi}}_\ell\\ \Delta\tilde{\boldsymbol{\pi}}_u\\ \Delta\tilde{\boldsymbol{\lambda}}\\ \Delta\tilde{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}-\boldsymbol{\pi}_\ell\\ -\boldsymbol{\pi}_u\\ \mathbf{h}(\mathbf{x})-\mathbf{h}_\ell-\mathbf{s}_\ell\\ -\mathbf{h}(\mathbf{x})+\mathbf{h}_u-\mathbf{s}_u\\ \mathbf{g}(\mathbf{x})\\ -\nabla f(\mathbf{x})+\nabla\mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T+\nabla\mathbf{h}(\mathbf{x})(\boldsymbol{\pi}_\ell{}^T-\boldsymbol{\pi}_u{}^T)\end{bmatrix} \tag{3.62}$$

Next, the step length which would be taken along the affine scaling direction if the latter was used is determined as for the PDIPA (3.50).
The affine complementarity gap $\tilde{\rho}$ is then estimated:

$$\tilde{\rho}^k=(\mathbf{s}_\ell^k+\tilde{\alpha}_p\Delta\tilde{\mathbf{s}}_\ell)^T(\boldsymbol{\pi}_\ell^k+\tilde{\alpha}_d\Delta\tilde{\boldsymbol{\pi}}_\ell)+(\mathbf{s}_u^k+\tilde{\alpha}_p\Delta\tilde{\mathbf{s}}_u)^T(\boldsymbol{\pi}_u^k+\tilde{\alpha}_d\Delta\tilde{\boldsymbol{\pi}}_u) \tag{3.63}$$

Finally, an estimate $\tilde{\mu}^k$ for μ^{k+1} is obtained from:

$$\tilde{\mu}^k=\min\left\{\left(\frac{\tilde{\rho}^k}{\rho^k}\right)^2,0.2\right\}\frac{\tilde{\rho}^k}{2(m+p)} \tag{3.64}$$

The aim of this adaptive scheme is based on the principle: the larger the decrease in complementarity gap from affine direction, $\tilde{\rho}^k\ll\rho^k$, the larger the decrease of affine barrier parameter $\tilde{\mu}^k$, and vice-versa.

3.10 The corrector step

The actual Newton step is computed from:

$$\mathbf{H}\begin{bmatrix}\Delta\mathbf{s}_\ell\\ \Delta\mathbf{s}_u\\ \Delta\boldsymbol{\pi}_\ell\\ \Delta\boldsymbol{\pi}_u\\ \Delta\boldsymbol{\lambda}\\ \Delta\mathbf{x}\end{bmatrix}=\begin{bmatrix}+\tilde{\mu}^k\mathbf{S}_\ell^{-1}\mathbf{e}-\boldsymbol{\pi}_\ell-\mathbf{S}_\ell^{-1}\Delta\tilde{\mathbf{S}}_\ell\Delta\tilde{\boldsymbol{\pi}}_\ell\\ +\tilde{\mu}^k\mathbf{S}_u^{-1}\mathbf{e}-\boldsymbol{\pi}_u-\mathbf{S}_u^{-1}\Delta\tilde{\mathbf{S}}_u\Delta\tilde{\boldsymbol{\pi}}_u\\ \mathbf{h}(\mathbf{x})+\mathbf{h}(\tilde{\alpha}\Delta\tilde{\mathbf{x}})-\mathbf{h}_\ell-\mathbf{s}_\ell\\ -\mathbf{h}(\mathbf{x})-\mathbf{h}(\tilde{\alpha}\Delta\tilde{\mathbf{x}})+\mathbf{h}_u-\mathbf{s}_u\\ \mathbf{g}(\mathbf{x})+\mathbf{g}(\tilde{\alpha}\Delta\tilde{\mathbf{x}})\\ -\nabla f(\mathbf{x})+\nabla\mathbf{g}(\mathbf{x})\boldsymbol{\lambda}^T+\nabla\mathbf{h}(\mathbf{x})(\boldsymbol{\pi}_\ell{}^T-\boldsymbol{\pi}_u{}^T)\end{bmatrix} \tag{3.65}$$

It is valuable to remark that the predictor-corrector procedure involves at every iteration the solution of two linear systems of equations with different right-hand sides while relying on the same matrix factorization (done on the predictor step). The extra computational burden with respect to the PDIPA is only one solution of the corrector system of equations with the matrix already factorized and the additional test to compute $\tilde{\mu}^k$. Generally, this increase of elapsed time per iteration is largely offset by a reduction of the overall computing time, thanks to a diminution of iterations count, as it will be shown in Section 4.
Note finally that, as suggested in Ref [25], higher-order terms $\mathbf{h}(\tilde{\alpha}\Delta\tilde{\mathbf{x}})$ and $\mathbf{g}(\tilde{\alpha}\Delta\tilde{\mathbf{x}})$ are computed only if these functions are quadratic, otherwise the usual gain in computational time comparatively to PDIPA could disappear.

3.11 Sparsity techniques

We end this Section about IPM with a critical implementation issue. The major computational effort in IPM is the solution of the large, sparse linear system (3.39 as for PDIPA, or (3.39) as for PCIPA). It is therefore crucial to use efficient methods for their solution. It is required the use of a good solver of such system on the one hand, and an adequate order of equations and unknowns of the system, on the other hand.
As regards the solver we rely on the sparse object-oriented library SPOOLES [42]. The latter, to best of our knowledge, is not widely used but exhibits excellent performance of its linear algebra kernel (see also Netlib web-site: www.netlib.org).
Second, as the factorization of augmented Hessian matrix is by far the most computational burden task of an IPM iteration, the order of equations and unknowns should be done such that to minimize the number of fill-ins during the factorization and consequently to speed up matrix factorization. To this end three types of data structures have been proposed in the literature [7, 32, 30]. Inspired by the former we place the control variables in the following order $[P_g, Q_g, \phi, b, r, a_1, a_2]$ at the top of the vector $\mathbf{x}$. The latter is then filled for each bus i with state variables f_i, e_i and $\lambda_{P_i}, \lambda_{Q_i}$, the Lagrange multipliers corresponding to active and reactive power balance (see eq. 2.6). $[f_i, e_i, \lambda_{P_i}, \lambda_{Q_i}\ i \in \mathcal{N}]$ constitutes a nodal block of 4x4, similar to the nodal admittance matrix.

4 Numerical results

In this section we present some numerical results of three OPF applications while comparing both PDIP and PCIP algorithms performance. The OPF has been coded in C++ and runs under Cygwin environment. It has been tested on three test systems, namely a 60bus system which is a modified variant of the Nordic32 system, and the standard IEEE118 and IEEE300 systems. A summary of these test systems is given in Table 1. All tests have been carried out on a Pentium IV PC of 1.7-GHz and 512-Mb RAM. All simulations have been performed with the following parameters: $\gamma = 0.99995$, $\sigma^0 = 0.2$, $\delta = 0.2$, while tolerances used when checking convergence are $\epsilon_1 = 10^{-4}$, $\epsilon_2 = 10^{-6}$ and $\epsilon_\mu = 10^{-6}$ (see Section *3.5*). Note finally that in all cases that follow we start from a converged load flow solution (see Section *3.6*).

Table 1: Summary of test systems

system	nn	ng	nc	nb	nl	nt	no	np	ns
Nordic32	60	23	22	81	57	31	4	0	12
IEEE118	118	54	91	186	175	11	9	0	14
IEEE300	300	69	198	411	282	129	50	1	14

4.1 Minimization of generation cost

We now focus on minimizing the overall cost of all generators active output (2.4). The control variables are the active power of generators and the phase shifter angle (for the IEEE300 system only). The equality constraints are the bus active and reactive power flow equations (2.6), imposed voltage of generators (2.8) and imposed ratio of phase shifter (2.9). The inequality constraints include limits on: generators active power (2.13), branch currents (2.11) and phase shifter angle (2.18).
For IEEE118 and IEEE300 systems we have considered a quadratic cost curve for all generators participating at optimization, while for Nordic32 system we have chosen a linear cost curve ($c_{2i} = 0,\ \forall i \in \mathcal{P}$ in 2.4).

Table 2 displays the number of iterations to convergence and the CPU time [2] in seconds of the two algorithms under study as well as the number and the type of binding constraints at optimum and the size of the OPF (see 3.19-3.22).

Table 2: Iterations, CPU time, active constraints and OPF size

	iterations		CPU time		active constraints			OPF size		
system	PDIPA	PCIPA	PDIPA	PCIPA	P_g	I	total	n	m	p
Nordic32	14	8	0.34	0.25	14	5	19	141	142	170
IEEE118	10	9	0.51	0.51	3	0	3	254	289	323
IEEE300	12	6	1.26	0.70	4	4	8	657	669	769

Both algorithms behave extremley well in all cases, as it is known that this is the mildest among OPF sub-problems. In order to assess the robustness of algorithms to tackle this problem we intentionally worsen the conditions of the tests. For instance, in the Nordic32 system we have chosen the generators cost curve such that at optimum more than a half of them reach their minimal or maximal active power output whereas 5 branch currents are at their maximum value, which explains at some extent the slightly higher number of iterations needed. Also for the IEEE300 system, at the initial point to be optimized 6 branches are overloaded but both algorithms cope with this congestion.
As expected, due to the account of higher-order terms which improve centrality and therefore convergence, the PCIPA algorithm outperforms the PDIPA one in terms of computational time while converging to the same optimum.
For the IEEE300 system the presence of the phase shifter among the control variables not only improves the value of the objective function but is also beneficial for the algorithms performance. Thus, if we remove the phase shifter angle from the control variables list, the PDIPA needs one extra iteration to converge while the PCIPA three.
As regards the initial value of the barrier parameter we set $\mu^0 = 0.1$ if generators cost curves are quadratic, and $\mu^0 = 1$ if generators cost curves are linear. The idea behind this choice is that, for a given value of μ^0, more linear is the objective function, faster constraints may become active, fact that worsens the convergence. In other words, to ensure a rather smooth approach of generators active output limits when generators cost curves are linear, we penalize further these constraints during the optimization process, than if generators cost curves are quadratic. Moreover, in systems containing phase shifters we set $\mu^0 = 100$ in order to avoid sharp phase shifter angle variations at early iterations.
Last but not least, dual variables at optimum yield very precious information. They are nothing but the sensitivity of the objective to a small constraint shift. For instance, the dual variable (Lagrange multiplier) associated with each power flow equation represents the variation of the overall generation cost for an increment of load at that bus. They are called *nodal prices* and are used as a method of pricing in some deregulated electricity markets [2]. As regards the dual variables corresponding to inequality constraints, they are all zero, unless the constraint is active (according to the complementarity conditions).

4.2 Minimization of active power losses

We now concentrate on another OPF sub-problem, the minimization of active power losses, counted as the sum of active losses over all branches of the system (2.5). The control variables are slack generator active power, generators reactive power, OLTC transformers ratio and shunt susceptance. The equality constraints are the bus active and reactive power balance (2.6). The

[2] CPU time concerns the optimization process only except of processing data and display of results

inequality constraints concern bounds on: slack generator active power, generators reactive power (2.14), voltage magnitudes (2.12), OLTC transformer ratio (2.15) and shunt susceptance (2.17).
As regards bounds on voltages, we imposed that the latter stay between 0.95 pu and 1.05 pu in all busses.
Table 3 presents the results obtained with the two algorithms, the binding constraints at the optimum and the size of the OPF (see 3.19-3.22).

Table 3: Iterations, CPU time, active constraints and OPF size

	iterations		CPU time		active constraints					OPF size		
system	PDIPA	PCIPA	PDIPA	PCIPA	V	Q_g	r	b	total	n	m	p
Nordic32	11	8	0.31	0.26	21	0	0	4	25	161	120	102
IEEE118	13	10	0.59	0.54	15	12	0	4	31	313	236	196
IEEE300	16	14	1.59	1.46	60	23	1	4	88	733	600	434

Once again both algorithms perform well also for this problem which, owing to the nonlinear behavior of reactive power, is unanimously recognized as being among the toughest OPF sub-problem to optimize.
As bus voltages are free to vary between bounds, in order to reduce active losses both algorithms tend to increase voltages to their upper bound which explains the quite high number of active voltage constraints. Conversely, generators hitting their minimal reactive power limit, the tap changers and shunt compensation attaining their limits prevent some voltages to be further increased. The first situation is the case of IEEE118 and IEEE300 systems, where many generators have a (very) narrow reactive capability.
The Nordic32 was the easiest system to optimize because at the initial point all voltages belong to their allowable variation interval and most generators have rather large reactive capability. In contrast, IEEE300 was the hardest system to optimize due to the (very) tight reactive capability of many generators as well as the very broad voltage profile at the initial point, voltages being in the range of 0.92-1.08 pu while the admissible bounds are 0.95-1.05 pu.
Initial value of losses in the systems is 164.76 MW for the Nordic32, 132.86 MW for IEEE118 and 408.55 MW for IEEE300 respectively. After optimization the reduction of losses is of 7.90% for the Nordic32, 12.31% for IEEE118 and 5.37% for IEEE300 respectively.
For this problem we set the initial value of the barrier parameter to $\mu^0 = 0.01$. A value of $\mu^0 = 0.1$ provides also quite similar results.

4.3 Optimization by a full OPF

We now aim at minimizing, by means of a full OPF, the generation cost (2.4) by playing on all control variables described in Section *4.1* and *4.2*. We moreover added the load curtailment percentage (ϕ_i, $\forall i \in \mathcal{C}$) as control variable allowing a shedding up to 10 %. This optimization is subject to equality constraints (2.6 and 2.9) and inequality constraints (2.11-2.18).
Let us remark that in this case the active losses are implicitly taken into account by the reactive control variables (OLTC ratio, shunt susceptance, etc.) and assigned to the cheapest generator(s).
No load is curtailed in any case due to their prohibitive cost we have chosen. Clearly, if load shedding compensation prices are diminished some of them will be curtailed.
Table 4 presents the number of iterations to convergence, the CPU time for the two algorithms, the number and type of binding constraints at optimum, and the dimension of the OPF.
The higher number of active constraints than when solving the sub-problems of Section *4.1* and *4.2* separately is due to the larger number of control variable used, an important contribution

Table 4: Iterations, CPU time, active constraints and OPF size

	iterations		CPU time		active constraints								OPF size		
system	PDIPA	PCIPA	PDIPA	PCIPA	P_g	Q_g	ϕ	I	V	r	b	total	n	m	p
Nordic32	24	14	0.92	0.65	15	1	22	5	26	0	4	73	201	142	230
IEEE118	18	11	1.28	0.90	3	8	91	3	25	0	5	135	304	289	491
IEEE300	22	16	3.17	2.78	3	26	181	4	59	3	2	278	791	669	903

having those relative to load curtailment (see column ϕ).
The observations made in Section *4.1* and *4.2* concerning the performance of PDIPA and PCIPA are equally valid for the tests carried out with the full OPF.
We finally comment on the CPU time. The development of any software is a trade-off between computational speed and the clarity of its code. Up to now we have encouraged the latter to the detriment of the former. By a more compact programming it is thus room to reduce the computational time with 15-20 %.

5 Conclusion

This paper has presented and compared the performances of two interior point based algorithms, PDIPA and PCIPA. They have been able to solve successfully various OPF problems on three test systems of reasonable size. Both algorithms can rather easily handle problems when a significant number of active constraints is active at optimum. Admittedly, the number of binding constraints at optimum may slightly increase the number of iterations to convergence, feature which is more pronounced for the PDIPA. It is noteworthy to observe that the number of iterations to convergence is (very) little sensitive to the size of the system.
Our experience with these two algorithms confirms other results from the literature, that is, most of the time PCIPA generally outperforms PDIPA in terms of CPU time (and thereby iterations count), both converging to the same optimum. The opposite situation sometimes happens, especially for rather simple problems. In such a case both algorithms need almost the same number of iterations to converge, the PDIPA being therefore a little bit faster.
We finally mention that the versatility of our OPF was also revealed when testing other three typical objectives, namely the minimization of the cost to remove congestions, the minimization of the cost (or amount) of load curtailment in an infeasible case or to alleviate congestions, and the maximization of the loadability limit respectively. The conclusions drawn above equally hold for all these tests.

6 Reference layout

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